An Improved S-Transform for Time-Frequency Analysis

Sitanshu Sekhar Sahu*, Ganapati Panda* and Nithin V George*
*Department of Electronics and Communication Engineering
National Institute of Technology, Rourkela,India
Email: sitanshusekhar@gmail.com,ganapati.panda@gmail.com and nithinvgeorge@gmail.com

***Abstract*—The time- frequency representation (TFR) has been used as a powerful technique to identify, measure and process the time varying nature of signals. In the recent past S-transform gained a lot of interest in time-frequency localization due to its superiority over all the existing identical methods. It produces the progressive resolution of the wavelet transform maintaining a direct link to the Fourier transform. The S-transform has an advantage in that it provides multi resolution analysis while retaining the absolute phase of each frequency component of the signal. But it suffers from poor energy concentration in the time-frequency domain. It gives degradation in time resolution at lower frequency and poor frequency resolution at higher frequency. In this paper we propose a modified Gaussian window which scales with the frequency in a efficient manner to provide improved energy concentration of the S-transform. The potentiality of the proposed method is analyzed using a variety of test signals. The results of the study reveal that the proposed scheme can resolve the time-frequency localization in a better way than the standard S-transform.**

I. Introduction

Now a days non-stationary time series such as speech, electrocardiograms, seismic signal and genomic signal are gaining more importance as they contain time-bounded events and artifacts. Indepth study of such events requires the determination of time-local spectra. During the last few years many methods of determining local spectra have been proposed. These include time-scale approach as in the continuous wavelet transform (CWT) and time-frequency approach as in the short-time Fourier transform (STFT). The main disadvantages of the STFT are its inability to detect and resolve low frequencies, and poor time resolution of high frequency events. Both these problems are due to the fixed width of the STFT window. Improved performance is observed by the wavelet transform, but it produces time-scale plots that are unsuitable for intuitive visual analysis. Also the absolute referenced phase information cannot be deduced from the wavelet transform. A superior method for time-frequency analysis known as S-transform has been evolved which enjoys the advantages of both STFT and wavelet transform. The basic idea is to obtain a time -frequency energy distribution of the signal so that we can isolate and process independently the components of the signal in the time-frequency plane. The S-transform, introduced by Stockwell, Mansinha, and Lowe [1], suitably combines the strengths of the STFT and wavelet transforms. It is the hybrid of short-time Fourier and Wavelet analyses. It employs a scalable and variable window length and uses the Fourier kernel to provide the phase information referenced to the time origin. Hence, it provides supplementary information about spectra which is not available from locally referenced phase obtained by the continuous wavelet transform. The S-transform has found extensive applications in many fields such as Geophysics [2] , Biomedical Engineering [7],Power transformer protection [5][6], Power quality analysis [12], oceanography , atmospheric physics , medicine, hydrogeology and mechanical engineering [11] .

One problem associated with S-transform is the nature of the Gaussian window, which leads to degradation of the time resolution of event onsets and offsets. It suffers from poor energy concentration in the time-frequency domain. Several improvements of the time-frequency representation of the S-transform have been reported. A variant of the original S-transform introduced by Mansinha et al.[1] has been dealt in many papers. A generalized S-transform proposed by Mc.Fadden et al. [11] provides a better control of the window function and indicates the use of non symmetric window for resolution. Pinnegar et al. proposed several different Gaussian windows that vary in shape and scale, modulates with amplitude and phase and another variant of Gaussian i.e hyperbolic window [13]- [15]. Pinnegar has also proposed a bi-Gaussian window by joining two non symmetric half Gaussian windows. All these variations provide the improvements in same respects i.e some results are better in time resolution but fails in frequency domain and vice-versa. In some other applications it demands better resolution in both time and frequency.In this paper we propose a method in the motive of improvement in the time-frequency resolution by optimizing the energy distribution in the time-frequency plane in order to minimize the smearing of signal components in both time and frequency. The improvement is achieved by introducing a new parameter in the original S-transform which has a greater control over the window width. The proposed scheme has been tested using a set of synthetic signals and compared with the standard S-transform and STFT. It reveals better performance than both the methods.

This paper is organized as follows. Section II deals with the standard and generalized S-transform. Section III proposes the modified scheme of S-transform window. Section IV contains the performance evaluation of the proposed scheme using some synthetic signals. Finally the conclusion of the paper is reported in section V.

II. The S-Transform

A. The Standard S-Transform

The S-transform is a time-frequency analysis technique proposed by Mansinha et al. combines both properties of the short time Fourier transform and wavelet transform. It provides frequency dependent resolution while maintaining a direct relationship with the Fourier spectrum. The S-Transform of a signal $x(t)$ is defined as

$$S(\tau, f) = \int_{-\infty}^{\infty} x(t) w(\tau - t) e^{-j2\pi f t} dt \quad (1)$$

where the window function is a scalable Gaussian window

$$w(t, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \quad (2)$$

and

$$\sigma(f) = \frac{1}{|f|} \quad (3)$$

Combining equation (2) and (3) gives

$$S(\tau, f) = \int_{-\infty}^{\infty} x(t) \left\{ \frac{|f|}{\sqrt{2\pi}} e^{-\frac{(\tau-t)^2 f^2}{2}} e^{-j2\pi f t} \right\} dt \quad (4)$$

The advantage of S-transform over the short time Fourier transform is that the standard deviation $\sigma(f)$ (window width) is a function of f rather than a fixed one as in STFT. In contrast to wavelet analysis the S-Transform wavelet is divided into two parts as shown within the braces of equation (4).One is the slowly varying envelope (the Gaussian window) which localizes the time and the other is the oscillatory exponential kernel $e^{-j2\pi f t}$ which selects the frequency being localized. It is the time localizing Gaussian that is translated while keeping the oscillatory exponential kernel stationary which is different from the wavelet kernel. As the oscillatory exponential kernel is not translating , it localizes the real and the imaginary components of the spectrum independently, localizing the phase as well as amplitude spectrum. Thus it retains absolute phase of the signal which is not provided by Wavelet Transform. The standard S-Transform provides unnecessary restrictions on the window function used. In fact the Gaussian window has no parameter to allow its width in time or frequency to be adjusted. Hence Mc Fadden et al. [11] and later Pinnegar [14] - [15] introduced a generalized S-Transform which has a greater control over the window function.

B. The Generalised S-Transform

The generalized S-transform is given by

$$S(\tau, f, \beta) = \int_{-\infty}^{\infty} x(t) w(\tau - t, f, \beta) e^{-j2\pi f t} dt \quad (5)$$

where w is the window function of the S-transform and β denotes the set of parameters that determine the shape and property of the window function. The window satisfies the normalized condition

$$\int_{-\infty}^{\infty} w(t, f, \beta) dt = 1 \quad (6)$$

The alternative expression of (5) by using the covolution theorem through the Fourier transform can be written as

$$S(\tau, f, \beta) = \int_{-\infty}^{\infty} X(\alpha + f) W(\alpha, f, \beta) e^{j2\pi\alpha\tau} d\alpha \quad (7)$$

where

$$X(\alpha + f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi(\alpha+f)t} dt \quad (8)$$

and

$$W(\alpha, f, \beta) = \int_{-\infty}^{\infty} w(t, f, \beta) e^{-j2\pi\alpha t} dt \quad (9)$$

The variable α and f in the above expression have the same units.

III. The proposed Scheme

In this scheme we retain the window function as the same Gaussian window because it satisfies the minimum value of the uncertainty principle. We have introduced an additional parameter δ into the Gaussian window where its width varies with frequency as follows

$$\sigma(f) = \frac{\delta}{|f|} \quad (10)$$

Hence the generalized S-transform becomes

$$S(\tau, f, \delta) = \int_{-\infty}^{\infty} x(t) \frac{|f|}{\sqrt{2\pi}\delta} e^{-\frac{(\tau-t)^2 f^2}{2\delta^2}} e^{-j2\pi f t} dt \quad (11)$$

where the Gaussian window becomes

$$w(t, f, \delta) = \frac{|f|}{\sqrt{2\pi}\delta} e^{-\frac{t^2 f^2}{2\delta^2}} \quad (12)$$

and its frequency domain representation is

$$W(\alpha, f, \delta) = e^{-\frac{2\pi^2\alpha^2\delta^2}{f^2}} \quad (13)$$

The proposed S-Transform can be computed similarly as in Eq. (7)

The adjustable parameter δ represents the number of periods of Fourier sinusoid that are contained within one standard deviation of the Gaussian window. The time resolution i.e. the event onset and offset time and frequency smearing is controlled by the factor δ. If δ is too small the Gaussian window retains very few cycles of the sinusoid. Hence the frequency resolution degrades at higher frequency. If δ is too high the window retains more sinusoids within it as a result the time resolution degrades at lower frequencies. It indicates that the δ value should be varied judiciously so that it would give better energy distribution in time-frequency plane. The tradeoff between the time-frequency resolution can be reuced by optimally varying the window width with the parameter δ.The variation of width of window with δ for a particular frequency componet (25) Hz is shown in Figure (1).

At lower δ value ($\delta < 1$) the window widens more with less sinusoids within it , thereby catches the low frequency components effectively. At higher δ value ($\delta > 1$) the window width decreases more with more sinusoids within it ,thereby resolves the high frequency components better. In this paper

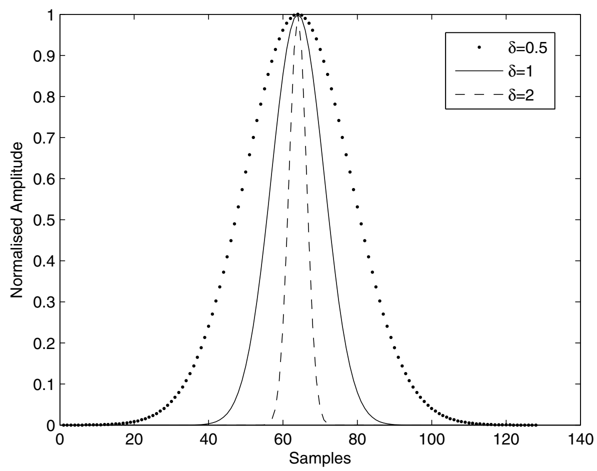

Fig. 1. Variation of window width with δ for a particular frequency$(25Hz)$

we varied the parameter δ linearly with frequency within a certain range as given by

$$\delta(f) \approx tf \tag{14}$$

where t is the slope of the linear curve.

The discrete version of (11) is used to compute the discrete S-Transform by taking the advantage of the efficiency of the fast Fourier transform (FFT) and the convolution theorem.

The Discrete S-Transform

Consider $x(kT), k = 0, 1, \cdots, N$ be the discrete time series corresponding to $x(t)$, with the sampling interval of T. So the discrete fourier transform of $x(kT)$ is given by

$$X\left[\frac{n}{NT}\right] = \frac{1}{N}\sum_{k=0}^{N-1} x\left[kT\right] e^{-\frac{j2\pi nk}{N}} \tag{15}$$

where $n = 0, 1, \cdots, N-1$

Using the discrete form of (11) the modified S-transform of the discrete signal $x(kT)$ is given by (letting $f \longrightarrow n/NT$ and $\tau \longrightarrow uT$)

$$S\left[\frac{n}{NT}, uT\right] = \sum_{m=0}^{N-1} X\left[\frac{m+n}{NT}\right] H(n,m) e^{-\frac{j2\pi mu}{N}} \tag{16}$$

when $n = 0$ the S-Transform $S(0, uT)$ is defined as

$$S\left[0, uT\right] = \frac{1}{N}\sum_{m=0}^{N-1} x\left[\frac{m}{NT}\right] \tag{17}$$

where u, m and $n = 0, 1, \cdots, N-1$ and $H(n,m) = e^{-\frac{2\pi^2 m^2}{n^2}}$,the Gaussian function. This is equal to the average of the time domain signal.

To compute the discrete form of the proposed S-transform the following steps are to be adapted.

1) Perform the discrete Fourier transform of the time series $x(kT)$(with N points and sampling interval T) to get $X\left[\frac{m}{NT}\right]$ using the FFT routine. This is computed once.
2) Calculate the localizing Gaussian $H\left[n, m\right]$ for the required frequency n/NT.
3) Shift the spectrum $X\left[\frac{m}{NT}\right]$ to $X\left[\frac{m+n}{NT}\right]$ for the frequency n/NT.
4) Multiply $X\left[\frac{m+n}{NT}\right]$ by $H\left[n, m\right]$ to get $B\left[\frac{n}{NT}, \frac{m}{NT}\right]$.
5) Inverse Fourier transform of $B\left[\frac{n}{NT}, \frac{m}{NT}\right]$ to give the row of $S\left[\frac{n}{NT}, uT\right]$ corresponding to the frequency $\frac{n}{NT}$.
6) Repeats steps 3, 4 and 5 until all the rows of $S\left[\frac{n}{NT}, uT\right]$ corresponding to all discrete frequencies $\frac{n}{NT}$ have been defined.

IV. Performance Analysis

In this section the performance of the proposed method is analysed using some synthetic test signals. In the first test a signal containing a low frequency (7 Hz), a medium frequency (25 Hz) and a high frequency (65 Hz) burst is taken. All these components are short lived and present in different time and also a zero signal component present at time t=0.23 sec. The signal is shown in Figure (2) and given by

$x_1 = zeros(1, 256)$
$t1 = 1 : 70$
$x_1(1 : 70) = cos(2 * pi * t1 * 7/256)$
$x_1(71 : 128) = 0$
$t2 = 1 : 128$
$x_1(129 : 256) = cos(2 * pi * t2 * 25/256)$
$t3 = 30 : 60$
$x_1(30 : 60) = x_1(30 : 60) + 0.5 * cos(2 * pi * t3 * 65/256)$

The spectrum of the signal obtained by STFT, the standard S-transform and the proposed scheme is shown in Figures (3)-(5).The STFT provides uniform frequency resolution but poor time resolution for all the frequency components. The standard S-transform results perfect time resolution at high frequency but fails in low frequency and good frequency resolution at low frequency but smears in higher frequency as seen from the vertical stretching of time-frequency signatures. All these defects are somehow overcomed by our proposed scheme of S-transform. It provides better energy concentration in both time and frequency direction.

In the second test a more complex signal is taken similar to that used by Sejdic [16] as shown in Figure (6) is given by

$$x_2(t) = cos(25\pi log(10t+1)) + cos(25\pi t + 80\pi t^2) \tag{18}$$

This signal is chosen because it contains hyperbolic and chrip frequency components which are crossed to each other.The frequency of the hyperbolic component decreases while the frequency of the linear chrip increases in time. So it is difficult to provide good resolution for both the components.The performance is compared for the three methods as shown in Figures (7) - (9).It reveals that both the STFT and standard S-Transform fails to provide good resolution at the high frequency of the hyperbolic and chrip component.

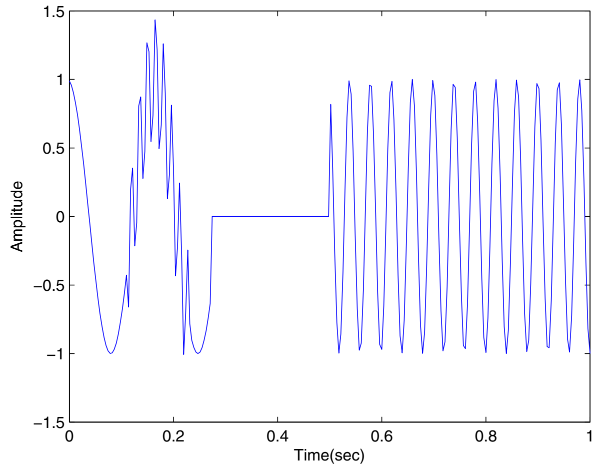

Fig. 2. First Test signal $x_1(t)$

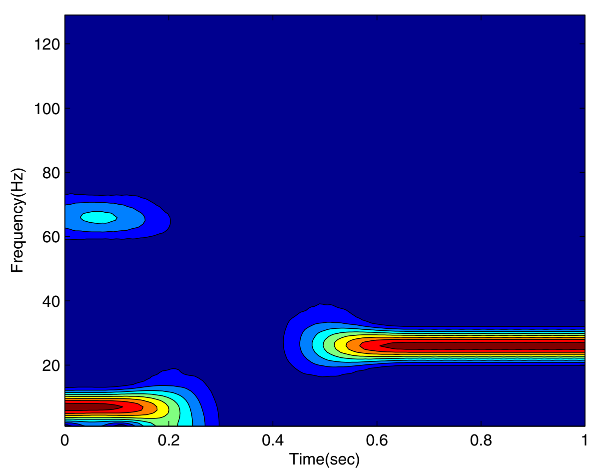

Fig. 3. STFT of $x_1(t)$

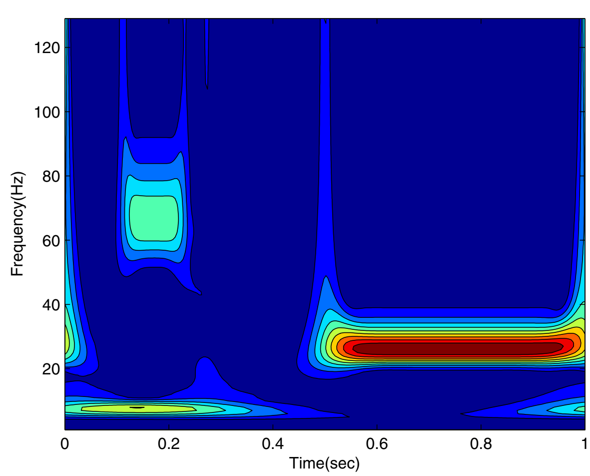

Fig. 4. Standard S-Transform of $x_1(t)$

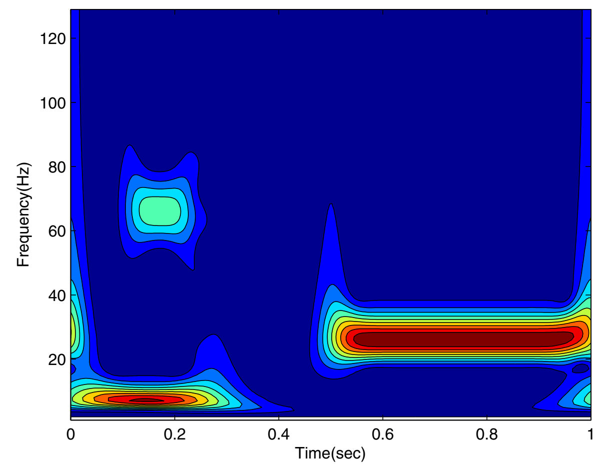

Fig. 5. Proposed S-Transform of $x_1(t)$

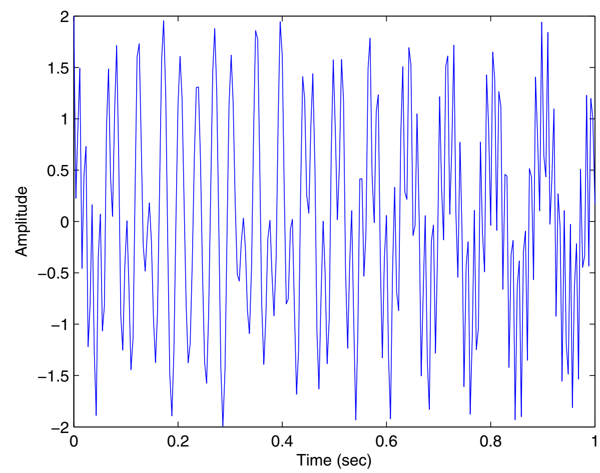

Fig. 6. Second Test signal $x_2(t)$

The proposed S-Transform shows better resolution at high frequency of both components.

V. Conclusion

In this paper we have proposed a modified S-Transform with improved time-frequency resolution. This has been achieved by introducing a modified Gaussian window which scales with the frequency in an efficient manner such that it provides improved energy concentration of the S-Transform. The effective variation of the width of the Gaussian window has a better control over the energy concentration of the S-Transform. This has been possible by introducing an additional parameter (δ) in the window which varies with frequency and thereby modulates the S-Transform kernel efficiently with the progress of frequency. The proposed scheme is evaluated and compared with the standard S-transform and STFT by using a set of synthetic test signals. The comparison shows that the proposed method is superior to the standard one as well as STFT providing a better time and frequency resolution. Hence the proposed S-transform can be widely used for analysis of all kinds of signal that need better time as well as frequency resolutions.

ACKNOWLEDGMENT

The authors were extremely benefited from frequent discussions with Dr. Lalu Mansinha of University of Western Ontario,Canada.

REFERENCES

[1] Stockwell RG, Mansinha L, Lowe RP *Localisation of the complex spectrum: the S transform*, IEEE Trans Signal Processing 1996;44(4),pp:998-1001.

[2] Stockwell, R. G. , *S-Transform Analysis of Gravity Wave Activity*, Ph.D. Dissertation, Dept. of Physics and Astronomy, The University of Western Ontario, London, Ontario, Canada,1999.

[3] Michael R. Portnoff, *Time-Frequency Representation of Digital Signals and Systems Based on - Short-Time Fourier Analysis*, IEEE Transactions On Acoustics, Speech, And Signal Processing, Vol. Asp–28, No. 1, February 1980.

[4] Ingrid Daubechies,*The Wavelet Transform, Time-Frequency Localization and Signal Analysis*, IEEE Trans. On Information Theory, Vol.36, No. 5,pp: 961–1005, Sep 1990.

[5] P.K. Dash, S.R. Samantaray, G. Panda and B.K. Panigrahi, *Power transformer protection using S-transform with complex window and pattern recognition approach*, IET Gener. Transm. Distrib., Vol. 1, pp:278–286, 2007.

[6] P. K. Dash, B. K. Panigrahi, D. K. Sahoo, and G. Panda, *Power Quality Disturbance Data Compression, Detection, and Classification Using Integrated Spline Wavelet and S-Transform*, IEEE Transactions On Power Delivery, Vol. 18, pp:595–600, April 2003.

[7] P Rakovi, E Sejdi, LJ Stankovi, J Jiang, *Time-Frequency Signal Processing Approaches with Applications to Heart Sound Analysis*, Computers in Cardiology ,Vol:33,pp:197200,2006.

[8] C.R. Pinnegar, *Timefrequency and timetime filtering with the S-transform and TT-transform*,Digital Signal Processing 15 (2005) 604-620.

[9] Lus B. Almeida, *The fractional Fourier transform and time-frequency representations*, IEEE Trans. Signal Processing 42 (11), pp:3084-3091,1994.

[10] K. F. Tiampo, Dawit Assefa, J. Ferndez,L. Mansinha, and H. Rasmussen,*Postseismic Deformation Following the 1994 Northridge Earthquake Identified Using the Localized Hartley Transform Filter*,Pure Applied Geophysics. 165 (2008) 1577-1602.

[11] P. D.McFadden, J. G. Cook, and L. M. Forster, *Decomposition of gear vibration signals by the generalized S-transform*,Mechanical Systems and Signal Processing, vol. 13, no. 5, pp.691-707, 1999.

[12] P. K. Dash, B. K. Panigrahi, and G. Panda,*Power quality analysis using S-transform*, IEEE Transactions on Power Delivery,vol. 18, no. 2, pp. 406411, 2003.

[13] C. R. Pinnegar and L. Mansinha, *Time-local Fourier analysis with a scalable, phase-modulated analyzing function: the Stransform with a complex window*, Signal Processing, vol. 84,no. 7, pp. 11671176, 2004.

[14] C. R. Pinnegar and L. Mansinha, *The bi-Gaussian Stransform*,SIAM Journal of Scientific Computing, vol. 24,no. 5, pp. 16781692, 2003.

[15] C. R. Pinnegar and L. Mansinha, *The S-transform with windows of arbitrary and varying shape*, Geophysics, vol. 68, no. 1,pp. 381385, 2003.

[16] I. Djurovic, E. Sejdic, and J. Jiang, *Frequency based window width optimization for S-transform*, AEU International Journal of Electronics and Communications, submitted.

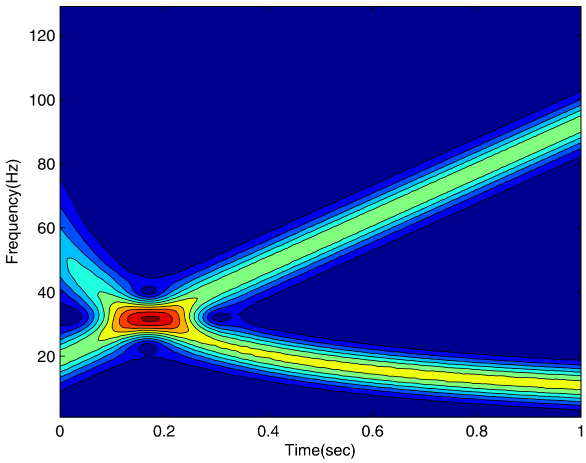

Fig. 7. STFT of $x_2(t)$

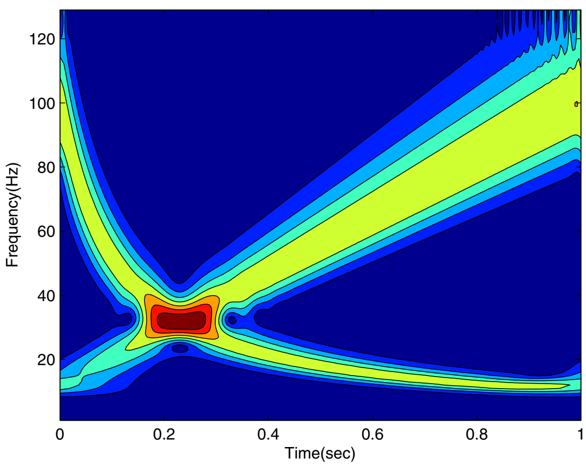

Fig. 8. Standard S-Transform of $x_2(t)$

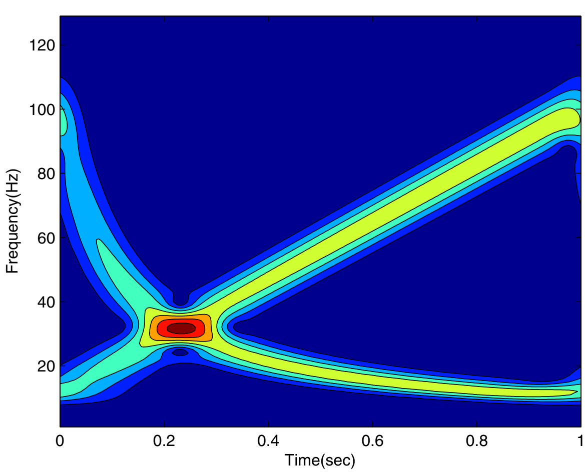

Fig. 9. Proposed S-Transform of $x_2(t)$