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DYNAMICS OF $\{\lambda \tanh(e^z) : \lambda \in \mathbb{R} \setminus \{0\}\}$

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Abstract. In this paper, the dynamics of transcendental meromorphic functions in the one-parameter family

$$\mathcal{M} = \{f_\lambda(z) = \lambda\, f(z)\ :\ f(z) = \tanh(e^z) \text{ for } z \in \mathbb{C} \text{ and } \lambda \in \mathbb{R} \setminus \{0\}\ \}$$

is studied. We prove that there exists a parameter value $\lambda^* \approx -3.2946$ such that the Fatou set of $f_\lambda(z)$ is a basin of attraction of a real fixed point for $\lambda > \lambda^*$ and, is a parabolic basin corresponding to a real fixed point for $\lambda = \lambda^*$. It is a basin of attraction or a parabolic basin corresponding to a real periodic point of prime period 2 for $\lambda < \lambda^*$. If $\lambda > \lambda^*$, it is proved that the Fatou set of f_λ is connected and, is infinitely connected. Consequently, the singleton components are dense in the Julia set of f_λ for $\lambda > \lambda^*$. If $\lambda \leq \lambda^*$, it is proved that the Fatou set of f_λ contains infinitely many pre-periodic components and each component of the Fatou set of f_λ is simply connected. Finally, it is proved that the Lebesgue measure of the Julia set of f_λ for $\lambda \in \mathbb{R} \setminus \{0\}$ is zero.

1. Introduction.

Let $f(z)$ be a transcendental meromorphic function in the complex plane $\mathbb{C}$. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. The central objects studied in the dynamics of a complex function f are its Fatou set and Julia set. The Fatou set of the function f, denoted by $\mathcal{F}(f)$, is defined as

$$\mathcal{F}(f) = \left\{ \begin{array}{lll} z \in \widehat{\mathbb{C}} & : & f^n(z) \text{ is defined for each } n = 0,\ 1,\ 2,\ \cdots \text{ and } \{f^n\}_{n=0}^{\infty} \\ & & \text{forms a normal family at a neighborhood of the point } z \\ & & \text{(in the sense of Montel)} \end{array} \right\}.$$

Then, the Julia set of f, denoted by $\mathcal{J}(f)$, is the complement of the Fatou set of f in the extended complex plane $\widehat{\mathbb{C}}$.

The dynamics of transcendental meromorphic functions in the one-parameter family

$$\mathcal{M} = \{f_\lambda(z) = \lambda\, f(z)\ :\ f(z) = \tanh(e^z) \text{ for } z \in \mathbb{C} \text{ and } \lambda \in \mathbb{R} \setminus \{0\}\ \}$$

is mainly studied in the present paper. The dynamics of the critically finite entire function λe^z, $(\lambda \in \mathbb{C} \setminus \{0\})$ have been extensively studied and a number of interesting properties of the Julia set of λe^z are proved [3–5, 8, 10, 11, 15, 17, 21–24].

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Devaney and Keen [12–14] studied the dynamics of meromorphic functions with constant/polynomial Schwarzian derivatives and in particular, the dynamics of functions in the one-parameter family $\{\lambda \tan z \ : \ \lambda \in \mathbb{R} \setminus \{0\} \}$. Jiang [19], Keen and Kotus [20] furthered the study of dynamics of the tangent family. It is worth to note that the dynamics of $\lambda \tanh z$ is essentially same as the dynamics of $\lambda \tan(z)$, since $\tanh z$ and $\tan z$ are conformally conjugate.

The real meromorphic function $\tan z$ maps the upper half-plane into itself and it simplifies the determination of the dynamics of $\lambda \tanh z$, $\lambda \in \mathbb{R} \setminus \{0\}$. But, the mapping properties of the real meromorphic function $\tanh(e^z)$ are comparatively more complicated. Further, the function $\tanh(e^z)$ is a meromorphic function having non-rational Schwarzian derivative and the Nevanlinna order is infinite. However, the functions λe^z and $\lambda \tanh z$ have constant Schwarzian derivatives and finite orders. Even though the order of $f_\lambda(z) = \lambda \tanh(e^z)$ is infinite, it has only 3 finite asymptotic values, namely $\pm\lambda$ and 0. The asymptotic values of e^z and $\tanh z$ are logarithmic that are the simplest kind of direct singularities of the respective inverse functions. Whereas the asymptotic values $\pm\lambda$ of f_λ are logarithmic, and 0 is the indirect singularity of the inverse function of f_λ. Thus, the properties of the function f_λ differ in many ways from that of λe^z and $\lambda \tanh z$ and that motivates us to explore the dynamics of f_λ.

It is well known that the bifurcation in the dynamics of functions in each of the families $\{T_\lambda(z) = \lambda \tanh z \ : \ \lambda \in \mathbb{R} \setminus \{0\}\}$ and $\{E_\lambda(z) = \lambda e^z \ : \ \lambda \in \mathbb{R} \setminus \{0\}\}$ occur at two critical parameter values. But, in the dynamics of functions in the family $\mathcal{M}$, we show that the bifurcation occurs only at one critical parameter. We mainly prove the following result on the dynamics of f_λ in Section 4.

Theorem 1. *Let $f_\lambda \in \mathcal{M}$. Let $\lambda^* = \frac{-1}{f'(x^*)}$ where x^* is the unique real root of $\frac{x}{f(x)} + \frac{1}{f'(x)} = 0$.*

1. *If $\lambda > \lambda^*$ then the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the basin of attraction $A(a_\lambda)$ where a_λ is the attracting real fixed point of f_λ.*
2. *If $\lambda = \lambda^*$ then the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the parabolic basin $P(x^*)$ where x^* is the rationally neutral real fixed point of f_λ.*
3. *If $\lambda < \lambda^*$ then the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle $\{a_{1\lambda},\ a_{2\lambda}\}$ of f_λ.*

For the function $T_\lambda(z) = \lambda \tanh z$, the Fatou set of T_λ is infinitely connected if $|\lambda| < 1$ and it contains only two simply connected components $\{z \in \mathbb{C} \ : \ \Re(z) < 0\}$ and $\{z \in \mathbb{C} \ : \ \Re(z) > 0\}$ if $|\lambda| \geq 1$. For the function $E_\lambda(z) = \lambda e^z$, every component of the Fatou set of E_λ is simply connected if $\lambda < -e$, and the Fatou set of E_λ itself is simply connected if $-e \leq \lambda \leq \frac{1}{e}$, and is empty if $\lambda > \frac{1}{e}$. In Section 5, the topological properties of the Fatou sets of f_λ are investigated and the following two theorems are proved.

Theorem 2. *Let $f_\lambda \in \mathcal{M}$ where $\lambda > \lambda^*$. Then, the Fatou set of f_λ is connected and, is infinitely connected.*

Theorem 3. *Let $f_\lambda \in \mathcal{M}$ where $\lambda \leq \lambda^*$. Then,*

1. *The Fatou set of f_λ contains infinitely many strictly pre-periodic (pre-periodic but not periodic) components.*
2. *Every component of the Fatou set is simply connected.*

In Section 6, it is shown that the measure of the Julia set of f_λ is zero.

2. **General properties of f_λ.** In this section, we prove some basic results on the functions $f_\lambda \in \mathcal{M}$ that are relevant in the study of their dynamics. The function $f(z) = \tanh(e^z)$ is periodic of minimal period $2\pi i$ and maps the real line $\mathbb{R}$ onto $(0,1)$. The poles of $f(z)$ are the zeros of $\cosh(e^z)$, and hence, they satisfy $e^{2e^z} = -1 = e^{i\pi(2k+1)}$ for $k \in \mathbb{Z}$. Therefore, the set of poles of $f(z)$ is $\{z = x + iy \in \mathbb{C} : x = \ln\left|\frac{\pi}{2}(2k+1)\right|$ and $y = \frac{\pi}{2}(2l+1)$ where $k \in \mathbb{Z}$ and $l \in \mathbb{Z}\}$. Further, all the poles are simple and lie in the right half-plane $\{z \in \mathbb{C} : \Re(z) \geq \ln(\frac{\pi}{2})\}$. One can show that the Nevanlinna order of the function $f(z) = \tanh(e^z)$ is infinite.

Lemma 1. *Let g be a non-constant critically finite meromorphic function in $\mathbb{C}$ and h be a non-constant critically finite entire function. Let $F(z) = g(h(z))$ be the composition function. If a is a finite asymptotic value of $F(z)$ then either a is an asymptotic value of g or there exists $b \in \mathbb{C}$ such that $g(b) = a$ and b is an asymptotic value of h. Consequently,* the number of finite asymptotic values of the composite function F is at most the sum of the number of finite asymptotic values of the individual functions g and h.

Proof. Let $\gamma : [0, \infty) \to \mathbb{C}$ be an asymptotic path corresponding to the asymptotic value a for the function $F(z)$. Let M denote the collection of all limit points of the set $\{h(\gamma(t_k)) : \{t_k\}$ is any sequence of positive real numbers which tends to ∞ as $k \to \infty\}$. Observe that $g(z) = a$ for every $z \in M$. Since g is a non-constant meromorphic function, the set M cannot have any limit point in $\mathbb{C}$. Therefore, $M \bigcap \mathbb{C}$ is a discrete subset of $\mathbb{C}$. Now we claim that M contains only one element in $\widehat{\mathbb{C}}$. If possible, the set M contains more than one element in $\widehat{\mathbb{C}}$. Suppose that m_1 and m_2 are in M with $m_1 \neq m_2$. Then, there exist open disks $B_1(m_1)$ and $B_2(m_2)$ such that $\overline{B_1(m_1)} \cap M = \{m_1\}$ and $\overline{B_2(m_2)} \cap M = \{m_2\}$. The curve $h(\gamma(t))$ intersects the disks $B_1(m_1)$ and $B_2(m_2)$ infinitely many times and also the boundaries $C_1 = \partial B_1(m_1)$ and $C_2 = \partial B_2(m_2)$ of these disks infinitely many times. Note that, if $\{h(\gamma(t)) : t \geq 0\} \cap C_i$ is a finite set S (say), then $S \bigcap M \neq \emptyset$ which is a contradiction to $\overline{B_i(m_i)} \cap M = \{m_i\}$ for $i = 1,\ 2$. Suppose that $\{h(\gamma(t)) : t \geq 0\} \cap C_i$ is an infinite set. Then, the intersecting points $\{h(\gamma(t)) : t \geq 0\} \cap C_i$ will have a limit point, l_i (say), since C_1 and C_2 are compact. This implies that $l_i \in M$ which is a contradiction to $\overline{B_i(m_i)} \cap M = \{m_i\}$ for $i = 1,\ 2$. Therefore, M is a singleton set in $\widehat{\mathbb{C}}$.

If $M = \{b\}$ where $b \in \mathbb{C}$ then $a = g(b)$ and b is an asymptotic value of $h(z)$. If $M = \{\infty\}$ then a is an asymptotic value of $g(z)$. Therefore, in both the cases, a corresponds either to an asymptotic value of h or to that of g. This completes the proof. □

The following proposition determines all the singular values of $f_\lambda \in \mathcal{M}$.

Proposition 1. *Let $f_\lambda \in \mathcal{M}$. Then $f_\lambda(z)$ has only three (finite) asymptotic values and no critical values.*

Proof. Since $f'_\lambda(z) = \lambda\, e^z\ \operatorname{sech}^2(e^z) \neq 0$ for any $z \in \mathbb{C}$, it follows that $f_\lambda(z)$ has no critical points and hence, it has no critical values.

Turning to asymptotic values, by Lemma 1, it follows that $f_\lambda(z)$ will have at most 3 finite asymptotic values, since e^z has only one finite asymptotic value, namely, 0 and $\lambda\tanh(z)$ has two finite asymptotic values, namely, λ and $-\lambda$.

If $\gamma_1(t) = -t$ for $t \in [0,\ \infty)$ then $\lim_{t\to\infty} f_\lambda(\gamma_1(t)) = 0$. If $\gamma_2(t) = t$ for $t \in [0,\ \infty)$ then $\lim_{t\to\infty} f_\lambda(\gamma_2(t)) = \lambda$. When $\gamma_3(t) = t + i\pi$ for $t \in [0,\ \infty)$, $\lim_{t\to\infty} f_\lambda(\gamma_3(t)) = -\lambda$. Therefore, 0 and $\pm\lambda$ are the three finite asymptotic values of $f_\lambda(z)$. □

Two meromorphic functions $f_1,\ f_2 : \mathbb{C} \to \widehat{\mathbb{C}}$ are called conformally conjugate if there is an analytic homeomorphism ψ on $\widehat{\mathbb{C}}$ such that $\psi(f_1(z)) = f_2(\psi(z))$ for all $z \in \widehat{\mathbb{C}}$. If a conformal conjugacy ψ exists between two transcendental meromorphic functions then the analytic homeomorphic map $\psi(z)$ will be of the form $\psi(z) = az+b$ where a and b are complex constants with $a \neq 0$. In the following, we show that no two functions f_{λ_1} and f_{λ_2} in $\mathcal{M}$ are conformally conjugate. Suppose that there exists an analytic homeomorphism $\psi(z) = az+b$ for all $z \in \widehat{\mathbb{C}}$ between two functions f_{λ_1} and f_{λ_2} in $\mathcal{M}$ with $\lambda_1 \neq \lambda_2$. In the proof of Proposition 1, it is shown that the function f_{λ_i} has three finite asymptotic values, namely, 0, λ_i and $-\lambda_i$. Note that $\pm\lambda_i$ are the exceptional values of f_{λ_i}. Now, the conjugacy map ψ is required to take the set $\{\lambda_1,\ -\lambda_1\}$ to $\{\lambda_2,\ -\lambda_2\}$. That is, either $\psi(\lambda_1) = \lambda_2$, $\psi(-\lambda_1) = -\lambda_2$ or $\psi(\lambda_1) = -\lambda_2$, $\psi(-\lambda_1) = \lambda_2$. It implies that $b = 0$ and $af_{\lambda_1}(z) = f_{\lambda_2}(az)$ for all $z \in \mathbb{C}$. Therefore, it follows that $af'_{\lambda_1}(0) = af'_{\lambda_2}(0)$ and $\lambda_1 = \lambda_2$ which is not true.

Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)}$. Rewriting it,

$$\phi(x) = \frac{xf'(x) + f(x)}{f(x)f'(x)} = \frac{1}{4e^x\ \tanh(e^x)}(4xe^x\ +\ e^{2e^x}\ -\ e^{-2e^x})\ .$$

Letting $\phi_1(x) = 4xe^x + e^{2e^x} - e^{-2e^x}$, we observe that $\phi_1'(x) = 2e^x(2x + 2 + e^{2e^x} + e^{-2e^x}) = 2e^x\phi_2(x)$, where $\phi_2(x) = 2x + 2 + e^{2e^x} + e^{-2e^x}$. The function $\phi_2'(x) = 2 + 2e^x(e^{2e^x} - e^{-2e^x}) > 0$ for $x < 0$. This implies that $\phi_2(x)$ is strictly increasing for $x < 0$. Since $\phi_2(x) \to -\infty$ as $x \to -\infty$ and $\phi_2(x) \to 2 + e^2 + e^{-2} > 0$ as $x \to 0$, there exists a point $x_2 < 0$ such that $\phi_2(x) < 0$ for $x < x_2$, $\phi_2(x_2) = 0$ and $\phi_2(x) > 0$ for $x_2 < x < 0$ and consequently, $\phi_1'(x) < 0$ for $x < x_2$, $\phi_1'(x_2) = 0$ and $\phi_1'(x) > 0$ for $x_2 < x < 0$. Therefore, $\phi_1(x)$ is decreasing for $x < x_2$ and, is increasing for $x_2 < x < 0$. This shows that the function $\phi_1(x)$ attains the minimum value at the point x_2 and the minimum value $\phi_1(x_2)$ is negative, because $\phi_1(x) \to 0$ as $x \to -\infty$. Since $\phi_1(x) \to e^2 - e^{-2} > 0$ as $x \to 0$, there exists a unique point x^* with $x_2 < x^* < 0$ such that $\phi_1(x) < 0$ for $x < x^*$, $\phi_1(x) = 0$ for $x = x^*$ and $\phi_1(x) > 0$ for $x^* < x < 0$; and consequently, $\phi(x) < 0$ for $x < x^*$, $\phi(x) = 0$ for $x = x^*$ and $\phi(x) > 0$ for $x^* < x < 0$. Observe that $\phi(x) > 0$ for $x \geq 0$.

Define

$$\lambda^* = \frac{x^*}{f(x^*)} = \frac{-1}{f'(x^*)} \tag{1}$$

where x^* is the unique real root of the equation $\phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} = 0$. Numerically it is found that $x^* \approx -1.0789$ and $\lambda^* \approx -3.2946$.

3. **Real periodic points.** In this section, the real periodic points of $f_\lambda \in \mathcal{M}$ are investigated. In Proposition 2, it is proved that f_λ cannot have a real periodic point of prime period more than two. The existence and nature of the real fixed points is proved in Proposition 3. The existence and nature of the real periodic points of prime period 2 is analyzed in Proposition 4.

Proposition 2. *Let $f_\lambda \in \mathcal{M}$. Then, f_λ has no real periodic point of prime period more than two.*

Proof. Let $g_\lambda(x) = f_\lambda(f_\lambda(x))$ for $x \in \mathbb{R}$. Then, the function $g_\lambda(x)$ is strictly increasing on $\mathbb{R}$, since $g'_\lambda(x) = \lambda f'(\lambda f(x))\lambda f'(x) > 0$ for all $x \in \mathbb{R}$. Suppose there exists a point x_0 such that x_0 is a real periodic point of prime period $p = 2$ for g_λ. Since $g_\lambda(x_0) \neq x_0$, either $g_\lambda(x_0) > x_0$ or $g_\lambda(x_0) < x_0$. If $g_\lambda(x_0) > x_0$, it implies that $g^k_\lambda(x_0) > g^{k-1}_\lambda(x_0)$ for all $k > 1$. It makes $g^k_\lambda(x_0) > x_0$ for every $k \in \mathbb{N}$ which is a contradiction to the fact that $g^p_\lambda(x_0) = x_0$. A similar contradiction can well be realized by assuming $g_\lambda(x_0) < x_0$. Therefore, $g_\lambda(x)$ has no real periodic point of prime period $p = 2$. Since $g_\lambda(x)$ cannot have a real periodic point of prime period 2, it follows that the function $f_\lambda(x)$ cannot have a real periodic point of prime period 4. Since f_λ is a continuous real valued function and not possessing a real periodic point of prime period 4 on $\mathbb{R}$, by Sarkovskii's theorem ([9], Page 62), we conclude that $f_\lambda(x)$ cannot have a real periodic point of prime period more than two. □

The function $f(x) = \tanh(e^x)$ is positive for all $x \in \mathbb{R}$. Since $f'(x) = e^x \operatorname{sech}^2(e^x)$ > 0 for all $x \in \mathbb{R}$, the function $f(x)$ is strictly increasing on $\mathbb{R}$. It is easy to see that $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 1$ as $x \to +\infty$. Now, we find the nature of the function $f''(x) = e^x \operatorname{sech}^2(e^x)(1 - 2e^x \tanh(e^x))$ on $\mathbb{R}$. Observe that the function $\frac{d}{dx}(1 - 2e^x \tanh(e^x)) = -2e^x(e^x \operatorname{sech}^2(e^x) + \tanh(e^x)) < 0$ for all $x \in \mathbb{R}$. Therefore, the function $\psi(x) = 1 - 2e^x \tanh(e^x)$ is strictly decreasing on $\mathbb{R}$. Since $\lim_{x\to-\infty} 1 - 2e^x \tanh(e^x) = 1$ and $\lim_{x\to 0} 1 - 2e^x \tanh(e^x) = 1 - 2\,\dfrac{e^2-1}{e^2+1} = \dfrac{3-e^2}{e^2+1} < 0$, it follows that there exists a point $\hat{x} < 0$ such that $\psi(x) > 0$ for $x < \hat{x}$, $\psi(x) = 0$ for $x = \hat{x}$ and $\psi(x) < 0$ for $x > \hat{x}$. Consequently $f''(x) = e^x \operatorname{sech}^2(e^x)(1 - 2e^x \tanh(e^x)) > 0$ for $x < \hat{x}$, $f''(x) = 0$ for $x = \hat{x}$ and $f''(x) < 0$ for $x > \hat{x}$. This shows that the function $f'(x)$ increases in the interval $(-\infty,\ \hat{x})$, decreases in the interval $(\hat{x},\ \infty)$ and attains the maximum value at the point $\hat{x}$. Also $f'(x) \to 0$ as $|x| \to \infty$. Define $\hat{\lambda}$ as $1/f'(\hat{x})$. It is numerically computed that $\hat{x} \approx -0.261$ and $\hat{\lambda} \approx 2.233$.

The existence and the nature of the real fixed points is proved in the following proposition.

Proposition 3. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, f_λ has a unique real fixed point a_λ (say) and that is attracting.*
2. *If $\lambda = \lambda^*$, f_λ has a unique rationally neutral real fixed point at $x = x^*$, where x^* is the unique real root of $\phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} = 0$.*
3. *If $\lambda < \lambda^*$, f_λ has a unique real fixed point r_λ (say) and that is repelling.*

Proof. Set $h_\lambda(x) = f_\lambda(x) - x = \lambda f(x) - x$ where $f(x) = \tanh(e^x)$ for $x \in \mathbb{R}$ and λ is a non-zero real parameter. Then, $h'_\lambda(x) = \lambda f'(x) - 1$ and $h''_\lambda(x) = \lambda f''(x)$.
For all λ,

$$\lim_{x\to-\infty} h_\lambda(x) = +\infty \qquad \text{and} \qquad \lim_{x\to+\infty} h_\lambda(x) = -\infty\ .$$

Since $h_\lambda(x)$ is a continuous function on $\mathbb{R}$, it has a real zero. Consequently, the function f_λ has a real fixed point x_λ (say). Since $f(x) > 0$ for all $x \in \mathbb{R}$, the real fixed point of f_λ has the same sign as that of λ. If $\lambda > 0$, the function $h'_\lambda(x)$ is increasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1$ in the interval $(-\infty, \hat{x}]$ and it is decreasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$ where $\hat{x}$ satisfies $f''(\hat{x}) = 0$. If $\lambda < 0$, the function $h'_\lambda(x)$ is decreasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 < 0$ in the interval $(-\infty, \hat{x}]$ and it is increasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$. For $\lambda < 0$, it follows that the function $h_\lambda(x)$ is strictly decreasing and consequently, the real fixed point x_λ of f_λ is unique.

Case (1): $\lambda > \lambda^*$

Subcase (a): $\lambda \geq \hat{\lambda}$

In this case, the function $h'_\lambda(x)$ is increasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 \geq \hat{\lambda} f'(\hat{x}) - 1 = 0$ in the interval $(-\infty, \hat{x}]$ and it is decreasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$. Therefore, there exist two points $x_{1,\lambda}$ and $x_{2,\lambda}$ (say) with $x_{1,\lambda} \leq x_{2,\lambda}$ such that $h'_\lambda(x) = 0$ for $x = x_{1,\lambda}$ and $x = x_{2,\lambda}$. Further, $h'_\lambda(x) < 0$ for $x \in (-\infty,\ x_{1,\lambda}) \cup (x_{2,\lambda},\ \infty)$ and $h'_\lambda(x) > 0$ for $x \in (x_{1,\lambda},\ x_{2,\lambda})$. If $x_{2,\lambda} \leq 0$, then $-1 < h'_\lambda(x) < 0$ for all $x > 0$. Therefore, it follows that the real fixed point x_λ (which is positive as $\lambda > 0$ in this case) of f_λ is unique and attracting. When $x_{2,\lambda} > 0$, the function h_λ attains the maximum value at $x = x_{2,\lambda}$ in $(0,\ \infty)$. Since $0 < h_\lambda(0) < h_\lambda(x_{2,\lambda})$ and $h_\lambda(x)$ is decreasing in the interval $(x_{2,\lambda},\ \infty)$, it follows that $x_{2,\lambda} < x_\lambda$. Therefore, the real fixed point x_λ of f_λ is unique and attracting. Let us rename the fixed point x_λ as a_λ when $\lambda \geq \hat{\lambda}$.

Subcase (b): $0 < \lambda < \hat{\lambda}$

If $0 < \lambda < \hat{\lambda}$, the maximum value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 < \hat{\lambda} f'(\hat{x}) - 1 = 0$. It follows that $-1 < h'_\lambda(x) = f'_\lambda(x) - 1 < 0$ for all $x \in \mathbb{R}$. Therefore, the real fixed point x_λ of f_λ is unique and it is an attracting fixed point for f_λ. Rename the real fixed point x_λ as a_λ.

Subcase (c): $-\hat{\lambda} < \lambda \ < 0$

If $-\hat{\lambda} < \lambda < 0$, the minimum value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 > -\hat{\lambda} f'(\hat{x}) - 1 = -2$. It follows that $-2 < h'_\lambda(x) = f'_\lambda(x) - 1 < -1$ for all $x \in \mathbb{R}$. Therefore, the real fixed point x_λ of f_λ is attracting for f_λ. In this case, we rename x_λ as a_λ.

Subcase (d): $\lambda^* < \lambda \leq -\hat{\lambda}$

The function $h'_\lambda(x)$ is decreasing from the value -1 to the value $h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 \leq -\hat{\lambda} f'(\hat{x}) - 1 \leq -2$ in the interval $(-\infty, \hat{x}]$ and it is increasing from the value $h'_\lambda(\hat{x})$ to -1 in the interval $[\hat{x},\ \infty)$. Since $h'_\lambda(\hat{x}) + 2 \leq 0$ for $\lambda^* < \lambda \leq -\hat{\lambda}$, there exist two points $y_{1,\lambda}$ and $y_{2,\lambda}$ (say) with $y_{1,\lambda} \leq y_{2,\lambda}$ such that $h'_\lambda(x) + 2 = 0$ for $x = y_{1,\lambda}$ and $x = y_{2,\lambda}$. Further, $h'_\lambda(x) + 2 > 0$ for $x \in (-\infty,\ y_{1,\lambda}) \cup (y_{2,\lambda},\ \infty)$ and $h'_\lambda(x) + 2 < 0$ for $x \in (y_{1,\lambda},\ y_{2,\lambda})$. Now, the parameter λ can be realized in two ways as $\lambda = \frac{-1}{f'(y_{1,\lambda})}$ and $\lambda = \frac{x_\lambda}{f(x_\lambda)}$ where $y_{1,\lambda}$ is the smaller root of $h'_\lambda(x) + 2 = 0$ and x_λ is the unique real fixed point of f_λ. It is noticed that $x^* < \hat{x} < 0$. Now we shall show that the points x_λ and $y_{1,\lambda}$ are in the interval $(x^*, \hat{x}]$ and $x_\lambda < y_{1,\lambda}$. Since $\lambda^* < \lambda \leq -\hat{\lambda}$, we have $\frac{-1}{f'(x^*)} < \frac{-1}{f'(y_{1,\lambda})} \leq \frac{-1}{f'(\hat{x})}$. Using the fact that $\frac{-1}{f'}$ is strictly increasing in $(-\infty, \hat{x})$, we get

$$x^* < y_{1,\lambda} \leq \hat{x}\ .$$

For all $x < 0$, $\frac{d}{dx}\left(\frac{x}{f(x)}\right) > 0$ implies that $\frac{x}{f(x)}$ is strictly increasing in $\mathbb{R}^- = \{x \in \mathbb{R}\ :\ x < 0\}$. So, the inequality $\lambda^* < \lambda \leq -\hat{\lambda}$ gives $\frac{x^*}{f(x^*)} < \frac{x_\lambda}{f(x_\lambda)} \leq \frac{-1}{f'(\hat{x})}$. Since $\hat{x} > x^*$, we have $\phi(\hat{x}) > 0$ and $\frac{\hat{x}}{f(\hat{x})} > \frac{-1}{f'(\hat{x})}$. Therefore, $\frac{x^*}{f(x^*)} < \frac{x_\lambda}{f(x_\lambda)} \leq \frac{-1}{f'(\hat{x})} < \frac{\hat{x}}{f(\hat{x})}$ which gives that

$$x^* < x_\lambda < \hat{x}\ .$$

Since $\phi(y_{1,\lambda}) > 0$, it follows that $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} > \frac{-1}{f'(y_{1,\lambda})} = \frac{x_\lambda}{f(x_\lambda)}$. Since the function $\frac{x}{f(x)}$ is increasing for $x < 0$, we get $x_\lambda < y_{1,\lambda}$. Now, the function $h'_\lambda(x) + 2 > 0$ for $x < y_{1,\lambda}$. So, it follows that $-1 < f'_\lambda(x) < 0$ for $x < y_{1,\lambda}$ and in particular,

$-1 < f'_\lambda(x_\lambda) < 0$. Therefore, the real fixed point x_λ is attracting and rename it as a_λ.

Case (2): $\lambda = \lambda^*$

By definition $\lambda^* = \frac{x^*}{f(x^*)} = \frac{-1}{f'(x^*)}$. Since the function $\frac{x}{f(x)}$ is one-to-one in the negative real axis, it follows that the real fixed point x_λ is equal to x^*. The real fixed point x^* is a rationally neutral fixed point, because $\lambda^* f'(x^*) = -1$.

Case (3): $\lambda < \lambda^*$

As in Subcase (d), the minimum value $h'_\lambda(\hat{x}) < -2$. Therefore, there exist two points $y_{1,\lambda}$ and $y_{2,\lambda}$ (say) with $y_{1,\lambda} < y_{2,\lambda}$ such that $h'_\lambda(x) + 2 = 0$ for $x = y_{1,\lambda}$ and $x = y_{2,\lambda}$. Further, $h'_\lambda(x) + 2 > 0$ for $x \in (-\infty,\ y_{1,\lambda}) \cup (y_{2,\lambda},\ \infty)$ and $h'_\lambda(x) + 2 < 0$ for $x \in (y_{1,\lambda},\ y_{2,\lambda})$. Here our intention is to show that the fixed point x_λ lies in $(y_{1,\lambda},\ y_{2,\lambda})$ where $|f'_\lambda(x)| > 1$. Arguing on the similar lines as in Subcase (d), one can get that $y_{1,\lambda} < x^* < \hat{x} < y_{2,\lambda}$ and $x_\lambda < x^* < \hat{x} < y_{2,\lambda}$ for $\lambda < \lambda^*$. Since $\phi(y_{1,\lambda}) < 0$, we get $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} < \frac{-1}{f'(y_{1,\lambda})}$. But, $\lambda = \frac{-1}{f'(y_{1,\lambda})} = \frac{x_\lambda}{f(x_\lambda)}$. Therefore, $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} < \frac{x_\lambda}{f(x_\lambda)}$ and consequently $y_{1,\lambda} < x_\lambda$. Therefore, the fixed point x_λ is repelling. Let us rename it as r_λ. □

The existence and the nature of the real periodic points of prime period 2 is explored in the following proposition.

Proposition 4. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$, f_λ^2 has only one real fixed point a_λ which is an attracting real fixed point of f_λ.*
2. *If $\lambda = \lambda^*$, f_λ^2 has only one real fixed point x^* which is a rationally neutral real fixed point of f_λ.*
3. *If $\lambda < \lambda^*$, f_λ^2 has exactly three real fixed points. One of the fixed points of f_λ^2 is r_λ which is a repelling real fixed point of f_λ. The other two fixed points of f_λ^2 are the periodic points of (prime) period 2 of f_λ and form an attracting or a parabolic 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ (say) with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$.*

Proof. **Case 1:** $\lambda > \lambda^*$

If $\lambda > \lambda^*$, by Proposition 3(1), $f_\lambda(x)$ has a unique attracting fixed point a_λ on the real line. The fixed point a_λ of f_λ is also a fixed point of f_λ^2. Now, we show that f_λ^2 has no other real fixed points.

For $\lambda > 0$, f_λ is strictly increasing on $\mathbb{R}$. If $f_\lambda(x) \neq x$ for a point $x \in \mathbb{R}$ then $f_\lambda^n(x) \neq x$ for any integer $n > 1$. Therefore, it follows that f_λ $(\lambda > 0)$ has no real periodic points of prime period $p \geq 2$.

Let $\lambda^* < \lambda < 0$. Suppose that there is a fixed point of f_λ^2 which is different from a_λ. As f_λ has only one real fixed point, any fixed point other than a_λ of f_λ^2 will be a 2-periodic cycle for f_λ. If f_λ has more than one 2-periodic cycles then the outermost 2-periodic cycle is chosen for consideration. This is possible, because, if f_λ has two different 2-periodic cycles $\{a,\ b\}$ with $a < b$ and $\{c,\ d\}$ with $c < d$, then it follows from the fact f_λ is strictly decreasing for $\lambda < 0$ that the two different 2-periodic cycles satisfy $c < a < a_\lambda < b < d$ or $a < c < a_\lambda < d < b$. In the first case $\{c,\ d\}$ and in the second case $\{a,\ b\}$ is called the outer cycle.

Let $\{d_{1\lambda},\ d_{2\lambda}\}$ be the outermost 2-periodic cycle of f_λ such that $f_\lambda(d_{1\lambda}) = d_{2\lambda}$ and $f_\lambda(d_{2\lambda}) = d_{1\lambda}$ with $d_{1\lambda} < d_{2\lambda}$. Set $D_1 = (-\infty, d_{1\lambda})$ and $D_2 = (d_{2\lambda}, \infty)$. Since $f_\lambda^2(x) > x$ for any $x \in D_1$, the sequence $\{f_\lambda^{2n}(x)\}$ will be a monotonically increasing sequence and $d_{1\lambda} = \sup\{f_\lambda^{2n}(x)\ :\ x \in D_1 \text{ and } n \in \mathbb{N}\}$. Therefore, $f_\lambda^{2n}(x) \to d_{1\lambda}$ as $n \to \infty$. Similarly, $\{f_\lambda^{2n}(x)\}$ is a monotonically decreasing sequence converging

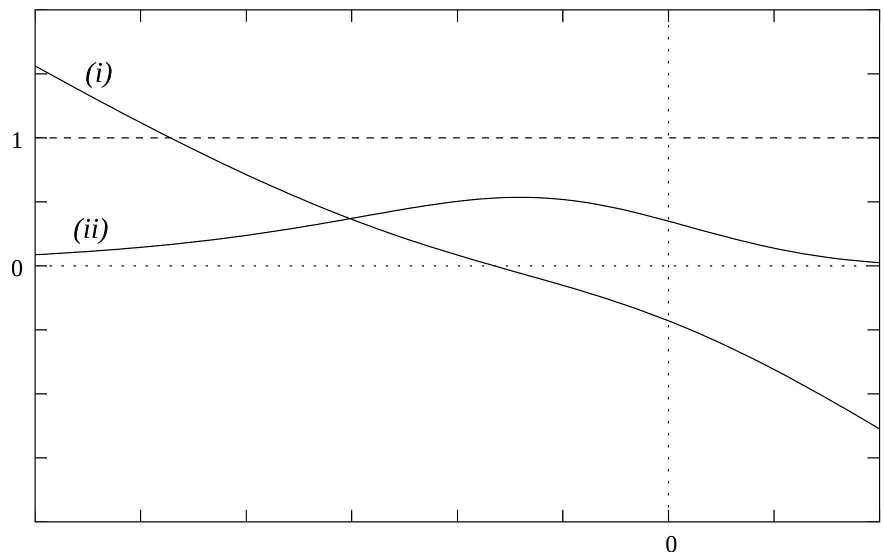

FIGURE 1. Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $\left(f_\lambda^2\right)'(x)$ for $\lambda > \lambda^*$

to $d_{2\lambda}$ for every $x \in D_2$, since $f_\lambda^2(x) < x$ for $x \in D_2$ and $d_{2\lambda} = \inf\{f_\lambda^{2n}(x) \; : \; x \in D_2 \text{ and } n \in \mathbb{N}\}$. This shows that the cycle $\{d_{1\lambda},\ d_{2\lambda}\}$ can be either an attracting or a parabolic cycle. Note that $\lambda < d_{1\lambda} < a_\lambda < d_{2\lambda} < 0 < -\lambda$. This implies that $f_\lambda^{2n}(\lambda) \to d_{1\lambda}$, $f_\lambda^{2n}(0) \to d_{2\lambda}$ and $f_\lambda^{2n}(-\lambda) \to d_{2\lambda}$ as $n \to \infty$. Thus, all the singular values are attracted by the 2-periodic cycle $\{d_{1\lambda},\ d_{2\lambda}\}$. It is shown in Proposition 3 that a_λ is a real attracting fixed point of f_λ for $\lambda > \lambda^*$. So, the basin of attraction $A(a_\lambda)$ of the attracting fixed point a_λ must contain at least one singular value which is a contradiction to the fact that all three singular values tend either to $d_{1\lambda}$ or to $d_{2\lambda}$ under iterations of f_λ^2. Therefore, f_λ^2 cannot have any real fixed point other than a_λ if $\lambda^* < \lambda < 0$ (See Figure 1).

Case 2: $\lambda = \lambda^*$:
If $\lambda = \lambda^*$, by Proposition 3(2), $f_\lambda(x)$ has a unique rationally neutral fixed point x^* on the real line. The fixed point x^* of f_{λ^*} is also a fixed point for $f_{\lambda^*}^2$. By similar arguments as in Case 1, one can show that $f_{\lambda^*}^2$ has no real periodic point of prime period 2 (See Figure 2).

Case 3: $\lambda < \lambda^*$:
If $\lambda < \lambda^*$, by Proposition 3(3), $f_\lambda(x)$ has a unique repelling fixed point r_λ on the real line. The fixed point r_λ of f_λ is also a fixed point for f_λ^2. Now, we show that including r_λ, the function f_λ^2 has 3 fixed points on $\mathbb{R}$.

Let $x < r_\lambda$. Suppose that $f_\lambda^2(x) > x$. Since $f_\lambda^2(x)$ is strictly increasing on $\mathbb{R}$, it follows that $f_\lambda^{2n}(x) > f_\lambda^{2(n-1)}(x)$ for all $n \in \mathbb{N}$. But, the sequence $\{f_\lambda^{2n}(x)\}_{n>0}$ is bounded above by r_λ. By Bolzano theorem, the sequence $\{f_\lambda^{2n}(x)\}$ converges to a point a (say). By the continuity of f_λ, it follows that the limit point a will be a periodic point of prime period at most two. As f_λ does not have any real fixed point other than r_λ, the limit point a must be a periodic point of prime period

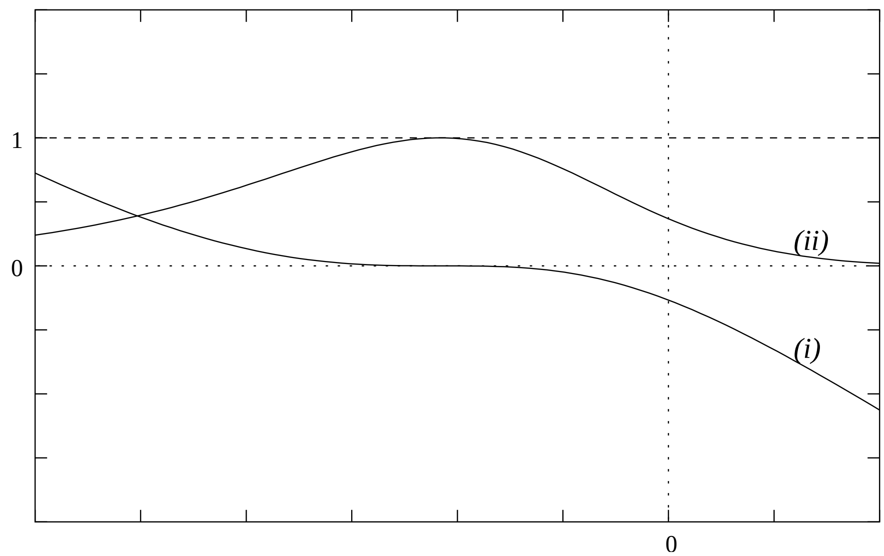

FIGURE 2. Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $\left(f_\lambda^2\right)'(x)$ for $\lambda = \lambda^*$

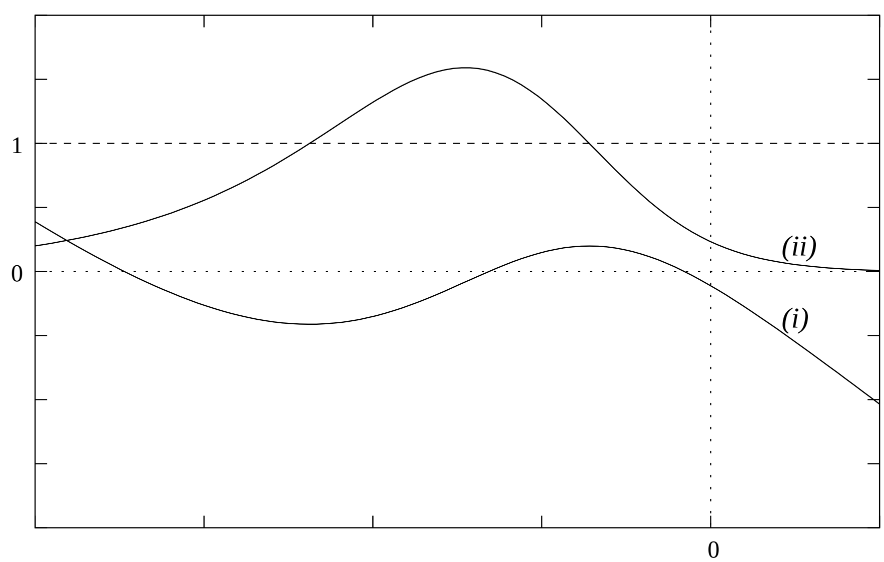

FIGURE 3. Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $\left(f_\lambda^2\right)'(x)$ for $\lambda < \lambda^*$

2. Similarly, the other possibility $f_\lambda^2(x) < x$ also leads to the same conclusion. Therefore, f_λ has a periodic point of prime period 2 on $\mathbb{R}$.

Now, we show that f_λ has a unique periodic point of prime period 2 on $\mathbb{R}$. Suppose that f_λ has more than one periodic point of prime period 2 on $\mathbb{R}$. Then, choose the outermost (in the sense defined earlier in **Case 1**) 2-periodic cycle of f_λ.

Let $\{o_{1\lambda},\ o_{2\lambda}\}$ be the outermost 2-periodic cycle of f_λ such that $f_\lambda(o_{1\lambda}) = o_{2\lambda}$ and $f_\lambda(o_{2\lambda}) = o_{1\lambda}$ with $o_{1\lambda} < o_{2\lambda}$. As shown in case of $\lambda \in (\lambda^*,\ 0)$, we can show that the 2-periodic cycle $\{o_{1\lambda},\ o_{2\lambda}\}$ is either an attracting cycle or a parabolic cycle of f_λ and the singular values 0 and $\pm\lambda$ are attracted by this cycle. Now, let us consider the innermost 2-periodic cycle $\{i_{1\lambda},\ i_{2\lambda}\}$ (say) of f_λ with $f_\lambda(i_{1\lambda}) = i_{2\lambda}$ and $f_\lambda(i_{2\lambda}) = i_{1\lambda}$ with $i_{1\lambda} < i_{2\lambda}$. Observe that $f_\lambda(x) \in (r_\lambda,\ i_{2\lambda})$ for $x \in (i_{1\lambda},\ r_\lambda)$ and $f_\lambda(x) \in (i_{1\lambda},\ r_\lambda)$ for $x \in (r_\lambda,\ i_{2\lambda})$. This gives that the sequence $\{f_\lambda^{2n}(x)\}$ is bounded for $x \in (i_{1\lambda},\ i_{2\lambda})$. Since f_λ^2 is strictly increasing on $\mathbb{R}$, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonic. Since r_λ is repelling, $\{f_\lambda^{2n}(x)\} \to i_{1\lambda}$ as $n \to \infty$ for $x \in (i_{1\lambda},\ r_\lambda)$ and $\{f_\lambda^{2n}(x)\} \to i_{2\lambda}$ as $n \to \infty$ for $x \in (r_\lambda,\ i_{2\lambda})$. This shows that the inner cycle $\{i_{1\lambda},\ i_{2\lambda}\}$ is also either attracting or parabolic. But, there is no singular value that can be attracted by the inner cycle $\{i_{1\lambda},\ i_{2\lambda}\}$, since all the singular values are already attracted by the outermost cycle. This rules out the existence of the innermost cycle $\{i_{1\lambda},\ i_{2\lambda}\}$. Therefore, the function f_λ has only one 2-periodic cycle $\{a_{1\lambda},\ a_{2\lambda}\}$ (say) that is either attracting or parabolic on $\mathbb{R}$ if $\lambda < \lambda^*$ (See Figure 3). This completes the proof. ∎

4. **Dynamics of** $f_\lambda \in \mathcal{M}$**.** The dynamics of the function $f_\lambda(z)$ for $z \in \mathbb{C}$ is investigated in the present section.

Proposition 5. *Let $f_\lambda \in \mathcal{M}$. Then, the Fatou set of f_λ does not contain wandering domain or Baker domain.*

Proof. For critically finite meromorphic functions, the non-existence of wandering domains is proved in [2] and the non-existence of Baker domains is proved in [6]. Since the function f_λ is critically finite by Proposition 1, it follows that the Fatou set of f_λ does not contain wandering domain or Baker domain. ∎

We determine the dynamics of f_λ on the real line in the following proposition.

Proposition 6. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$ then $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for all $x \in \mathbb{R}$ where a_λ is the attracting real fixed point of f_λ.*
2. *If $\lambda = \lambda^*$ then $f_\lambda^n(x) \to x^*$ as $n \to \infty$ for all $x \in \mathbb{R}$ where x^* is the rationally neutral real fixed point of f_λ.*
3. *If $\lambda < \lambda^*$ then $f_\lambda^{2n}(x) \to a_{1\lambda}$ as $n \to \infty$ for $x < r_\lambda$ and $f_\lambda^{2n}(x) \to a_{2\lambda}$ as $n \to \infty$ for $x > r_\lambda$ where $\{a_{1\lambda},\ a_{2\lambda}\}$ is the attracting or parabolic real 2-periodic cycle and r_λ is the repelling real fixed point of f_λ.*

Proof. **Case 1:** $\lambda > \lambda^*$
By Proposition 3(1) and Proposition 4(1), the function $f_\lambda(z)$ has a unique real attracting fixed point a_λ and f_λ^2 has no fixed point other than a_λ on the real line. It is noted that f_λ^2 is strictly increasing and bounded on $\mathbb{R}$. Observe that $f_\lambda^2(x) > x$ for $x < a_\lambda$. It implies that $\{f_\lambda^{2n}(x)\}$ is a monotonically increasing bounded sequence and hence convergent. By continuity of f_λ^2, it follows that the limit point of $\{f_\lambda^{2n}(x)\}$ is a fixed point of f_λ^2 and therefore it equals to the only such point, namely, a_λ. Therefore, $f_\lambda^{2n}(x) \to a_\lambda$ as $n \to \infty$ for $x < a_\lambda$. Similarly, the same conclusion follows for $x > a_\lambda$ since $f_\lambda^2(x) < x$ for $x > a_\lambda$. Therefore, $\lim\limits_{n\to\infty} f_\lambda^{2n}(x) = a_\lambda$ for all $x \in \mathbb{R}$. Since a_λ is an attracting fixed point of the continuous function f_λ, it is concluded that $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for all $x \in \mathbb{R}$.
Case 2: $\lambda = \lambda^*$
By Proposition 3(2) and Proposition 4(2), the function $f_\lambda(z)$ has a unique rationally

neutral real fixed point x^* and f_λ^2 has no fixed point other than x^* on the real line. Since f_λ^2 is a strictly increasing, bounded function on $\mathbb{R}$ with $f_\lambda^2(x) > x$ for $x < x^*$ and $f_\lambda^2(x) < x$ for $x > x^*$, it follows by similar arguments as in the previous case that $f_\lambda^{2n}(x) \to x^*$ as $n \to \infty$ for $x \in \mathbb{R}$. Since f_λ is continuous and x^* is a fixed point of f_λ, it is concluded that $f_\lambda^n(x) \to x^*$ as $n \to \infty$ for all $x \in \mathbb{R}$.

Case 3: $\lambda < \lambda^*$

On the real line $\mathbb{R}$, the function f_λ has a unique repelling real fixed point r_λ by Proposition 3(3), and has a unique attracting or parabolic 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$ by Proposition 4(3). Observe that $f_\lambda^2(x) > x$ for $x < a_{1\lambda}$. Since f_λ^2 is strictly increasing on $\mathbb{R}$ and $f_\lambda^2(a_{1\lambda}) = a_{1\lambda}$, it follows that the sequence $\{f_\lambda^{2n}(x)\}$ is a monotonically increasing sequence and $\sup\{f_\lambda^{2n}(x) \ : \ n \in \mathbb{N}\} = a_{1\lambda}$ for $x \leq a_{1\lambda}$. This gives that $\lim\limits_{n\to\infty} f_\lambda^{2n}(x) = a_{1\lambda}$ for $x \leq a_{1\lambda}$. Since $f_\lambda^2(x) < x$ for $a_{1\lambda} < x < r_\lambda$, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonically decreasing and bounded below by $a_{1\lambda}$. Therefore, $f_\lambda^{2n}(x) \to a_{1\lambda}$ as $n \to \infty$ for $a_{1\lambda} < x < r_\lambda$. For $x \in (r_\lambda,\ a_{2\lambda})$, the function f_λ^2 satisfies $f_\lambda^2(x) > x$. Consequently, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonically increasing and converges to $a_{2\lambda}$. When $x \geq a_{2\lambda}$, the sequence $\{f_\lambda^{2n}\}$ is decreasing and bounded below by $a_{2\lambda}$, since $f_\lambda^2(x) < x$ for $x > a_{2\lambda}$ and $f_\lambda^2(a_{2\lambda}) = a_{2\lambda}$. Therefore, $f_\lambda^{2n}(x) \to a_{2\lambda}$ as $n \to \infty$ for $x \geq a_{2\lambda}$ which completes the proof. □

In the following, we present the proof of Theorem 1.

Proof of Theorem 1.

Case 1: $\lambda > \lambda^*$

By Proposition 3(1), the function $f_\lambda(z)$ has a unique real attracting fixed point a_λ on the real line. Let $A(a_\lambda) = \{z \in \widehat{\mathbb{C}} \ : \ f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$ be the basin of attraction of the real attracting fixed point a_λ. Since by Proposition 6(1), the real line $\mathbb{R}$ is in the basin of attraction $A(a_\lambda)$ and in particular, all the singular values $\{\lambda, -\lambda, 0\}$ and their forward orbits are in $A(a_\lambda)$.

The Fatou set of $f_\lambda(z)$ has no basin of attraction other than $A(a_\lambda)$. To see this, assume, if possible, $A(z_\lambda)$ is a basin of attraction of an attracting periodic point $z_\lambda \neq a_\lambda$. Obviously, $A(z_\lambda) \bigcap A(a_\lambda) = \emptyset$. But, $A(z_\lambda)$ contains at least one singular value and its forward orbit. This contradicts the fact that all the singular values and their forward orbits are contained in $A(a_\lambda)$, since $A(z_\lambda) \bigcap A(a_\lambda) = \emptyset$ for $z_\lambda \neq a_\lambda$.

The Fatou set of $f_\lambda(z)$ cannot contain a parabolic domain. For, if the Fatou set of $f_\lambda(z)$ contains a parabolic domain U, then U must contain at least one singular value, which leads to a contradiction to the fact that all the singular values are in $A(a_\lambda)$.

Again, the Fatou set of $f_\lambda(z)$ cannot contain a Siegel disk or a Herman ring. For, if possible, the Fatou set of $f_\lambda(z)$ contains a Siegel disk or a Herman ring, then the boundary of Siegel disk / Herman ring is contained in the closure of the forward orbits of all singular values of $f_\lambda(z)$. But all the singular values and their forward orbits are contained in $A(a_\lambda)$, giving a contradiction.

By Proposition 5, the Fatou set of $f_\lambda(z)$ does not contain Baker domains or wandering domains. Therefore, the Fatou set of $f_\lambda(z)$ is equal to the basin of attraction $A(a_\lambda)$ of the attracting real fixed point a_λ if $\lambda > \lambda^*$.

Case 2: $\lambda = \lambda^*$

The function $f_\lambda(z)$ has a unique rationally neutral real fixed point x^* on the real line by Proposition 3(2). Let $P(x^*) = \{z \in \widehat{\mathbb{C}} \ : \ f_\lambda^n(z) \to x^* \text{ as } n \to \infty\}$ be the parabolic basin corresponding to the rationally neutral real fixed point x^*. By

Proposition 6(2), it follows that the real line $\mathbb{R}$ and in particular, all the singular values $\{\lambda, -\lambda, 0\}$ and their forward orbits are in the parabolic basin $P(x^*)$. Now, the Fatou set of $f_\lambda(z)$ for $\lambda = \lambda^*$ does not contain any other parabolic domain U other than $P(x^*)$. If the Fatou set of $f_{\lambda^*}(z)$ contains any other parabolic domain U $(\neq P(x^*))$, then U must contain at least one singular value which is not possible.

Since all singular values are in $P(x^*)$, the Fatou set of $f_{\lambda^*}(z)$ cannot contain a basin of attraction. The proofs of the fact that the Fatou set of $f_\lambda(z)$ for $\lambda = \lambda^*$ does not contain Sigel disk, Herman ring, Baker domain and wandering domain are similar to that of Case 1. Thus, all the possible stable domains other than the components of $P(x^*)$ are ruled out and hence the Fatou set of $f_{\lambda^*}(z)$ equals the parabolic basin $P(x^*)$ corresponding to the rationally neutral real fixed point x^*.

Case 3: $\lambda < \lambda^*$

By Proposition 4(3), the function $f_\lambda(z)$ has an attracting or a parabolic real 2-periodic cycle $\{a_{1\lambda},\ a_{2\lambda}\}$ with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$ where r_λ is the unique repelling real fixed point. Let us denote the basin of attraction of the attracting real 2-periodic cycle or the parabolic basin corresponding to the real parabolic 2-periodic cycle as

$$\mathsf{A} = \{z \in \widehat{\mathbb{C}} \ : \ f_\lambda^{2n}(z) \to a_{1\lambda} \text{ or } f_\lambda^{2n}(z) \to a_{2\lambda} \text{ as } n \to \infty\} \,.$$

By Proposition 6(3), it follows that the real line $\mathbb{R}$ except the point r_λ and in particular, all the singular values $\{\lambda, -\lambda, 0\}$ and their forward orbits are in A. By proceeding in the same lines of arguments as in Case 1 or Case 2, we get that Fatou set of $f_\lambda(z)$ does not contain Herman ring, Sigel disk, Baker domain, wandering domain or any basin of attraction or parabolic basin other than A. Therefore, the Fatou set of $f_\lambda(z) = \mathsf{A}$ for $\lambda < \lambda^*$. □

Theorem 1 gives the following characterization of the Julia set of $f_\lambda(z)$ which is computationally useful to generate the pictures of the Julia sets.

Corollary 1. *Let $f_\lambda \in \mathcal{M}$.*

1. *If $\lambda > \lambda^*$ then the Julia set $\mathcal{J}(f_\lambda)$ is the complement of the basin of attraction $A(a_\lambda)$ where a_λ is the attracting real fixed point of f_λ.*
2. *If $\lambda = \lambda^*$ then the Julia set $\mathcal{J}(f_\lambda)$ is the complement of the parabolic basin $P(x^*)$ where x^* is the rationally neutral real fixed point of f_λ.*
3. *If $\lambda < \lambda^*$ then the Julia set $\mathcal{J}(f_\lambda)$ is the complement of the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle $\{a_{1\lambda},\ a_{2\lambda}\}$.*

5. **Topology of the Fatou components.** In the present section, the proofs of Theorems 2 and 3 are mainly provided. Some preliminary observations on the Fatou set of f_λ are made in Propositions 7 and 8.

Proposition 7. *Let $f_\lambda \in \mathcal{M}$. Then,*

1. *The Fatou set of f_λ contains the left half-plane $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M_\lambda\}$ where M_λ is a real number depending on λ.*
2. *The Fatou set of f_λ contains the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi \ : x \in \mathbb{R}\}$ for every integer k. Further, there exists a real number $\delta \in (0,\ \frac{\pi}{2})$ depending upon λ such that the strip $S_{2k+1} = \{z \in \mathbb{C} \ : \ |\Im(z) - (2k+1)\pi| < \delta,\ \Re(z) \geq M_\lambda\}$ is contained in the Fatou set.*

Proof. 1. For every $f_\lambda \in \mathcal{M}$, the point $z = 0$ is always either in the basin of attraction or in the parabolic domain by Proposition 6. Since $z = 0$ is in the Fatou set of f_λ, there exists a disk $D_r(0) = \{z \in \mathbb{C} : |z| < r\}$ for some $r > 0$ such that $D_r(0) \subset \mathcal{F}(f_\lambda)$.

Since e^z maps the left half-plane $\{z \in \mathbb{C} : \Re(z) < a\}$ where $a \in \mathbb{R}$ onto a punctured disk $D^*(0) = \{z \in \mathbb{C} : 0 < |z| < e^a\}$ and $\tanh(0) = 0$, we can find a real number M_λ depending on λ such that the left half-plane $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M_\lambda\}$ is mapped inside the open ball $D_r(0) \subset \mathcal{F}(f_\lambda)$ by the map $w = f_\lambda(z)$. Therefore, the Fatou set of f_λ contains the left half-plane $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M_\lambda\}$.

2. The function e^z maps the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi : x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$, onto the negative real axis $\{x \in \mathbb{R} : x < 0\}$. The function $\lambda\tanh(x)$ maps the negative real axis into a subset of the real axis. By Proposition 6, if $\lambda > \lambda^*$, the real line $\mathbb{R}$ is contained in the Fatou set of f_λ. Therefore, it follows that the horizontal lines $L_{2k+1} = \{x+i(2k+1)\pi : x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$, are in the Fatou set of f_λ for $\lambda > \lambda^*$.

If $\lambda \le \lambda^* < 0$, the function $\lambda\tanh(x)$ maps the negative real axis into a subset of the positive real axis. By Proposition 6, if $\lambda \le \lambda^*$, the positive real axis is contained in the Fatou set of f_λ. This gives that the horizontal lines $L_{2k+1} = \{x + i(2k+1)\pi : x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$, are in the Fatou set of f_λ for $\lambda \le \lambda^*$.

It is already shown that $-\lambda$ lies in the Fatou set of f_λ. So, there exists a disk $D_r(-\lambda)$ with center at $-\lambda$ and radius r such that $D_r(-\lambda)$ is a subset of the Fatou set. One can find a $\tilde{M_\lambda} < 0$ depending on λ so that $\lambda\tanh z$ maps the half-plane $\tilde{H} = \{z : \Re(z) < \tilde{M_\lambda}\}$ into $D_r(-\lambda)$. Now, we choose $\delta^* \in (0, \frac{\pi}{2})$ and $M_\lambda^* > 0$ depending on $\tilde{M_\lambda}$ such that the image of the strip $\{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \delta^*, \Re(z) > M_\lambda^*\}$ under e^z is an angular region $\{z \in \mathbb{C} : |\arg(z) - \pi| < \delta^*, |z| > e^{M_\lambda^*}\}$ lying in the left half-plane $\tilde{H}$. Therefore, $\{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \delta^*, \Re(z) > M_\lambda^*\}$ is in the Fatou set of f_λ. As the line segment $\{z \in L_{2k+1} : M_\lambda \le \Re(z) \le M_\lambda^*\}$ is in the Fatou set, there exists a $\hat{\delta} \in (0, \frac{\pi}{2})$ such that the rectangular region $\{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \hat{\delta}, M_\lambda \le \Re(z) \le M_\lambda^*\}$ is in the Fatou set of f_λ. Choosing δ to be the minimum of δ^* and $\hat{\delta}$, it follows that $S_{2k+1} = \{z \in \mathbb{C} : |\Im(z) - (2k+1)\pi| < \delta, \Re(z) \ge M_\lambda\}$ is contained in the Fatou set.

□

Remark 1. For $\lambda > \lambda^*$, it also follows that the Fatou set of f_λ contains the horizontal lines $L_{2k} = \{x + i\ 2k\pi : x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$ by the same arguments used in proving the second part of the above proposition.

A maximally connected subset of the Julia set is called a component of the Julia set. We prove in the following proposition that the Julia set of f_λ for $\lambda > \lambda^*$ cannot contain an unbounded component.

Proposition 8. *Let $f_\lambda \in \mathcal{M}$. If $\lambda > \lambda^*$ then every component of $\mathcal{J}(f_\lambda)\bigcap\mathbb{C}$ is bounded.*

Proof. Let, on the contrary, γ be an unbounded component of $\mathcal{J}(f_\lambda)\bigcap\mathbb{C}$. Then a sequence t_n can be found on γ such that $\lim_{n\to\infty} t_n = \infty$. It follows from Proposition 7 and Remark 1 that γ lies in a horizontal strip bounded by L_k and L_{k+1} for some $k \in \mathbb{Z}$ and the set $\{\Re(t_n) \ : \ n \in \mathbb{N}\}$ is unbounded. Now, observe that the image $\gamma_1 = e^{\gamma}$ of γ is unbounded under the mapping $w = e^z$.

If γ_1 intersects L_k for some $k \in \mathbb{Z}$ then the map $\lambda \tanh(z)$ takes each such intersecting point to a real number which is in the Fatou set. In this way, there is a point common to γ and the Fatou set of f_λ which is a contradiction. Therefore, γ_1 lies in some horizontal strip bounded by two consecutive L_k's. Since γ_1 is unbounded, there exists a sequence s_n on γ_1 such that $\lim_{n\to\infty} \Re(s_n) = \infty$ or $\lim_{n\to\infty} \Re(s_n) = -\infty$. But in both the cases, $\lambda \tanh(s_n)$ tends to an asymptotic value of f_λ as $n \to \infty$. Since all the three asymptotic values lie in the Fatou set, there is a sequence $\{z_n\}_{n>0}$ on γ such that $e^{z_n} = s_n$ and $f_\lambda(z_n)$ is a subset of the Fatou set for sufficiently large n. By the complete invariance of the Fatou set, there are points z_n on γ which are in the Fatou set. It gives a contradiction. Therefore, any component of $\mathcal{J}(f_\lambda) \bigcap \mathbb{C}$ is bounded. □

In the following, the proof of Theorem 2 is given.

Proof of Theorem 2.

Let V be a component of the Fatou set of f_λ different from the immediate basin of attraction $IM(a_\lambda)$ of the attracting fixed point a_λ. Then, there exists a natural number k such that $f_\lambda^k(V) \subseteq IM(a_\lambda)$. Let $W = f_\lambda^{k-1}(V)$. If U_1 and U_2 are two Fatou components of a meromorphic function f such that $f : U_1 \to U_2$, then $U_2 \setminus f(U_1)$ contains at most two points [18]. The two exceptional values $\pm\lambda$ of f_λ lie in $IM(a_\lambda)$. Therefore, it follows that $f_\lambda(W) = IM(a_\lambda) \setminus \{\lambda, \ -\lambda\}$. Let $D_r(\lambda)$ be a disk of radius $r > 0$ with center λ such that $D_r(\lambda)$ is contained in $IM(a_\lambda)$. Let $U(r)$ be a component of $f_\lambda^{-1}(D_r(\lambda))$ in W. If $r_2 < r_1 < r$ then there are components $U(r_2)$ of $f_\lambda^{-1}(D_{r_2}(\lambda))$ and $U(r_1)$ of $f_\lambda^{-1}(D_{r_1}(\lambda))$ in $U(r) \subset W$ such that $U(r_2) \subset U(r_1)$. Note that $U(r)$ is unbounded, since there is only one logarithmic singularity of f_λ^{-1} over λ and for that $\bigcap_{r>0}(U(r)) = \emptyset$ [7]. Thus, there are at least two unbounded components, namely, W and $IM(a_\lambda)$ of the Fatou set. Consequently, the boundary of any of these two unbounded components is an unbounded component of $\mathcal{J}(f_\lambda) \bigcap \mathbb{C}$. But, it is not possible by Proposition 8. Therefore, the Fatou set of f_λ for $\lambda > \lambda^*$ contains only one component and hence, the Fatou set is connected.

It is shown in [1] that the connectivity of an invariant Fatou component is either 1, 2 or ∞, 2 being the case for Herman rings. For $\lambda > \lambda^*$, the Fatou set of f_λ is equal to the basin of attraction of the attracting fixed point a_λ and the connectivity of the Fatou set is either 1 or ∞. If the connectivity of the Fatou set is 1, then the Julia set is connected and there is an unbounded component of $\mathcal{J}(f_\lambda) \bigcap \mathbb{C}$. But this is impossible by Proposition 8. Therefore, the Fatou set of f_λ for $\lambda > \lambda^*$ is infinitely connected. □

As a consequence of Proposition 8 and the infinite connectivity of $\mathcal{F}(f_\lambda)$ for $\lambda > \lambda^*$, we make the following remark on the Julia set of f_λ for $\lambda > \lambda^*$.

Remark 2. Let w be a pre-pole of f_λ. If it is not a singleton component of the Julia set then there will be a component γ of the Julia set that contains w and $f_\lambda^k(\gamma) \subset \mathcal{J}(f_\lambda)$ is a component containing the point $z = \infty$ for some natural number k. But, it is not possible for $\lambda > \lambda^*$ by Proposition 8. Thus, every pre-pole is a singleton component of the Julia set of f_λ for $\lambda > \lambda^*$. Since pre-poles are dense in $\mathcal{J}(f_\lambda)$, we conclude that the singleton components of the Julia set are dense in the Julia set of f_λ for $\lambda > \lambda^*$. It can also be concluded from the previous theorem and using Theorem (A) in [16]. It is not known that non-singleton components exist or not in the Julia set of f_λ for $\lambda > \lambda^*$.

Let I_1 be a component of the Fatou set containing the interval $(-\infty, a_{1\lambda})$ when $\lambda < \lambda^*$. When $\lambda = \lambda^*$, let I_1 denote the Fatou component containing the interval $(-\infty, x^*)$. Let I_2 denote the Fatou component containing $f_\lambda(I_1)$. We use these notations in the following lemma which is required to prove Theorem 3.

Lemma 2. *Let $f_\lambda \in \mathcal{M}$ with $\lambda \leq \lambda^*$. Let V be a component of the Fatou set $\mathcal{F}(f_\lambda)$ of f_λ. If γ is a Jordan curve in V and the bounded component B of $\widehat{\mathbb{C}} \setminus \gamma$ intersects the Julia set then B does not contain any pole of f_λ.*

Proof. Since V is a Fatou component, $f_\lambda(V)$ is contained in a Fatou component, say V_1. Let γ be a Jordan curve in V and the bounded component B of $\widehat{\mathbb{C}} \setminus \gamma$ intersects the Julia set.

Suppose that V_1 is different from I_1. Let us assume, on the contrary that B contains a pole. Then, $B_1 = f_\lambda(B)$ contains $\{z : |z| > M\}$ for some $M > 0$. Since I_1 is unbounded, B_1 intersects I_1. It means that there are points in B whose f_λ-images belong to I_1. Consequently, there is a Fatou component, say W in B such that $f_\lambda(W) \subseteq I_1$. In [18], it has been proved that for any meromorphic function $f : A_1 \to A_2$, the cardinality of the set $A_2 \setminus f(A_1)$ is at most two where A_1 and A_2 are two Fatou components of f. Using this result, it follows that $E = I_1 \setminus f_\lambda(W)$ contains at most two points. Since $\lambda \in I_1$, there exists a neighborhood N_λ of the point λ which is completely contained in I_1 and $N_\lambda \bigcap E = \{\lambda\}$. Therefore, there is a component of $f_\lambda^{-1}(N_\lambda)$ in W. As there is only one singularity lying over $-\lambda$ and it is logarithmic, every component of $f_\lambda^{-1}(N_\lambda)$ is unbounded [7]. Consequently, W is unbounded and B is also unbounded which is not true. Therefore, it follows that B contains no pole of f_λ.

Suppose that V is a Fatou component such that $f_\lambda(V) \subset I_1$. Since I_2 is unbounded, $-\lambda \in I_2$ and, there is only one singularity lying over $-\lambda$ and it is logarithmic, the same arguments given in the previous paragraph with I_1 replaced by I_2 are applied to conclude that B contains no pole of f_λ. It completes the proof. □

Now, we present the proof of Theorem 3.

Proof of Theorem 3. 1. By Theorem 1, it follows that the Fatou set $\mathcal{F}(f_\lambda)$ for $\lambda < \lambda^*$ is equal to the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ of f_λ. Let $\{a_{1\lambda}, a_{2\lambda}\}$ be attracting cycle. Let $IM(a_{1\lambda})$ be the component of the Fatou set containing the point $a_{1\lambda}$ and $IM(a_{2\lambda})$ be the component of the Fatou set containing the point $a_{2\lambda}$. Then, $(-\infty, r_\lambda) \subset IM(a_{1\lambda})$ and $(r_\lambda, \infty) \subset IM(a_{2\lambda})$. Let $L_{2k} = \{x + i\,2k\pi : x \in \mathbb{R}\}$ where $k \in \mathbb{Z}$ and $k \neq 0$. Then, $f_\lambda : L_{2k} \to (\lambda, 0)$ is a bijection and it maps $L_{2k}^- = \{x + i\,2k\pi : -\infty < x < r_\lambda = f_\lambda^{-1}(r_\lambda)\}$ and $L_{2k}^+ = \{x + i\,2k\pi : r_\lambda = f_\lambda^{-1}(r_\lambda) < x < \infty\}$ to $(r_\lambda, 0)$ and (λ, r_λ) respectively. It gives that L_{2k}^+ and L_{2k}^- lie in two different components of the Fatou set. It is clear that some left half-plane H_λ, all horizontal lines $L_{2k+1} = \{x + i\,(2k+1)\pi : x \in \mathbb{R}\}$ for $k \in \mathbb{Z}$ and $L_{2k}^- = \{x + i\,2k\pi : -\infty < x < r_\lambda\}$ are in $IM(a_{1\lambda})$. Further, L_{2k}^+ lies in a component, W_k (say) of the Fatou set which is different from $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$. For each non-zero integer k, we can find such component W_k which contains the line L_{2k}^+ and $W_k \bigcap W_l = \emptyset$ for $k \neq l$. These components W_k's are pre-periodic but not periodic.

If $\{a_{1\lambda}, a_{2\lambda}\}$ is a parabolic cycle then there will be two different components of Fatou set containing $(-\infty, a_{1\lambda})$ and $(a_{2\lambda}, \infty)$. Considering them as $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$, it can be observed that $IM(a_{1\lambda})$ contains some half-plane H_λ and horizontal lines L_{2k+1}. Similar arguments as in previous paragraph gives the existence of infinitely many Fatou components W_k containing $L_{2k}^{+} = \{x + i\ 2k\pi\ : f_\lambda^{-1}(a_{1\lambda}) < x < \infty\}$ for non-zero integer k which are different from $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$. These components are pre-periodic but not periodic.

For $\lambda = \lambda^*$, the proof is similar to the case $\lambda < \lambda^*$.

2. Let V be any component of the Fatou set of f_λ for $\lambda \leq \lambda^*$. Suppose that V is not simply connected. Let γ be a Jordan curve in V for which the bounded component U of $\widehat{\mathbb{C}} \setminus \gamma$ contains at least one component of $\widehat{\mathbb{C}} \setminus V$. Set $U_n = f_\lambda^n(U)$ for $n = 0, 1, 2, \cdots$. By Lemma 2, it follows that U does not contain any pole. Since the boundary of U also does not contain any pole, the component $U_1 = f_\lambda(U)$ is a bounded domain. Also, the boundary of U_1 is a subset of $f_\lambda(\partial U)$. Since the boundary ∂U of U is the Jordan curve γ which is in the Fatou set, the image $f_\lambda(\partial U)$ is in a Fatou component, and hence, ∂U_1 is in a Fatou component. If U_1 does not contain a pole, the boundary of U_2 lies in a Fatou component by repeating the above arguments. As $U \bigcap \mathcal{J}(f_\lambda) \neq \emptyset$, after finite number of steps, we can find a natural number n_0 for which U_{n_0} contains a pole which gives a contradiction to Lemma 2. Therefore, it is concluded that the component V of the Fatou set of f_λ for $\lambda \leq \lambda^*$ is simply connected.

□

Remark 3. For $\lambda \leq \lambda^*$, all the singular values of f_λ are in the immediate basin of attraction or in the petals of the parabolic domain which are not completely invariant.

6. **Measure of $\mathcal{J}(f_\lambda)$.** In this section, the (Lebesgue) measure of the Julia set of $f_\lambda \in \mathcal{M}$ is computed.

Let $m(A)$ denote the measure of $A \subset \widehat{\mathbb{C}}$ and $D_r(z)$ denote the disk of radius r with center at z. A subset E of $\widehat{\mathbb{C}}$ is said to be thin at ∞ if its density is bounded away from 1 in all sufficiently large disks, that is, if there exist positive R_0 and ϵ such that, for all complex z and every disk $D_r(z) = \{w : |w - z| < r\}$, $r > R_0$,

$$density(E, D_r(z)) = \frac{m(E \bigcap D_r(z))}{m(D_r(z))} < 1 - \epsilon.$$

For a given meromorphic function f, let

$$P_f^* = \{z\ :\ \text{for some } n \in \mathbb{N} \text{ some branch of } f^{-n} \text{ has a singularity at } z\}$$

and $P_f = P_f^* \setminus \{\infty\}$.

The following proposition is due to Stallard [25].

Proposition 9. *Let f be a meromorphic function and $d(\overline{P_f}, \mathcal{J}(f)) > 0$ where $\overline{P_f}$ is the closure of P_f in $\mathbb{C}$. If E is a measurable completely invariant subset of $\mathcal{J}(f)$ such that E is thin at ∞, then $m(E) = 0$. In particular, the Julia set has measure zero if it is thin at ∞.*

Theorem 4. *Let $f_\lambda \in \mathcal{M}$. Then, the Julia set of f_λ has measure zero.*

Proof. It is already shown that each singular value of f_λ is in an attracting basin or a parabolic domain. It gives that $d(\overline{P_{f_\lambda}},\ \mathcal{J}(f_\lambda)) > 0$. In view of Proposition 9, it is enough to show that $\mathcal{J}(f_\lambda)$ is thin at ∞ in order to show that the measure of $\mathcal{J}(f_\lambda)$ is zero.

Let $M \equiv M(\lambda)$ and $\delta \equiv \delta(\lambda)$ be two real numbers such that $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M\}$ and $S_{2k+1} = \{z \in \mathbb{C} :\ |\Im(z) - (2k+1)\pi| < \delta,\ \Re(z) \geq M\}$ are in the Fatou set of f_λ which is possible by Proposition 7.

Now, consider the square $S(z,r) = \{w :\ |\Re(w) - \Re(z)| < \frac{r\sqrt{2}}{2}, |\Im(w) - \Im(z)| < \frac{r\sqrt{2}}{2}\}$ having its sides parallel to the co-ordinate axes and it is inscribed in the disk $D(z,r)$ with center at z and radius r. For a rectangle R having its sides parallel to co-ordinate axes with vertical side length 2π and horizontal side length h, $R \bigcap \mathcal{F}(f_\lambda) \supset R \bigcap (\bigcup_{k\in\mathbb{Z}} S_{2k+1})$. It implies that $m(R \bigcap \mathcal{F}(f_\lambda)) > m(R \bigcap (\bigcup_{k\in\mathbb{Z}} S_{2k+1})) > 2\delta h$. If $j = [\frac{r\sqrt{2}}{2\pi}]$ is the greatest integer not exceeding $\frac{r\sqrt{2}}{2\pi}$ then $S(z,r)$ will contain j different rectangles each having its sides parallel to co-ordinate axes with vertical side length 2π and horizontal side length $r\sqrt{2}$. It gives that $m(\mathcal{F}(f_\lambda) \bigcap S(z,r)) > j2\delta r\sqrt{2} \geq (\frac{r\sqrt{2}}{2\pi} - 1)(2\delta r\sqrt{2}) = \frac{2\delta r^2}{\pi} - 2\delta r\sqrt{2}$. Consequently, $m(\mathcal{F}(f_\lambda) \bigcap D_r(z)) > \frac{2\delta r^2}{\pi} - 2\delta r\sqrt{2} = 2\delta(\frac{r^2}{\pi} - r\sqrt{2})$ and

$$density(\mathcal{F}(f_\lambda), D_r(z)) = \frac{m(\mathcal{F}(f_\lambda) \bigcap D_r(z))}{m(D_r(z))} > \frac{2\delta}{\pi r^2}\left(\frac{r^2}{\pi} - r\sqrt{2}\right).$$

Now, $density(\mathcal{F}(f_\lambda),\ D_r(z)) > \frac{2\delta}{\pi}(\frac{1}{\pi} - \frac{\sqrt{2}}{r}) > \frac{\delta}{\pi^2}$ for $r > 2\sqrt{2}\pi$.

Letting $\epsilon = \frac{\delta}{\pi^2}$ and $R_0 = 2\sqrt{2}\pi$, it is concluded that $density(\mathcal{F}(f_\lambda),\ D_r(z)) > \epsilon$ for all $z \in \mathbb{C}$ and all $r > R_0$. Since $density(\mathcal{F}(f_\lambda),\ D_r(z)) + density(\mathcal{J}(f_\lambda),\ D_r(z)) = 1$, it follows that $density(\mathcal{J}(f_\lambda),\ D_r(z)) < 1 - \epsilon$ for all $z \in \mathbb{C}$ and all $r > R_0$. Therefore, the Julia set of f_λ is thin at ∞ which completes the proof. □

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