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Iteration of certain meromorphic functions with unbounded singular values

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Abstract. Let $\mathcal{M} = \{f(z) = (z^m/\sinh^m z)$ for $z \in \mathbb{C} \mid$ either m or $m/2$ is an odd natural number$\}$. For each $f \in \mathcal{M}$, the set of singularities of the inverse function of f is an unbounded subset of the real line $\mathbb{R}$. In this paper, the iteration of functions in one-parameter family $\mathcal{S} = \{f_\lambda(z) = \lambda f(z) \mid \lambda \in \mathbb{R} \setminus \{0\}\}$ is investigated for each $f \in \mathcal{M}$. It is shown that, for each $f \in \mathcal{M}$, there is a critical parameter $\lambda^* > 0$ depending on f such that a period-doubling bifurcation occurs in the dynamics of functions f_λ in $\mathcal{S}$ when the parameter $|\lambda|$ passes through λ^*. The non-existence of Baker domains and wandering domains in the Fatou set of f_λ is proved. Further, it is shown that the Fatou set of f_λ is infinitely connected for $0 < |\lambda| \leq \lambda^*$ whereas for $|\lambda| \geq \lambda^*$, the Fatou set of f_λ consists of infinitely many components and each component is simply connected.

1. *Introduction*

Let $f : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be a non-constant transcendental meromorphic function. The set of points $z \in \widehat{\mathbb{C}}$ for which the sequence of iterates $\{f^n(z)\}_{n=0}^{\infty}$ is defined and forms a normal family is called the Fatou set of f and is denoted by $\mathcal{F}(f)$. The Julia set, denoted by $\mathcal{J}(f)$, is the complement of the Fatou set of f in $\widehat{\mathbb{C}}$. It is well known that the Fatou set is open and the Julia set is a perfect set. Let $\text{sing}(f^{-1})$ denote the set of finite singularities of the inverse function f^{-1} of the function f (also called singular values of f). Then, $\text{sing}(f^{-1})$ is the set of critical and finite asymptotic values of f and finite limit points of these values. Denote by $\text{sing}(f^{-p})$ the set of finite singularities of the inverse function of f^p. Let $A_k(f) = \{z \in \mathbb{C} \mid f^k$ is not analytic at $z\}$ and define

$$S_p(f) = \bigcup_{k=0}^{p-1} f^k(\text{sing}(f^{-1}) \setminus A_k(f)) \quad \text{and} \quad P(f) = \bigcup_{p=1}^{\infty} S_p(f). \tag{1}$$

It is easy to see that $\text{sing}(f^{-p}) \subseteq S_p(f) \subseteq S_{p+1}(f)$ and the set $P(f)$ consists of the forward orbits of all points in $\text{sing}(f^{-1})$ as long as they are defined and finite. Let B denote the class of all meromorphic functions f for which $\text{sing}(f^{-1})$ is a bounded set.

The existence of Baker domains and wandering domains is one of the important dynamical aspects of transcendental meromorphic functions and has been investigated [**1, 5, 6, 8, 15, 16, 18, 20, 23**]. Rippon and Stallard proved the non-existence of Baker domains with period p in the Fatou set of transcendental meromorphic functions f for which the set $S_p(f)$ is bounded [**19**]. Non-existence of wandering domains for meromorphic functions f of finite type (i.e., f for which $\text{sing}(f^{-1})$ is a finite set) is established by Baker *et al* [**3**]. A number of one-parameter families of meromorphic functions of finite type are investigated by Keen and Kotus [**9**], Keen *et al* [**14**], Jiang [**13**] and Prasad *et al* [**11**]. Zheng [**22, 23**] investigated the relations between $P(f)$ and the limit functions of iterates $\{f^n\}_{n>0}$ in a Fatou component and proved the non-existence of Baker domains and wandering domains for certain meromorphic functions in the class B. However, the dynamics of meromorphic functions outside the class B is largely unexplored.

Let

$$\mathcal{M} = \left\{ f(z) = \frac{z^m}{\sinh^m z} \text{ for } z \in \mathbb{C} \;\middle|\; m \text{ or } m/2 \text{ is an odd natural number} \right\}.$$

For each $f \in \mathcal{M}$, consider the one-parameter family of functions

$$\mathcal{S} = \{f_\lambda(z) = \lambda f(z) \mid \lambda \in \mathbb{R} \setminus \{0\}\}.$$

In this paper, the iteration of functions f_λ in the one-parameter family $\mathcal{S}$ is investigated.

Observe that $f_\lambda(z)$ is an even function. If $\lambda \in \mathbb{R} \setminus \{0\}$ then $f_\lambda(z) = -f_{-\lambda}(-z)$ and $f_\lambda^n(z) = -f_{-\lambda}^n(-z)$ for $z \in \mathbb{C}$ and $n \in \mathbb{N}$. It shows that the functions f_λ and $f_{-\lambda}$ are conformally conjugate and the dynamics of f_λ and $f_{-\lambda}$ are essentially same. Therefore, we prove the results on the dynamics of the functions $f_\lambda \in \mathcal{S}$ for $\lambda > 0$.

In §2, it is mainly shown that $\text{sing}(f_\lambda^{-1})$ is an unbounded subset of the real line. The dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is investigated in §3. We show that there is a critical parameter $\lambda^* > 0$ (depending on f) such that a period-doubling bifurcation occurs in the dynamics of functions f_λ in $\mathcal{S}$ when the parameter $|\lambda|$ passes through λ^*. In §4, the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ is studied. The non-existence of Baker domains and wandering domains in the Fatou set of f_λ is also proved. There is a change in topology of the Fatou components effectuated by the above mentioned bifurcation which is described in §5.

2. *Properties of* f_λ

The function $f_\lambda(z) = \lambda(z^m/\sinh^m z)$ is meromorphic with poles at $\{i\pi k \mid k \in \mathbb{Z} \setminus \{0\}\}$. All the poles are multiple if $m > 1$ and simple if $m = 1$. Further, the function $f_\lambda(z)$ is even and not periodic. In Proposition 2.1, we prove that the Julia set of f_λ is symmetric with respect to both the real and imaginary axes. The point $z = 0$ is an omitted value of f_λ and hence an asymptotic value of $f_\lambda(z)$. More importantly, it is shown that $\text{sing}(f_\lambda^{-1})$ is an unbounded subset of the real line in Proposition 2.2.

PROPOSITION 2.1. *Let $f_\lambda \in \mathcal{S}$. If $z \in \mathcal{J}(f_\lambda)$ then $-z \in \mathcal{J}(f_\lambda)$ and $\bar{z} \in \mathcal{J}(f_\lambda)$.*

Proof. Let $z \in \mathcal{J}(f_\lambda)$. Since $f_\lambda(-z) = f_\lambda(z)$ for all $z \in \mathbb{C}$ and $\mathcal{J}(f_\lambda)$ is completely invariant, $-z \in \mathcal{J}(f_\lambda)$. Observe that $f_\lambda(\overline{z}) = \overline{f_\lambda(z)}$ and consequently, $f_\lambda^n(\overline{z}) = \overline{f_\lambda^n(z)}$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$. For $z \in \mathcal{J}(f_\lambda)$, the sequence $\{f_\lambda^n\}_{n>0}$ is not normal at z. It follows that $\{\overline{f_\lambda^n}\}_{n>0}$ is also not normal at z. Therefore, $\{f_\lambda^n\}_{n>0}$ is not normal at $\overline{z}$ and $\overline{z} \in \mathcal{J}(f_\lambda)$. □

PROPOSITION 2.2. *Let $f_\lambda \in \mathcal{S}$. Then, the set of all the critical values of f_λ is an unbounded subset of $\mathbb{R} \setminus (-|\lambda|, |\lambda|)$ and 0 is the only finite asymptotic value of f_λ.*

Proof. Observe that

$$f_\lambda'(z) = \lambda \frac{mz^{m-1}}{\sinh^{m-1} z}\left\{\frac{\sinh z - z\cosh z}{\sinh^2 z}\right\} \quad \text{and} \quad \frac{mz^{m-1}}{\sinh^{m-1} z} \neq 0 \quad \text{for } z \in \mathbb{C}.$$

Further, the point $z = 0$ is the only common zero of $\sinh z - z\cosh z$ and $\sinh^2 z$ and is a zero of $(\sinh z - z\cosh z)/\sinh^2 z$. Therefore, the solutions of $f_\lambda'(z) = 0$ are precisely the solutions of $\sinh z - z\cosh z = 0$ i.e., the solutions of $\tanh z = z$. It is easy to see that the set of all the solutions of $\tanh z = z$ is an unbounded subset of the imaginary axis. If $\tanh(iy) = iy$ for some $y \in \mathbb{R}$ then $\tanh(-iy) = -\tanh(iy) = -iy$. Therefore, the set of all the critical points of $f_\lambda(z)$ is symmetric with respect to the origin and is an unbounded subset of the imaginary axis. Let $\{iy_k\}_{k>0}$ be the sequence of critical points in the positive imaginary axis arranged in the increasing order of their moduli. Then $-iy_k$ is also a critical point of $f_\lambda(z)$ for each k. Since $f_\lambda(z)$ is an even function,

$$\lim_{k\to\infty} |f_\lambda(iy_k)| = \lim_{k\to\infty} |f_\lambda(-iy_k)| = \lim_{k\to\infty}\left|\lambda \frac{i^m y_k^m}{i^m \sin^m y_k}\right| = \infty.$$

Therefore, the set of all the critical values of f_λ is unbounded. Every critical point iy_k of $f_\lambda(z)$ satisfies $\tanh(iy_k) = iy_k$ and consequently,

$$\frac{iy_k}{\sinh(iy_k)} = \frac{1}{\cosh(iy_k)}.$$

The critical value

$$f_\lambda(iy_k) = \lambda\left(\frac{iy_k}{\sinh(iy_k)}\right)^m = \lambda\left(\frac{1}{\cosh(iy_k)}\right)^m = \lambda\left(\frac{1}{\cos y_k}\right)^m$$

is real. Since $|\cos y| \le 1$ for all $y \in \mathbb{R}$, it follows that $|f_\lambda(iy_k)| \ge |\lambda|$. Therefore, the set of all the critical values of $f_\lambda(z)$ is an unbounded subset of $\mathbb{R} \setminus (-|\lambda|, |\lambda|)$.

In order to determine the asymptotic values of f_λ, first we find all the asymptotic values of $(\sinh z/z)$. All the critical points of $(\sinh z/z)$, i.e., the roots of $(z\cosh z - \sinh z)/z^2$ are purely imaginary and form an unbounded set. Since

$$\lim_{|y|\to\infty} \frac{\sinh iy}{iy} = \lim_{|y|\to\infty} \frac{\sin y}{y} = 0,$$

0 is an asymptotic value of $(\sinh z/z)$ and is the only limit point of all the critical values of $(\sinh z/z)$. Since the order of $(\sinh z/z)$ is one, it can have at most two finite asymptotic values. Further, if there are exactly two finite asymptotic values of $(\sinh z/z)$ then both the asymptotic values are indirect singularities of the inverse function of $(\sinh z/z)$ [**17**]. If f is

a meromorphic function of finite order and a is an asymptotic value of f then, a is a limit point of critical values $a_k \neq a$ or all singularities of f^{-1} are logarithmic (a special case of direct singularity) [**7**]. Therefore, if there is a finite asymptotic value $\hat{w}$ of $(\sinh z/z)$ other than 0 then both 0 and $\hat{w}$ are indirect singularities of inverse function of $(\sinh z/z)$ and the limit points of critical values of $(\sinh z/z)$. Since the critical values of $(\sinh z/z)$ accumulate only at 0, $\hat{w}$ can not be an asymptotic value of $(\sinh z/z)$. Thus, 0 is the only finite asymptotic value of $(\sinh z/z)$. Since $(\sinh z/z)$ is an entire function, ∞ is also an asymptotic value. It implies that the function $(z/\sinh z)$ has only one finite asymptotic value, namely 0. Hence, 0 is the only finite asymptotic value of $f_\lambda(z) = \lambda(z^m/\sinh^m z)$ for $m \in \mathbb{N}$. □

Remark 2.1. For $z = x + iy \neq 0$,

$$\left|\frac{z^m}{\sinh^m z}\right| = \frac{|z|^m}{|\sinh z|^m} = \left\{\left(\frac{x^2+y^2}{\sinh^2 x + \sin^2 y}\right)^{1/2}\right\}^m.$$

If $\gamma : [0, \infty) \to \mathbb{C}$ is a path for which $\{\Im(z) \mid z \in \gamma\}$ is bounded and $\lim_{t\to\infty} |\Re(\gamma(t))| = \infty$ then $\lim_{t\to\infty} f_\lambda(\gamma(t)) = 0$. Further, if γ is a path for which $\{\Re(z) \mid z \in \gamma\}$ is bounded and $\lim_{t\to\infty} |\Im(\gamma(t))| = \infty$ then $\lim_{t\to\infty} f_\lambda(\gamma(t)) = \infty$.

3. *Dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$*

In this section, the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is studied. In Theorem 3.1, the existence and nature of real fixed points of f_λ are explored. The change in the nature and existence of real periodic points leads to a bifurcation in the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ at a critical parameter value and is proved in Theorem 3.2.

Consider the function

$$\begin{aligned}\phi(x) = xf'(x) + f(x) &= x\frac{mx^{m-1}}{\sinh^{m+1} x}(\sinh x - x\cosh x) + \frac{x^m}{\sinh^m(x)}\\ &= \frac{x^m}{\sinh^{m+1}(x)}((m+1)\sinh x - mx\cosh x) \quad \text{for } x \geq 0.\end{aligned}$$

Let $p(x) = (m+1)\sinh x - mx\cosh x$. Then $p'(x) = \cosh x - mx\sinh x$ and

$$p''(x) = (1-m)\sinh x - mx\cosh x.$$

Observe that $p''(x) < 0$ for $x \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$, since $m \geq 1$. Therefore, the function $p'(x)$ is decreasing on $\mathbb{R}^+$. Since $p'(0) = 1$ and $\lim_{x\to+\infty} p'(x) = -\infty$, by continuity of $p'(x)$, it follows that there is a unique $\hat{x} > 0$ such that $p'(x) > 0$ for $0 \leq x < \hat{x}$, $p'(\hat{x}) = 0$ and $p'(x) < 0$ for $x > \hat{x}$. Therefore, $p(x)$ increases in $[0, \hat{x})$, attains its maximum at $\hat{x}$ and decreases thereafter. It follows from the facts $p(0) = 0$ and $\lim_{x\to+\infty} p(x) = -\infty$ that, there is a unique positive $x^* > \hat{x}$ such that $p(x) > 0$ for $0 < x < x^*$, $p(x^*) = 0$ and $p(x) < 0$ for $x > x^*$. Since $(x^m/\sinh^{m+1} x) > 0$ for all $x > 0$, it follows that

$$\phi(x) = \frac{x^m}{\sinh^{m+1} x}p(x)\begin{cases} > 0 & \text{for } 0 < x < x^*,\\ = 0 & \text{for } x = x^*,\\ < 0 & \text{for } x > x^*.\end{cases} \tag{2}$$

Define

$$\lambda^*(m) = \lambda^* = \frac{x^*}{f(x^*)} \tag{3}$$

where x^* is the unique positive real root of the equation $\phi(x) = xf'(x) + f(x) = 0$.

Remark 3.1. For the function $f(x) = (x^m/\sinh^m x)$, let $x^*(m)$ denote the positive real root of the equation $\phi(x) = xf'(x) + f(x) = 0$ and let

$$\lambda^*(m) = \frac{x^*(m)}{f(x^*(m))}$$

denote the corresponding critical parameter. For $m = 1$, 2 and 3, it is numerically computed that $x^*(1) \approx 1.915$, $x^*(2) \approx 1.2878$, $x^*(3) \approx 1.034\ 02$ and $\lambda^*(1) \approx 3.3198$, $\lambda^*(2) \approx 2.1772$, $\lambda^*(3) \approx 1.7926$.

The following theorem shows that f_λ has a unique real fixed point for each $\lambda > 0$. However, the nature of the fixed point changes when the parameter λ passes through the critical parameter λ^*.

THEOREM 3.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. Then, the function f_λ has a unique real fixed point x_λ. Furthermore, the following cases hold.*

(1) *The fixed point x_λ is attracting for $0 < \lambda < \lambda^*$.*

(2) *The fixed point x_λ is rationally indifferent for $\lambda = \lambda^*$.*

(3) *The fixed point x_λ is repelling for $\lambda > \lambda^*$.*

Proof. Since $f_\lambda(x) > 0$ for all $x \in \mathbb{R}$, each real periodic point of f_λ is positive. The function

$$f'_\lambda(x) = \lambda \frac{mx^{m-1}}{\sinh^{m+1} x}(\sinh x - x \cosh x) < 0 \quad \text{for } x > 0$$

and hence $f_\lambda(x)$ is decreasing on $\mathbb{R}^+$. Let $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Since $f'_\lambda(x) < 0$ for $x > 0$, $g'_\lambda(x) = f'_\lambda(x) - 1 < 0$ and consequently, $g_\lambda(x)$ is decreasing on $\mathbb{R}^+$. Now, $g_\lambda(0) = \lambda > 0$, $\lim_{x\to+\infty} g_\lambda(x) = -\infty$ and $g_\lambda(x)$ is continuous on $\mathbb{R}^+$. By the intermediate-value theorem, there exists a unique positive x_λ such that $g_\lambda(x_\lambda) = 0$. In other words, $f_\lambda(x)$ has a unique positive fixed point x_λ and $\lambda = (x_\lambda/f(x_\lambda))$. Note that the function $(x/f(x))$ is increasing on $\mathbb{R}^+$, since

$$\frac{d}{dx}\left(\frac{x}{f(x)}\right) = \frac{f(x) - xf'(x)}{(f(x))^2} > 0 \quad \text{for } x > 0.$$

(1) For $0 < \lambda < \lambda^*$, $(x_\lambda/f(x_\lambda)) < (x^*/f(x^*))$ which gives $x_\lambda < x^*$. By equation (2), $\phi(x_\lambda) > 0$. This implies that

$$\frac{\phi(x_\lambda)}{f(x_\lambda)} = \frac{xf'(x_\lambda) + f(x_\lambda)}{f(x_\lambda)} = f'_\lambda(x_\lambda) + 1 > 0.$$

Since $f'_\lambda(x)$ is negative on $\mathbb{R}^+$, it follows that $-1 < f'_\lambda(x_\lambda) < 0$ and the fixed point x_λ is attracting for $0 < \lambda < \lambda^*$.

(2) For $\lambda = \lambda^*$, it follows that $x_\lambda = x^*$ and $\phi(x_\lambda) = 0$ by arguments similar to those used in case (1). Now, by equation (2), it follows that $(\phi(x_\lambda)/f(x_\lambda)) = 0$ implying $f'_{\lambda^*}(x_\lambda) = -1$. Therefore, the fixed point $x_\lambda = x^*$ is rationally indifferent if $\lambda = \lambda^*$.

(3) For $\lambda > \lambda^*$, it follows that $x_\lambda > x^*$ by arguments similar to those used in case (1). Again by equation (2) and by the fact $x_\lambda > x^*$, we have $\phi(x_\lambda) < 0$. It shows that $(\phi(x_\lambda)/f(x_\lambda)) = f'_\lambda(x_\lambda) + 1 < 0$ and hence $f'_\lambda(x_\lambda) < -1$. Therefore, x_λ is a repelling fixed point of f_λ for $\lambda > \lambda^*$. □

Now, we investigate the possibility of the real periodic points of f_λ with minimal period greater than one. The function $f_\lambda(x)$ is decreasing on $\mathbb{R}^+$, $f_\lambda(\mathbb{R}) = (0, \lambda]$ and f_λ has a unique real fixed point x_λ by Theorem 3.1. It is easy to see that $f_\lambda(0) = \lambda > f_\lambda(x) > x_\lambda$ for $0 < x < x_\lambda$ and $f_\lambda(x) < x_\lambda < f_\lambda(0) = \lambda$ for $x > x_\lambda > 0$. In other words, $f_\lambda((0, x_\lambda)) = (x_\lambda, \lambda)$ and $f_\lambda(x_\lambda, \infty) = (0, x_\lambda)$. It follows that $f^n_\lambda(x) \neq x$ for any $x \in \mathbb{R}^+ \setminus \{x_\lambda\}$ and odd n. Therefore, $f_\lambda(x)$ does not have any real periodic point of odd period other than x_λ. Observe that $f_\lambda(x) > 0$ and $f'_\lambda(x) < 0$ for $x > 0$ and $\lambda > 0$. So $(f^2_\lambda)'(x) = f'_\lambda(f_\lambda(x))f'_\lambda(x) > 0$ and $f^2_\lambda(x)$ is increasing on $\mathbb{R}^+$. Consequently, if $f^2_\lambda(x) > x$ (or $f^2_\lambda(x) < x$) for some $x \in \mathbb{R}^+$ then $f^{2n}_\lambda(x) > f^{2(n-1)}_\lambda(x)$ (or $f^{2n}_\lambda(x) < f^{2(n-1)}_\lambda(x)$) for all n. It shows that the function $f^2_\lambda(x)$ does not have any real periodic points of period greater than one, and hence $f_\lambda(x)$ has no real periodic point of even period greater than two. Therefore, a real periodic point of f_λ other than x_λ is of minimal period exactly equal to two, if it exists. Also, each cycle $\{x_{1\lambda}, x_{2\lambda}\}$ of real 2-periodic points satisfies $x_{1\lambda} < x_\lambda < x_{2\lambda}$. Let us assume that f_λ has two different 2-periodic real cycles $\{a, b\}$ with $0 < a < b$ and $\{c, d\}$ with $0 < c < d$. Since $f_\lambda(x)$ is strictly decreasing on $\mathbb{R}^+$ for $\lambda > 0$, it follows that $c < a < x_\lambda < b < d$ or $a < c < x_\lambda < d < b$. In the first case $\{c, d\}$ and in the second case $\{a, b\}$ is called the outer cycle. In the first case $\{a, b\}$ and in the second case $\{c, d\}$ is called the inner cycle. The following proposition shows that whenever such a 2-periodic cycle exists, it is attracting or rationally indifferent and all the singular values of $f_\lambda(z)$ tend to this cycle under iteration of f^2_λ.

PROPOSITION 3.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. If f_λ has a real 2-periodic cycle, then $\lim_{n\to\infty} f^{2n}_\lambda(x) = y_{1\lambda}$ or $y_{2\lambda}$ for all $x \in [0, y_{1\lambda}] \cup [y_{2\lambda}, +\infty)$ where $\{y_{1\lambda}, y_{2\lambda}\}$ is the outermost 2-periodic cycle. In particular, the cycle $\{y_{1\lambda}, y_{2\lambda}\}$ is either attracting or rationally indifferent and all the singular values of f_λ tend to $\{y_{1\lambda}, y_{2\lambda}\}$ under iteration of f^2_λ.*

Proof. It is observed earlier that any periodic point of the function f_λ is of minimal period one or two and each 2-periodic cycle $\{a, b\}$ satisfies $a < x_\lambda < b$ where x_λ is the fixed point of f_λ. Since $\{y_{1\lambda}, y_{2\lambda}\}$ is the outermost 2-periodic cycle, $f^2_\lambda(x) \neq x$ for all $x > y_{2\lambda}$. If possible, let $f^2_\lambda(x) > x$ for some $x > y_{2\lambda}$. Then, the sequence $\{f^{2n}_\lambda(x)\}_{n>0}$ is increasing and bounded above by λ, and hence $f^{2n}_\lambda(x)$ converges to l, say. Obviously, $l > y_{2\lambda}$. By the continuity of f^2_λ it follows that the point l must be a periodic point of f_λ of period at most two. This contradicts the fact that $\{y_{1\lambda}, y_{2\lambda}\}$ is the outermost 2-periodic cycle. Therefore, we conclude that $f^2_\lambda(x) < x$ for all $x > y_{2\lambda}$. Since $f^2_\lambda(x)$ is increasing, the sequence $\{f^{2n}_\lambda(x)\}_{n>0}$ is decreasing and bounded below by $y_{2\lambda}$ and consequently, $\lim_{n\to\infty} f^{2n}_\lambda(x) = y_{2\lambda}$ for $x > y_{2\lambda}$. Similarly, it can be proved that $f^2_\lambda(x) > x$ and $\lim_{n\to\infty} f^{2n}_\lambda(x) = y_{1\lambda}$ for all $0 \leq x < y_{1\lambda}$. Therefore, $\lim_{n\to\infty} f^{2n}_\lambda(x) = y_{1\lambda}$ or $y_{2\lambda}$ for all $x \in [0, y_{1\lambda}] \cup [y_{2\lambda}, +\infty)$.

Each interval containing $y_{1\lambda}$ contains points tending to $y_{1\lambda}$ under iteration of f^2_λ. Therefore, $y_{1\lambda}$ cannot be a repelling periodic point of f^2_λ and is either attracting or rationally indifferent. Thus, $\{y_{1\lambda}, y_{2\lambda}\}$ is either attracting or rationally indifferent. As $(-y_{2\lambda}, y_{2\lambda}) \subset (-\lambda, \lambda)$ and f_λ is an even function, $\lim_{n\to\infty} f_\lambda^{2n}(x) = y_{1\lambda}$ or $y_{2\lambda}$ for all $x \in \mathbb{R} \setminus (-\lambda, \lambda)$. Since all the critical values of f_λ are in $\mathbb{R} \setminus (-\lambda, \lambda)$ and the finite asymptotic value 0 is mapped to λ by f_λ, it is concluded that all the singular values of f_λ tend to $\{y_{1\lambda}, y_{2\lambda}\}$ under iteration of f_λ^2. □

The dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ is determined in the following theorem.

THEOREM 3.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$.*

(1) *If $\lambda < \lambda^*$ then $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for all $x \in \mathbb{R}$ where a_λ is the unique real attracting fixed point of f_λ.*

(2) *If $\lambda = \lambda^*$ then $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for all $x \in \mathbb{R}$ where x^* is the unique real rationally indifferent fixed point of f_λ.*

(3) *If $\lambda > \lambda^*$ then $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in \mathbb{R} \setminus \{r_\lambda, -r_\lambda\}$ where r_λ is the unique real repelling fixed point of f_λ and $\{a_{1\lambda}, a_{2\lambda}\}$ is the real attracting or rationally indifferent 2-periodic cycle.*

Proof. All the singular values of $f_\lambda(z)$ are in $(\mathbb{R} \setminus (-\lambda, \lambda)) \cup \{0\}$ by Proposition 2.2. If there is a 2-periodic cycle then the cycle is in $(0, \lambda)$ and by Proposition 3.1, all the singular values tend to the outermost 2-cycle under iteration of f_λ^2.

(1) Let $f_\lambda^2(x) > x$ (or $f_\lambda^2(x) < x$) for some $x > 0$. Since $f_\lambda^2(x)$ is increasing on $\mathbb{R}^+$, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ (or decreasing and bounded below by 0). Therefore, $f_\lambda^{2n}(x)$ converges to $\hat{x}$, say. Now, by continuity of f_λ, the point $\hat{x}$ is a periodic point of $f_\lambda(x)$ of period one or two. If possible, let $\hat{x}$ be a periodic point of f_λ with prime period two. Then, there is an outermost 2-periodic cycle of f_λ and all the singular values of f_λ tend to the outermost 2-periodic cycle under iteration of f_λ^2 which is a contradiction to the fact that the basin of attraction of a_λ must contain at least one singular value of f_λ. Therefore, $\hat{x}$ is not a 2-periodic point and is a fixed point. Since f_λ has only one real fixed point a_λ for $0 < \lambda < \lambda^*$, $\hat{x} = a_\lambda$ and $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_\lambda$ for all $x \in \mathbb{R}^+$. By continuity of f_λ, it follows that $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for all $x \in \mathbb{R}^+$. Since

$$f_\lambda(\mathbb{R}^- \cup \{0\}) \subset \mathbb{R}^+, \quad \lim_{n\to\infty} f_\lambda^n(x) = a_\lambda \quad \text{for all } x \in \mathbb{R}.$$

(2) Let $f_\lambda^2(x) > x$ (or $f_\lambda^2(x) < x$). Since $f_\lambda^2(x)$ is increasing on $\mathbb{R}^+$, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ (or decreasing and bounded below by 0). Proceeding as in case (1), it is easy to see that $\{f_\lambda^{2n}(x)\}_{n>0}$ converges to x^* for all $x \in \mathbb{R}^+$. By continuity of f_λ, it follows that $\lim_{n\to\infty} f_\lambda^n(x) = x^*$ for all $x \in \mathbb{R}^+$. Since

$$f_\lambda(\mathbb{R}^- \cup \{0\}) \subset \mathbb{R}^+, \quad \lim_{n\to\infty} f_\lambda^n(x) = x^* \quad \text{for all } x \in \mathbb{R}.$$

(3) If $\lambda > \lambda^*$, then the unique real fixed point of f_λ is repelling. Therefore, we can find a real number x sufficiently close to the fixed point r_λ such that $f_\lambda^2(x) > x$. Since $f_\lambda^2(x)$ is increasing on $\mathbb{R}^+$, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is increasing and bounded above by λ. Therefore, $\{f_\lambda^{2n}(x)\}_{n>0}$ converges to $\hat{x}$, say. By continuity of f_λ^2, it follows that $\hat{x}$ is a 2-periodic point of f_λ. If possible, let there be more than one 2-periodic cycle of periodic points. If $\{i_{1\lambda}, i_{2\lambda}\}$ is the innermost real cycle of 2-periodic points of f_λ then $i_{1\lambda} < r_\lambda < i_{2\lambda}$ and, $f_\lambda(x) \in (r_\lambda, i_{2\lambda})$ for all $x \in (i_{1\lambda}, r_\lambda)$ and $f_\lambda(x) \in (i_{1\lambda}, r_\lambda)$

for all $x \in (r_\lambda, i_{2\lambda})$. Furthermore, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ converges either to $i_{1\lambda}$ or to $i_{2\lambda}$ for $x \in (i_{1\lambda}, i_{2\lambda}) \setminus r_\lambda$ by the same arguments as used in the previous cases. Therefore, $\{i_{1\lambda}, i_{2\lambda}\}$ is either an attracting or a rationally indifferent cycle and at least one singular value of f_λ tends to this cycle under iteration of f_λ^2. But all the singular values of f_λ^2 tend to the outermost 2-cycle under iteration of f_λ by Proposition 3.1 leading to a contradiction. Hence, f_λ has exactly one 2-periodic cycle. Let it be $\{a_{1\lambda}, a_{2\lambda}\}$. By Proposition 3.1, $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in [0, a_{1\lambda}] \cup [a_{2\lambda}, +\infty)$. If $x \in (r_\lambda, a_{2\lambda}]$, then $f_\lambda^2(x) > x$ and $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{2\lambda}$. Similarly, it is easily seen that $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ for all $x \in [a_{1\lambda}, r_\lambda)$. Since $f_\lambda(z)$ is an even function, it follows that $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in \mathbb{R}^- \setminus \{-r_\lambda\}$. Therefore, if $\lambda > \lambda^*$ it is concluded that $\lim_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ or $a_{2\lambda}$ for all $x \in \mathbb{R} \setminus \{r_\lambda, -r_\lambda\}$ where r_λ is the repelling fixed point of f_λ and $\{a_{1\lambda}, a_{2\lambda}\}$ is the attracting or rationally indifferent 2-periodic cycle. □

The above theorem exhibits the occurrence of a period-doubling bifurcation at $\lambda = \lambda^*$ in the dynamics of functions f_λ in the one-parameter family $\mathcal{S}$.

Remark 3.2. All the singular values of f_λ, $\lambda > 0$ are in $\mathbb{R}$ and tend to either an attracting or a rationally indifferent periodic point under iteration of f_λ^2. Therefore, the set $P(f_\lambda)$ is contained in the Fatou set of f_λ for $\lambda > 0$. In particular, the point 0 is in the Fatou set $\mathcal{F}(f_\lambda)$ for $\lambda > 0$.

Remark 3.3. Note that $f_\lambda(iy) = (y^m/\sin^m y)$ and the image of any point on the imaginary axis is either infinity or a real number. By Theorem 3.2, each of the real numbers except at most two are in an attracting or a parabolic domain of f_λ corresponding to a real periodic point. Therefore, any Fatou component U of f_λ other than an attracting or parabolic domain (and their pre-images) intersects neither the real nor the imaginary axis. Thus, such a Fatou component U is contained completely in one of the four quadrants of the complex plane.

4. *Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$*

The dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ is studied in this section. The non-existence of Baker domains and wandering domains in the Fatou set of $f_\lambda \in \mathcal{S}$ for $\lambda > 0$ is proved in Theorem 4.1 and Theorem 4.2 respectively. The dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ is described in Theorem 4.3.

THEOREM 4.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. Then, the Fatou set of f_λ has no Baker domain.*

Proof. Suppose, on the contrary that the Fatou set of f_λ has a Baker domain B of minimal period p. All the singular values of f_λ are real by Proposition 2.2 and $f_\lambda(\mathbb{R}) = (0, \lambda]$. Therefore, $S_p(f_\lambda)$ is bounded for each $p > 1$ and the Fatou set of f_λ cannot have a Baker domain of minimal period greater than one [**19**]. Therefore, $p = 1$. That is, B is an invariant Baker domain. By the definition of an invariant Baker domain, there is a point z^* in the boundary of B such that $\lim_{n\to\infty} f_\lambda^n(z) = z^*$ for all $z \in B$ and $f_\lambda(z^*)$ is not defined. Since the point at infinity is the only point in $\widehat{\mathbb{C}}$ where the function $f_\lambda(z)$ is not defined, $z^* = \infty$. Now, $\lim_{n\to\infty} f_\lambda^n(z) = \infty$ and $f_\lambda^n(z) \in B$ for $z \in B$ and $n \in \mathbb{N}$ gives that the domain B is unbounded. Since $f_\lambda(\overline{z}) = \overline{f_\lambda(z)}$ for all $z \in \mathbb{C}$ and B is contained in one of the four quadrants by Remark 3.3, $\overline{B} = \{\overline{z} \in \mathbb{C} \mid z \in B\}$ is also an invariant Baker domain of f_λ. Clearly, one of B and $\overline{B}$ contains points with positive imaginary parts. Let it be B, i.e., $\Im(z) > 0$ for each $z \in B$.

We assert that the set $\{\Im(z) \mid z \in B\}$ is unbounded. To see it, suppose on the contrary that $\{\Im(z) \mid z \in B\}$ is bounded. Then $\{\Re(z) \mid z \in B\}$ must be unbounded as B is itself unbounded. Now, let $\{z_k\}_{k>0}$ be a sequence in B such that $\lim_{k\to\infty} |\Re(z_k)| = \infty$. Then

$$f_\lambda(z_k) = \frac{\lambda 2^m z_k^m}{(e^{z_k} - e^{-z_k})^m} \to 0 \quad \text{as } k \to \infty$$

by Remark 2.1. The point 0 is in the attracting or parabolic domain for each $\lambda > 0$ by Remark 3.2. Let $N(0)$ be a neighbourhood of $z = 0$ completely lying in the Fatou set. Then, there is a natural number $\hat{k}$ such that $f_\lambda(z_k) \in N(0)$ for all $k > \hat{k}$. Consequently, z_k is in a Fatou component U such that $f_\lambda(U)$ is contained in an attracting domain or a parabolic domain and hence, not in B for $k > \hat{k}$. It contradicts the invariance of B. Thus the set $\{\Im(z) \mid z \in B\}$ is unbounded.

Let B be in the first quadrant of the plane. If B is in the second quadrant, the proof follows similarly. For $\theta \in (0, (\pi/2))$, let $S_\theta = \{z \in \mathbb{C} \mid \theta < \operatorname{Arg}(z) < \pi/2\}$ and $S_{\theta'} = \{z \in \mathbb{C} \mid 0 < \operatorname{Arg}(z) \le \theta\}$ where $0 < \operatorname{Arg}(z) < 2\pi$. Let $L_k = \{z \in \mathbb{C} \mid \Im(z) = \pi k\}$ and $L_k^+ = \{z \in L_k \mid \Re(z) > 0\}$ for $k \in \mathbb{Z}$. We now show that the set $\{\Im(z) \mid z \in B \cap S_\theta\}$ is unbounded for each $\theta \in (0, \pi/2)$. In view of the conclusion obtained in the previous paragraph, it is sufficient to prove that the set $\{\Im(z) \mid z \in B \cap S_{\theta'}\}$ is bounded. Suppose the set $\{\Im(z) \mid z \in B \cap S_{\theta'}\}$ is unbounded for some θ. Then a sequence $\{s_n\}_{n>0}$ of points can be found in $B \cap S_{\theta'}$ such that $\Im(s_n) \le (\tan\theta)\Re(s_n)$ for all $n \in \mathbb{N}$ and $\Im(s_n) \to \infty$ as $n \to \infty$. Consequently, $\Re(s_n) \to \infty$ and

$$\left|\frac{s_n}{\sinh(s_n)}\right| \le 2\frac{|\Re(s_n) + i\Im(s_n)|}{e^{\Re(s_n)} - e^{-\Re(s_n)}} \le 2\frac{|(1+\tan\theta)\Re(s_n)|}{e^{\Re(s_n)} - e^{-\Re(s_n)}} \to 0 \quad \text{as } n \to \infty.$$

It follows that there is an $n_0 \in \mathbb{N}$ such that $f_\lambda(s_n) \in N(0)$ for $n > n_0$. Consequently, the set $\{s_n \mid n > n_0\}$ is not in the Baker domain, which is a contradiction. Therefore, the set $\{\Im(z) \mid z \in B \cap S_{\theta'}\}$ is bounded, and hence the set $\{\Im(z) \mid z \in B \cap S_\theta\}$ is unbounded. Furthermore, $B \cap S_\theta$ has an unbounded connected subset. In particular, there exists an integer k_0 such that the set $B \cap S_\theta$ intersects L_k^+ for all $k \ge k_0$. Choose θ in such a way that for all $\delta, \beta \in (\theta, \pi/2)$, $|m(\delta - \beta)| < (\pi/4)$ where $f_\lambda(z) = \lambda(z^m/\sinh^m z)$.

Case 1. m is odd.

Note that

$$f_\lambda(x + i\pi k) = \lambda\frac{(x+i\pi k)^m}{\sinh^m(x+i\pi k)} = \begin{cases} -\lambda\dfrac{(x+i\pi k)^m}{\sinh^m x} & \text{for odd } k, \\ \lambda\dfrac{(x+i\pi k)^m}{\sinh^m x} & \text{for even } k. \end{cases} \tag{4}$$

Let $z_1 = x_1 + i\pi k$, $z_2 = x_2 + i\pi(k+1) \in B \cap S_\theta$ for some $k \ge k_0$. If $\operatorname{Arg}(z_1) = \theta_1$ and $\operatorname{Arg}(z_2) = \theta_2$ then $\theta_1, \theta_2 \in (\theta, \pi/2)$ and $|\operatorname{Arg}(z_1^m) - \operatorname{Arg}(z_2^m)| = |m(\theta_1 - \theta_2)| < \pi/4$. Therefore, the two points z_1^m and z_2^m belong either to the same quadrant or to two consecutive quadrants. This means either the real parts or the imaginary parts of z_1^m

and z_2^m have same sign. Let the first possibility hold i.e., $(\Re(z_1^m)/\Re(z_2^m)) > 0$. One of k and $k+1$ is even and the other is odd. Also note that $(\lambda/\sinh^m x) > 0$ for $x > 0$. Using equation (4), we have $\Re(f_\lambda(z_1))/\Re(f_\lambda(z_2)) < 0$. In other words, $\Re(f_\lambda(z_1))$ and $\Re(f_\lambda(z_2))$ have opposite sign. Thus $f_\lambda(B) = B$ intersects the imaginary axis which contradicts Remark 3.3. For $\Im(z_1^m)/\Im(z_2^m) > 0$, arguing similarly, we can get $\Im(f_\lambda(z_1))/\Im(f_\lambda(z_2)) < 0$, which also results in a similar contradiction to Remark 3.3.

Case 2. $m/2$ is odd.

Note that

$$\sinh^m\left(x + i\left(\frac{\pi}{2} + 2\pi k\right)\right) = -\cosh^m x \quad \text{for } k \in \mathbb{N}.$$

Since the line

$$L_{(\pi/2)+2\pi k} = \left\{ z \in \mathbb{C} \;\middle|\; \Im(z) = \frac{\pi}{2} + 2\pi k \right\}$$

intersects $B \cap S_\theta$ for all sufficiently large k, there is an even $k' \in \mathbb{N}$ such that the points $z_3 = x_3 + i((\pi/2) + 2\pi k')$ and $z_4 = x_4 + i(2\pi k')$ are in $B \cap S_\theta$ for some $x_3,\ x_4 > 0$ where θ is so chosen that $|\mathrm{Arg}(z_3^m) - \mathrm{Arg}(z_4^m)| < \pi/4$. Now,

$$f_\lambda(z_3) = -\lambda \frac{(x_3 + i((\pi/2) + 2\pi k'))^m}{\cosh^m x_3} \quad \text{and} \quad f_\lambda(z_4) = \lambda \frac{(x_4 + i2\pi k')^m}{\sinh^m x_4}.$$

Arguing exactly in the same manner as in Case 1, it is found that either

$$\frac{\Re(f_\lambda(z_3))}{\Re(f_\lambda(z_4))} < 0 \quad \text{or} \quad \frac{\Im(f_\lambda(z_3))}{\Im(f_\lambda(z_4))} < 0.$$

Both of these possibilities contradict Remark 3.3.

Therefore, the Fatou set of f_λ does not contain any Baker domain. □

THEOREM 4.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$. Then, the Fatou set of f_λ has no wandering domain.*

Proof. By Remark 3.2, the set $P(f_\lambda) \setminus \{\infty\}$ is in the Fatou set of f_λ. Since ∞ is in the derived set $P(f_\lambda)'$ of $P(f_\lambda)$, we have $\mathcal{J}(f_\lambda) \cap P(f_\lambda)' = \{\infty\}$. If a point z_0 is in a wandering domain of f_λ then, every limit point of $\{f_\lambda^n(z_0)\}_{n>0}$ is infinity [**22**]. Since $S_2(f_\lambda)$ is bounded, $f_\lambda^{2n}(z_0)$ does not tend to infinity as $n \to \infty$. Then, we can find a subsequence $\{n_k\}_{k>0}$ of $\{2n\}_{n>0}$ such that $\{f_\lambda^{n_k}(z_0)\}_{k>0}$ is bounded. Let us consider $\{f_\lambda^{n_k}\}_{k>0}$. Since $\{f_\lambda^n\}_{n>0}$ is normal at z_0, there is a subsequence $\{f_\lambda^{n_{k,m}}\}_{m>0}$ of $\{f_\lambda^{n_k}\}_{k>0}$ such that $\lim_{m\to\infty} f_\lambda^{n_{k,m}}(z_0) = \infty$. However, it is not possible because $\{n_{k,m}\}_{m>0}$ is a subsequence of $\{n_k\}_{k>0}$. Therefore, the Fatou set of f_λ does not contain any wandering domain. □

THEOREM 4.3. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > 0$.*

(1) *For $\lambda < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the basin of attraction of the unique real attracting fixed point a_λ of f_λ.*

(2) *For $\lambda = \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the parabolic basin corresponding to the unique real rationally indifferent fixed point x^* of f_λ.*

(3) *For $\lambda > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the basin of attraction or parabolic basin corresponding to a cycle of real 2-periodic points $\{a_{1\lambda}, a_{2\lambda}\}$ of f_λ*

Proof. We know that the boundary of any rotational domain of a meromorphic function f is contained in the closure of the set $P(f)$ [**4**]. Thus, the Fatou set of f_λ does not contain any rotational domain. By Theorems 4.1 and 4.2, the Fatou set of f_λ also does not contain any Baker domain and wandering domain for $\lambda > 0$.

If U is an attracting domain or parabolic domain of period p and z_u is the corresponding attracting or rationally indifferent periodic point of f_λ, then there is a singular value s of f_λ such that $f_\lambda^{np}(f_\lambda^k(s)) \to z_u$ as $n \to \infty$ for some k, $0 < k \le p$. Since all the singular values and their forward orbits (whenever defined) are in $\mathbb{R}$, z_u is real. Therefore, any attracting or parabolic domain of f_λ corresponds to a real attracting or rationally indifferent periodic point.

(1) For $0 < \lambda < \lambda^*$, f_λ has only one real periodic point which is the attracting fixed point a_λ. Therefore, $\mathcal{F}(f_\lambda)$ is the basin of attraction of a_λ.
(2) For $\lambda = \lambda^*$, f_λ has only one real periodic point which is the rationally indifferent fixed point x^*. Therefore, $\mathcal{F}(f_\lambda)$ is the parabolic basin corresponding to x^*.
(3) For $\lambda > \lambda^*$, f_λ has a repelling fixed point r_λ and a cycle of real 2-periodic points $\{a_{1\lambda}, a_{2\lambda}\}$ which is either attracting or rationally indifferent. Therefore, $\mathcal{F}(f_\lambda)$ is the attracting basin or parabolic basin corresponding to $\{a_{1\lambda}, a_{2\lambda}\}$. □

Since f_λ and $f_{-\lambda}$ are conformally conjugate, the dynamics of f_λ for $\lambda < 0$ is as follows.

COROLLARY 4.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda < 0$.*
(1) *For $-\lambda^* < \lambda < 0$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the basin of attraction of the unique real attracting fixed point of f_λ.*
(2) *For $\lambda = -\lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the parabolic basin corresponding to the unique real rationally indifferent fixed point of f_λ.*
(3) *For $\lambda < -\lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is the basin of attraction or parabolic basin corresponding to a cycle of real 2-periodic points of f_λ.*

5. *Topology of Fatou components*

Topology of the Fatou components of f_λ, $\lambda > 0$ is investigated in this section. It is observed from Theorem 4.3 that the Fatou set of f_λ contains components with period one and two. The connectivity of a periodic Fatou component of a meromorphic function is either one, two or infinity whereas the connectivity of a pre-periodic Fatou component can be any finite number [**2**]. In Theorem 5.1, it is proved that the Fatou set of f_λ, $0 < \lambda < \lambda^*$ is infinitely connected. The existence of pre-periodic Fatou components is established and the connectivity of all the Fatou components of f_λ is determined for $\lambda > \lambda^*$ in Theorem 5.2.

THEOREM 5.1. *Let $f_\lambda \in \mathcal{S}$ and $0 < \lambda < \lambda^*$. Then, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ is connected. Furthermore, the Fatou set $\mathcal{F}(f_\lambda)$ is infinitely connected.*

Proof. By Theorem 3.2(1), $\lim_{n\to\infty} f_\lambda^n(x) = a_\lambda$ for $x \in \mathbb{R}$ and $0 < \lambda < \lambda^*$ where a_λ is the attracting fixed point of f_λ. The Fatou set of f_λ is the attracting basin

$$A(a_\lambda) = \{z \in \mathbb{C} \mid f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\} \quad \text{for } 0 < \lambda < \lambda^*.$$

Let $I(a_\lambda)$ be the immediate basin of attraction of a_λ. By definition, $I(a_\lambda)$ is a forward invariant connected subset of the Fatou set $\mathcal{F}(f_\lambda)$. Note that $A(a_\lambda) = I(a_\lambda)$ if $I(a_\lambda)$ is backward invariant. Since $I(a_\lambda)$ is connected, in order to prove the connectedness of $\mathcal{F}(f_\lambda)$, it is sufficient to show that $I(a_\lambda)$ is backward invariant.

Let, if possible, V be a component of $f_\lambda^{-1}(I(a_\lambda))$ different from $I(a_\lambda)$. Since 0 is an omitted value of f_λ, each singularity of f_λ^{-1} lying over 0 is transcendental. It means that V contains an asymptotic path γ corresponding to the asymptotic value 0 and by Remark 2.1, the set $\{\Re(z) \mid z \in \gamma\}$ is unbounded. Therefore, the set $\{\Re(z) \mid z \in V\}$ is unbounded. The function f_λ is even and $f_\lambda(\overline{z}) = \overline{f_\lambda(z)}$ for all $z \in \mathbb{C}$. In view of Remark 3.3, it is assumed without loss of generality that, the set V is in the upper half plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. Let $\{w_n\}_{n>0}$ be a sequence on γ such that $\Re(w_n) \to \infty$ as $n \to \infty$. Then $\lim_{n\to\infty} f_\lambda(w_n) = 0$. Each of the vertical lines $l_n = \{z \in \mathbb{C} \mid \Re(z) = \Re(w_n) \text{ and } 0 \le \Im(z) < \Im(w_n)\}$ joins a point of V and a point of $\mathbb{R} \cap I(a_\lambda)$ and we get that l_n intersects the boundary ∂V of V for each n. Let $z_n \in l_n \cap \partial V$. Then $z_n \in \mathcal{J}(f_\lambda)$ and $\Im(z_n) < \Im(w_n)$ for all n. Furthermore,

$$\begin{aligned} |f_\lambda(z_n)| &= \lambda\left\{\left(\frac{\Re(z_n)^2 + \Im(z_n)^2}{\sinh^2\Re(z_n) + \sin^2\Im(z_n)}\right)^{1/2}\right\}^m \\ &< \lambda\left\{\left(\frac{\Re(w_n)^2 + \Im(w_n)^2}{\sinh^2\Re(w_n) + \sin^2\Im(z_n)}\right)^{1/2}\right\}^m. \end{aligned} \tag{5}$$

Since the sequence $\{\sin^2(\Im(z_n))\}_{n>0}$ is bounded, the right-hand side of equation (5) is equal to $|f_\lambda(w_n)|$ when $n \to \infty$. Therefore, $\lim_{n\to\infty} f_\lambda(z_n) = 0$. Let $D_r(0) = \{z \in \mathbb{C} : |z| < r\} \subset I(a_\lambda)$. Then, there exists an n_0 such that $f_\lambda(z_n) \in D_r(0)$ for all $n > n_0$. It means that z_n is in the Fatou set of f_λ for $n > n_0$, which is a contradiction. Therefore, each component of $f_\lambda^{-1}(I(a_\lambda))$ intersects $I(a_\lambda)$ and hence is a subset of $I(a_\lambda)$. Thus $I(a_\lambda)$ is backward invariant.

Since $\mathcal{F}(f_\lambda)$ is connected and contains an attracting fixed point, it is invariant. The connectivity of any invariant Fatou component of a meromorphic function is one, two or infinity, two being the case when the component is an Herman ring. Since the Fatou set $\mathcal{F}(f_\lambda)$ is an attracting domain for $0 < \lambda < \lambda^*$, the connectivity of $\mathcal{F}(f_\lambda)$ is either one or infinity. If possible, let $\mathcal{F}(f_\lambda)$ be simply connected. Then, the Julia set $\mathcal{J}(f_\lambda)$ is connected. As the point at infinity and a pole w^* lying on the imaginary axis are in $\mathcal{J}(f_\lambda)$, there is an unbounded connected subset J_{w^*} of the Julia set containing w^*. Now, $\overline{-J_{w^*}} = \{z \in \mathbb{C} \mid -\overline{z} \in J_{w^*}\}$ is also in the Julia set by Proposition 2.1. Thus $J = J_{w^*} \cup \overline{-J_{w^*}}$ is in the Julia set and the set $\widehat{\mathbb{C}} \setminus J$ has at least two components each intersecting the Fatou set of f_λ. This contradicts the fact that $\mathcal{F}(f_\lambda)$ is connected. Therefore, $\mathcal{F}(f_\lambda)$ is infinitely connected for $0 < \lambda < \lambda^*$. □

Remark 5.1. Since the Fatou set is connected with connectivity greater than three for $0 < \lambda < \lambda^*$, singleton components of $\mathcal{J}(f_\lambda)$ are dense in $\mathcal{J}(f_\lambda)$ [**10**].

It is seen in Theorem 5.1 that the Fatou set of f_λ is connected and hence unbounded for $0 < \lambda < \lambda^*$. The next proposition shows that there are at least three Fatou components of f_λ, two of which are unbounded for $\lambda > \lambda^*$.

PROPOSITION 5.1. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > \lambda^*$. If U^+, U^- and U_0 denote the Fatou components containing $(a_{2\lambda}, +\infty)$, $(-\infty, -a_{2\lambda})$ and 0 respectively where $\{a_{1\lambda}, a_{2\lambda}\}$ is the 2-cycle of real periodic points of f_λ, then the Fatou components U^+, U^- and U_0 are mutually disjoint. Further, the components U^+ and U^- are unbounded.*

Proof. Observe that both U^+ and U^- are mapped into U_0 and U_0 is mapped into U^+ by f_λ for $\lambda > \lambda^*$. Since U_0 and U^+ form a cycle of 2-periodic Fatou components, $U_0 \neq U^+$. If U_0 intersects U^- then $U_0 = U^-$ will become invariant, which is not true. Therefore, U_0 is different from U^+ and U^-. If U^+ and U^- are the same component of $\mathcal{F}(f_\lambda)$ then $U^+ = U^-$ intersects the imaginary axis. Then, since all the points in the imaginary axis are mapped onto $\mathbb{R} \setminus (-\lambda, \lambda) \subset (-\infty, -a_{2\lambda}) \cup (a_{2\lambda}, +\infty)$, the points of the set $U^+ \cap \{iy \mid y \in \mathbb{R}\}$ are mapped into U^+ and consequently, U^+ is invariant, leading to a contradiction. Therefore, U_0, U^+ and U^- are mutually disjoint components of $\mathcal{F}(f_\lambda)$ for $\lambda > \lambda^*$. The components U^- and U^+ are unbounded by definition. □

THEOREM 5.2. *Let $f_\lambda \in \mathcal{S}$ and $\lambda > \lambda^*$. Then, the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ contains infinitely many pre-periodic components and each component of $\mathcal{F}(f_\lambda)$ is simply connected.*

Proof. It is clear from Theorem 3.2 that the point $0 \in \mathcal{F}(f_\lambda)$ for all λ. Let U_0 be the Fatou component containing 0. If $\lambda > \lambda^*$ and $\{a_{1\lambda}, a_{2\lambda}\}$ is the 2-cycle of real periodic points of f_λ then by Theorem 3.2, $(-\infty, -a_{2\lambda})$ and $(a_{2\lambda}, +\infty)$ are in the Fatou set of f_λ. Let U^- and U^+ be the Fatou components of f_λ containing $(-\infty, -a_{2\lambda})$ and $(a_{2\lambda}, +\infty)$ respectively. If a pre-image of a point of U^- lies in U^- then $U^- \cap f_\lambda(U^-) \neq \emptyset$ which shows that $U^- = f_\lambda(U^-)$ since $f_\lambda(U^-)$ is connected. It means that U^- is forward invariant. But it is already known that U^- is not forward invariant. Therefore, no pre-image of any point of U^- lies in U^-. In other words, U^- is not backward invariant. Since none of U_0 and U^+ is mapped into U^- by f_λ, each component of $f_\lambda^{-1}(U^-)$ is different from U_0 and U^+, and consequently is a pre-periodic Fatou component. Repeating the same arguments for each component of $f_\lambda^{-1}(U^-)$ and continuing the process, we can find infinitely many pre-periodic Fatou components.

Let U be any Fatou component of f_λ. Suppose, on the contrary that U is multiply connected. Let γ be a simple closed curve in U such that the bounded component $B(\gamma^c)$ of $\gamma^c = \widehat{\mathbb{C}} \setminus \gamma$ intersects the Julia set $\mathcal{J}(f_\lambda)$. Set $B_j = f_\lambda^j(B(\gamma^c))$ for $j \in \mathbb{N}$. If $B(\gamma^c)$ does not contain a pole of f_λ then $f_\lambda(z)$ is analytic on $\overline{B(\gamma^c)}$, the closure of $B(\gamma^c)$, and $B_1 = f_\lambda(B(\gamma^c))$ is bounded. Further, the function $f_\lambda(z)$ maps the interior of $B(\gamma^c)$ (which intersects the Julia set) into the interior of B_1 and, by the complete invariance of $\mathcal{J}(f_\lambda)$, it follows that $B_1 \cap \mathcal{J}(f_\lambda) \neq \emptyset$. If B_1 does not contain any pole of f_λ then consider $B_2 = f_\lambda(B_1)$ and repeat the above arguments. Since the pre-images of all the poles of f_λ are dense in $\mathcal{J}(f_\lambda)$, $B(\gamma^c)$ contains a point $\tilde{w}$ such that $f_\lambda^n(\tilde{w})$ is a pole of f_λ for a natural number n. Let n^* the minimum of all such natural numbers, minimum being taken over all points in the backward orbit of ∞ which lie in $B(\gamma^c)$. Then, the set B_{n^*} contains a pole. Since all the poles of f_λ are on the imaginary axis, the boundary of B_{n^*} intersects the imaginary axis. Therefore, the set $B_{n^*+1} = f_\lambda(B_{n^*})$ contains a neighbourhood of ∞ and the unboundedness of U^+ and U^- gives that B_{n^*+1} intersects both U^+ and U^-. Since $f_\lambda(iy) \in \mathbb{R}$ and $|f_\lambda(iy)| \geq \lambda$ for all $y \in \mathbb{R}$, the f_λ-image of ∂B_n^* intersects at least one of U^+ or U^-. Note that $\partial B_{j+1} \subseteq f_\lambda(\partial B_j)$ for $j = 1, 2, 3, \ldots, n^*$.

TABLE 1. Comparison between the dynamics of $\lambda\tanh(e^z)$ and $\lambda(z^m/\sinh^m z)$.

Dynamics of $g_\lambda(z)=\lambda\tanh(e^z)$, $\lambda\neq 0$	Dynamics of $f_\lambda(z)=\lambda z^m/\sinh^m z$, $\lambda\neq 0$, where m or $m/2$ is an odd natural number
g_λ is periodic with period $2\pi i$.	f_λ is not periodic.
g_λ is neither even nor odd.	f_λ is even.
g_λ has no critical values.	f_λ has infinitely many critical values.
g_λ has three (finite) asymptotic values 0, λ and $-\lambda$.	f_λ has only one (finite) asymptotic value 0.
The set of all singular values of g_λ is finite.	The set of all singular values of f_λ is unbounded.
Bifurcation in the dynamics of g_λ occurs at one critical parameter $\lambda^*\approx -3.2946$.	Bifurcation in the dynamics of f_λ occurs at two critical parameters $\pm\lambda^*(m)$ whose values depend on f.
The Fatou set of g_λ has infinitely many components and each component is simply connected for $\lambda\leq\lambda^*$.	The Fatou set of f_λ has infinitely many components and each component is simply connected for $\|\lambda\|\geq\lambda^*(m)$.
The Fatou set of g_λ is infinitely connected for $\lambda>\lambda^*$.	The Fatou set of f_λ is infinitely connected for $\|\lambda\|<\lambda^*(m)$.

Therefore, $\partial B_{n^*+1}\subseteq f_\lambda(\partial B_{n^*})\subseteq\cdots\subseteq f_\lambda^{n^*+1}(\gamma)\subset\mathcal{F}(f_\lambda)$ and consequently, ∂B_{n^*+1} lies either in U^+ or in U^-. Since neither U^+ nor U^- intersects the imaginary axis, ∂B_{n^*+1} cannot wind around U_0. Now, U_0 is a subset of B_{n^*+1} and each singularity of f_λ^{-1} lying over 0 is transcendental. This means that B_{n^*} contains an asymptotic path corresponding to the asymptotic value 0 which contradicts the boundedness of B_{n^*}. Therefore, U is simply connected. □

Remark 5.2. Theorem 5.2 is true for $\lambda=\lambda^*$ and the proof is similar.

The function $\lambda(z^m/\sinh^m z)$ differs in many fundamental properties from the meromorphic function $\lambda\tanh(e^z)$, but these functions exhibit similar bifurcations in their dynamics. The iteration of $\lambda\tanh(e^z)$ is studied in [**11**]. Table 1 provides a comparison between the dynamical properties of these two functions.

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