

Differential Operator Representations of Affine Kac-Moody Lie Algebras $A_2^{(1)}$ and $A_2^{(2)}$

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Abstract

Kac-Moody Lie algebras are infinite-dimensional Lie algebras whose representation theory is very similar to those of simple Lie algebras. Kac-Moody Lie algebras are of three types, namely finite type, indefinite type, and affine type based on the characteristics of the generalized Cartan matrix. Here we work with affine Kac-Moody algebras which are algebraic structures with determinant value of its generalized Cartan matrix zero. In this paper, we give a brief description of the root systems and the Dynkin diagrams of untwisted and twisted Kac-Moody algebras. Finally, we have calculated the irreducible representations of two affine untwisted and twisted Kac-Moody Lie algebras, namely $A_2^{(1)}$ and $A_2^{(2)}$ by first-order differential operators. Our results have potential implementations in two-dimensional conformal field theory, string theory, supersymmetric gauge theory, and many others.

References

- [1] V.G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, 1990.
- [2] R. V. Moody, *Lie algebras associated with generalized Cartan matrices*, 1967.

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Introduction and Literature Survey

- Kac [1] and Moody [2], in the year 1967, independently pioneered the development of a new class of infinite-dimensional Lie algebras, now referred to as Kac-Moody Lie algebras.
- In 1990, Kac published [3] a book, infinite-dimensional Lie algebras where he described the classification, structure and representations of all infinite-dimensional Lie algebras.
- The study of irreducible representations of these Kac-Moody Lie algebras helps in deeper understanding of the symmetries, physical properties, and mathematical structures associated with it.
- In 2022, Morozov et al. gave an algorithm for all the irreducible representations of simple Lie algebras by differential operators [4].
- We describe here a method to calculate the irreducible highest weight representation of affine Kac-Moody algebra using differential operators by considering two examples untwisted Kac-Moody algebra $A_2^{(1)}$ and twisted Kac-Moody algebra $A_2^{(2)}$.

Generalized Cartan matrix [3]

We call a matrix $A = (a_{ij})_{i,j \in \{1, \dots, n\}}$ is a generalized Cartan matrix (GCM) or a Kac-Moody Cartan matrix if it satisfies:

- ① $a_{ij} \leq 0$ for $i \neq j$;
- ② $a_{ii} = 2$ for $i = 1, \dots, n$;
- ③ $a_{ij} = 0$ iff $a_{ji} = 0$.

Kac-Moody algebras [3]

Suppose A be a GCM of rank r , then for every A , there exists a unique up to isomorphism a *realization* of A , that is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$.

Let $\mathfrak{g}_{aux}(A)$ denote the associated auxiliary Lie algebra defined by the generators e_i, f_i and $h_i \in \mathfrak{h}$ and having relations

$$[h_i, h_j] = 0$$

$$[ad_{e_i}]^{1-a_{ij}} e_j = 0$$

$$[h_i, e_j] = a_{ij} e_j$$

$$[ad_{f_i}]^{1-a_{ij}} f_j = 0$$

$$[h_i, f_j] = -a_{ij} f_j$$

$$[e_i, f_j] = \delta_{ij} h_j$$

for $i, j = 1, \dots, n$. Then there exists an embedding $\mathfrak{h} \rightarrow \mathfrak{g}_{aux}(A)$. Assume $\hat{I}$ to be the maximal ideal of $\mathfrak{g}_{aux}(A)$ that intersects $\mathfrak{h}$ trivially, that is, $\hat{I} \cap \mathfrak{h} = 0$.

Define a quotient algebra, $\mathfrak{g}(A) := \mathfrak{g}_{aux}(A)/\hat{I}$. The Lie algebra $\mathfrak{g}(A)$ is known as the **Kac-Moody algebra** associated with GCM A .

Types of Kac-Moody algebras [3]

The Kac-Moody algebras are divided into three classes:

- 1 **Finite:** if the GCM is positive definite.
- 2 **Affine:** if the GCM is positive semi-definite.
- 3 **Indefinite:** if the GCM is neither positive definite nor positive semi-definite.

Chevalley-Serre relations

Let $A = (a_{ij})_{i,j \in \{0, \dots, n\}}$ be the GCM of Kac-Moody algebra $\mathfrak{g}(A)$ having rank r . Then $\mathfrak{g}(A)$ can be constructed from a $3(r+1)$ Chevalley basis consisting of h_i, e_i , and f_i , where $i = 0, \dots, r$, which satisfies the following Cartan-Chevalley commutation relations:

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ij} e_j \\ [h_i, f_j] &= -a_{ij} f_j \\ [e_i, f_j] &= \delta_{ij} h_j. \end{aligned} \tag{1}$$

Also, these generators satisfy the following Serre relations:

$$\begin{aligned} [ad_{e_i}]^{1-a_{ij}} e_j &= 0 \\ [ad_{f_i}]^{1-a_{ij}} f_j &= 0. \end{aligned} \tag{2}$$

Highest weight representation [5]

Let $\rho : \mathfrak{g}(A) \rightarrow \text{End}(V)$ be a representation of a Kac-Moody algebra $\mathfrak{g}(A)$ on a vector space V over $\mathbb{C}$, and let λ be a linear functional on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$. Then the *weight space* of V with weight λ is the subspace V_λ given by:

$$V_\lambda = \{v \in V : \rho(H)(v) = \lambda(H)v \mid \forall H \in \mathfrak{h}\}.$$

- A *weight* of the representation V is a linear functional λ such that the corresponding weight space V_λ is nonzero.
- The nonzero elements of the weight space are called *weight vectors*.
- A weight λ for representation ρ is said to be the *highest weight* if for all weights λ_k of ρ , $\lambda \geq \lambda_k$.
- An irreducible highest weight representation is uniquely determined by its highest weight.

Representation of $A_2^{(1)}$

The affine untwisted Kac-Moody algebra $A_2^{(1)}$ has rank 2 and infinite dimension. The GCM of $A_2^{(1)}$ is

$$A_{ij} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

for $i, j = 0, 1, 2$. The positive root system Δ_+ of $A_2^{(1)}$ is

$$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, n\delta \pm \alpha_1, n\delta \pm \alpha_2, n\delta \pm (\alpha_1 + \alpha_2), n\delta \mid n = 1, 2, \dots\}.$$

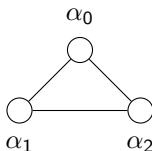
The simple roots of $A_2^{(1)}$ is $\{\alpha_0, \alpha_1, \alpha_2\}$ described as

$$\begin{aligned}\alpha_0 &= -(\alpha_1 + \alpha_2) + \delta, \\ \alpha_1 &= \varepsilon_1 - \varepsilon_2, \\ \alpha_2 &= \varepsilon_2 - \varepsilon_3.\end{aligned}\tag{3}$$

where $\{\alpha_1, \alpha_2\}$ are the simple root of Lie algebra $A_2 = \mathfrak{sl}_3$.

Representation of $A_2^{(1)}$

The Dynkin diagram for $A_2^{(1)}$ is



- The $A_2^{(1)}$ is generated by elements $\{e_0, e_1, e_2, f_0, f_1, f_2, h_0, h_1, h_2\}$ where $\{e_0, f_0, h_0\}$ are the affine Chevalley generators of $A_2^{(1)}$.
- Representations are determined by its highest weight λ .
- As rank of A_2 is 2, so its representation is described by two Dynkin labels (λ_1, λ_2) .
- The algebra has three positive root generators $e_1, e_2, e_3 = [e_1, e_2]$ and three negative root generators $f_1, f_2, f_3 = [f_2, f_1]$.
- Of them, the simple roots of A_2 are $\{e_1, e_2, f_1, f_2\}$.

Representation of $A_2^{(1)}$

A method is described to construct the representation of $A_2^{(1)}$ using differential operators which is derived from [4].

- **STEP 1:** The number of variables required for representation of $A_2^{(1)}$ is $|\Delta_+| = \frac{1}{2}(\dim A_2 - \text{rank } A_2) = 3$. Then set of variables are $\{X_1, X_2, X_3\}$. The two simple positive and negative root generators have the form as:

$$\begin{aligned}e_1 &= \frac{\partial}{\partial X_1} + A_{12} \frac{\partial}{\partial X_2} + A_{13} \frac{\partial}{\partial X_3} \\e_2 &= A_{21} \frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2} + A_{23} \frac{\partial}{\partial X_3} \\f_1 &= \lambda_1 X_1 - X_1^2 \frac{\partial}{\partial X_1} + B_{12} \frac{\partial}{\partial X_2} + B_{13} \frac{\partial}{\partial X_3} \\f_2 &= \lambda_2 X_2 + B_{21} \frac{\partial}{\partial X_2} - X_2^2 \frac{\partial}{\partial X_2} + B_{23} \frac{\partial}{\partial X_3}.\end{aligned}\tag{4}$$

where A_{ij} and B_{ij} are polynomial functions.

- **STEP 2:** Here, the highest vectors are set to 1:

$$|0\rangle_1 = 1$$

$$|0\rangle_2 = 1$$

Here we assume $i = 3 = \text{rank}(A_2) + 1 = |\Delta_+|$. So, we have to set some state to be X_3 . Possible yet undefined candidates are $f_3 |0\rangle_1$ or $f_3 |0\rangle_2$. They are both acceptable since all states $e_i f_3 |0\rangle_j$ are already defined:

$$e_1 f_3 |0\rangle_1 = f_2 |0\rangle_1 = 0 \qquad e_1 f_3 |0\rangle_2 = f_2 |0\rangle_2 = X_2 \qquad (5)$$

$$e_2 f_3 |0\rangle_1 = -f_1 |0\rangle_1 = -X_1 \qquad e_2 f_3 |0\rangle_2 = -f_1 |0\rangle_2 = 0 \qquad (6)$$

$$e_3 f_3 |0\rangle_1 = |0\rangle_1 = 1 \qquad e_3 f_3 |0\rangle_2 = -|0\rangle_2 = -1 \qquad (7)$$

Assuming $f_3 |0\rangle_2 = X_3$, while $f_3 |0\rangle_1$ is determined uniquely at the following stages of the algorithm.

Representation of $A_2^{(1)}$

- **STEP 3:** To find the form of e_1, e_2 we use the actions of the corresponding operators. We use the following conditions:

$$e_1 f_2 |0\rangle_2 = 0 \quad \Rightarrow A_{12}(X) = 0 \quad (8)$$

$$e_1 f_3 |0\rangle_2 = -f_2 |0\rangle_2 = -X_2 \quad \Rightarrow A_{13}(X) = -X_2 \quad (9)$$

$$e_2 f_1 |0\rangle_1 = 0 \quad \Rightarrow A_{21}(X) = 0 \quad (10)$$

$$e_2 f_3 |0\rangle_2 = 0 \quad \Rightarrow A_{23}(X) = 0 \quad (11)$$

We have completely determined the positive simple roots:

$$e_1 = \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_3} \quad (12)$$

$$e_2 = \frac{\partial}{\partial X_2} \quad (13)$$

Representation of $A_2^{(1)}$

- **STEP 4:** Here we determine $f_3 |0\rangle_1$ by solving following system of equations:

$$e_1 f_3 |0\rangle_1 = f_2 |0\rangle_1 = 0 \quad \Rightarrow \left(\frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_3} \right) (f_3 |0\rangle_1) = 0 \quad (14)$$

$$e_2 f_3 |0\rangle_1 = -f_1 |0\rangle_1 = -X_1 \quad \Rightarrow \left(\frac{\partial}{\partial X_2} \right) (f_3 |0\rangle_1) = X_1 \quad (15)$$

$$e_3 f_3 |0\rangle_1 = |0\rangle_1 = 1 \quad \Rightarrow \left(\frac{\partial}{\partial X_3} \right) (f_3 |0\rangle_1) = 1 \quad (16)$$

The answer to this system reads: $f_3 |0\rangle_1 = X_3 + X_1 X_2$.

- **STEP 5:** In this step, we find B_{ij} by following operations.

$$f_1(f_2 |0\rangle_2) = -X_3 \quad \Rightarrow B_{12}(X) = -X_3 \quad (17)$$

$$f_1(f_3 |0\rangle_2) = 0 \quad \Rightarrow B_{13}(X) = 0 \quad (18)$$

$$f_2(f_1 |0\rangle_1) = f_3 |0\rangle_1 = X_3 + X_1 X_2 \quad \Rightarrow B_{21}(X) = X_3 + X_1 X_2 \quad (19)$$

$$f_2(f_3 |0\rangle_2) = 0 \quad \Rightarrow B_{23}(X) = -X_2 X_3 \quad (20)$$

- **STEP 6:** The representation of simple negative root operators of $A_2^{(1)}$ are

$$f_1 = \lambda_1 X_1 - X_1^2 \frac{\partial}{\partial X_1} - X_3 \frac{\partial}{\partial X_2} \quad (21)$$

$$f_2 = \lambda_2 X_2 - X_2^2 \frac{\partial}{\partial X_2} + X_3 \frac{\partial}{\partial X_1} + X_1 X_2 \frac{\partial}{\partial X_1} - X_2 X_3 \frac{\partial}{\partial X_3} \quad (22)$$

where the non-simple roots are $e_3 = [e_1, e_2]$ and $f_3 = [f_2, f_1]$.

- **STEP 7:** We consider the affine Chevalley generators of $A_2^{(1)}$ as the following:

$$\begin{aligned} e_0 &= -f_3, \\ f_0 &= -e_3, \\ h_0 &= [e_0, f_0]. \end{aligned} \quad (23)$$

Representation of $A_2^{(1)}$

Therefore, the representation of the affine untwisted Kac-Moody algebra $A_2^{(1)}$ generators via differential operators is:

$$e_0 = -\lambda_1 X_3 - \lambda_1 X_1 X_2 - \lambda_2 X_3 + (X_1^2 X_2 + X_1 X_3) \frac{\partial}{\partial X_1} + X_2 X_3 \frac{\partial}{\partial X_2} + X_3^2 \frac{\partial}{\partial X_3} \quad (24)$$

$$f_0 = -\frac{\partial}{\partial X_3} \quad (25)$$

$$h_0 = -\lambda_1 - \lambda_2 + X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} + 2X_3 \frac{\partial}{\partial X_3} \quad (26)$$

$$e_1 = \frac{\partial}{\partial X_1} - X_2 \frac{\partial}{\partial X_3} \quad (27)$$

$$f_1 = \lambda_1 X_1 - X_1^2 \frac{\partial}{\partial X_1} - X_3 \frac{\partial}{\partial X_2} \quad (28)$$

$$e_2 = \frac{\partial}{\partial X_2} \quad (29)$$

$$f_2 = \lambda_2 X_2 + (X_3 + X_1 X_2) \frac{\partial}{\partial X_1} - X_2^2 \frac{\partial}{\partial X_2} - X_2 X_3 \frac{\partial}{\partial X_3}. \quad (30)$$

All these representations satisfy the Chevalley-Serre commutation relations given in (1) and (2).

Representation of $A_2^{(2)}$

The affine twisted Kac-Moody algebra $A_2^{(2)}$ has rank 1. The GCM of $A_2^{(2)}$ is

$$A'_{ij} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

for $i, j = 0, 1$. The two simple root of $A_2^{(2)}$ is α_0 and α'_1 . The positive root system Δ_+ of $A_2^{(2)}$ is

$$\{2n\alpha'_1 + n\alpha_0, 4n\alpha'_1 + (2n-1)\alpha_0, 4(n-1)\alpha'_1 + (2n-1)\alpha_0, (2n-1)\alpha'_1 + n\alpha_0, (2n-1)\alpha'_1 + (n-1)\alpha_0 \mid n = 1, 2, \dots\}$$

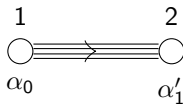
and the entire root system is given by $\Delta(A_2^{(2)}) = \Delta_+(A_2^{(2)}) \cup (-\Delta_+(A_2^{(2)}))$.

Representation of $A_2^{(2)}$

To construct the differential operator realizations of $A_2^{(2)}$, consider a graph automorphism σ that satisfies $\sigma^2 = 1$. The graph automorphism σ acts on the simple roots of A_2 as follows:

$$\sigma(\alpha_1) = \alpha_2, \sigma(\alpha_2) = \alpha_1. \quad (31)$$

The Dynkin Diagram for $A_2^{(2)}$ is given as



Here the generators of $A_2^{(2)}$ are $\{E_0, E_1, E_2, F_0, F_1, F_2, H_0, H_1, H_2\}$ which are obtained from $\{e_1, e_2, f_1, f_2, h_1, h_2\}$ of A_2 in a similar way as discussed above only an automorphism is involved which reverses α_1 and α_2 .

Representation of $A_2^{(2)}$

Therefore, the differential operator representation of the affine twisted Kac-Moody algebra $A_2^{(2)}$ is given below:

$$E_0 = \lambda_1 X_3 + \lambda_1 X_1 X_2 + \lambda_2 X_3 - (X_1^2 X_2 + X_1 X_3) \frac{\partial}{\partial X_1} - X_2 X_3 \frac{\partial}{\partial X_2} - X_3^2 \frac{\partial}{\partial X_3} \quad (32)$$

$$F_0 = \frac{\partial}{\partial X_3} \quad (33)$$

$$H_0 = -\lambda_1 - \lambda_2 + X_1 \frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} + 2X_3 \frac{\partial}{\partial X_3} \quad (34)$$

$$E_1 = \sqrt{2} \left(\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2} - X_2 \frac{\partial}{\partial X_3} \right) \quad (35)$$

$$F_1 = \sqrt{2} \left(\lambda_1 X_1 + \lambda_2 X_2 + (X_1 X_2 + X_3 - X_1^2) \frac{\partial}{\partial X_1} - (X_2^2 + X_3) \frac{\partial}{\partial X_2} - X_2 X_3 \frac{\partial}{\partial X_3} \right) \quad (36)$$

$$H_1 = 2\lambda_1 + 2\lambda_2 - 2X_1 \frac{\partial}{\partial X_1} - 2X_2 \frac{\partial}{\partial X_2} - 4X_3 \frac{\partial}{\partial X_3} \quad (37)$$

$$E_2 = \sqrt{2} \left(\lambda_1 X_1 - \lambda_2 X_2 - (X_1^2 + X_1 X_2 + X_3) \frac{\partial}{\partial X_1} + (X_2^2 - X_3) \frac{\partial}{\partial X_2} + X_2 X_3 \frac{\partial}{\partial X_3} \right) \quad (38)$$

$$F_2 = \sqrt{2} \left(-\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2} + X_2 \frac{\partial}{\partial X_3} \right) \quad (39)$$

$$H_2 = \lambda_1 - \lambda_2 - 3X_1 \frac{\partial}{\partial X_1} + 3X_2 \frac{\partial}{\partial X_2}. \quad (40)$$

Conclusion

- The differential operator representations provided satisfy the Chevalley-Serre commutation relations establishing the correctness of the result.
- These explicit representations fill an important gap in our understanding of the structures of these Kac-Moody algebras.
- In a similar fashion, we can calculate the representations of other affine untwisted and twisted Kac-Moody Lie algebras.

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References



V. G. Kac.

Simple graduated Lie algebras of finite growth.

Funktsional'nyi Analiz i ego Prilozheniya, 1(4):82–83, 1967.



R. V. Moody.

Lie algebras associated with generalized cartan matrices.

Bulletin of the American Mathematical Society, 1967.



V. G. Kac.

Infinite-dimensional Lie algebras.

Cambridge University Press, Cambridge, UK, 1990.



A. Morozov, M. Reva, N. Tselousov, and Y. Zenkevich.

Irreducible representations of simple lie algebras by differential operators.

The European Physical Journal C, 81(10):898, 2021.



L. Frappat, A. Sciarrino, and P. Sorba.

Dictionary on lie algebras and superalgebras.

HAL, 2000.

