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Bright and dark spatial solitons in coupled photorefractive media

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Abstract. We consider beam propagation in photorefractive media, where the pulse dynamics is governed by a set of coupled nonlinear Schrödinger-type equations. We construct anharmonic oscillator equations using the Stokes parameters and find a solution for bright spatial solitons in photorefractive media. We then reduce the anharmonic oscillator to the well-known undamped and unforced Duffing oscillator equation. From this equation, we present the necessary conditions for the formation of both bright and dark screening solitons, and obtain solutions for both of them.

1. Introduction

In recent years, soliton researchers have paid much attention to the investigation of spatial solitons after realizing their potential applications, such as all-optical switching, three-dimensional optical interconnects [1] and waveguide applications [2, 3]. These spatial solitons relied on the optical Kerr effect wherein the refractive index of the material increases in proportion to the intensity of the light. The resulting self-trapped beams are called Kerr spatial solitons. These solitons are found to have two major drawbacks. The first is that the creation of such spatial solitons in Kerr-type material requires a high light intensity, i.e. high power. The second is that these solitons are stable only in one dimension. Attempts to overcome these drawbacks gave birth to a new class of solitons called 'photorefractive solitons'.

Researchers [4–7] have shown recently that spatial solitons can be created with very low laser power in photorefractive (PR) materials. Segev *et al*. [4] successfully predicted the photorefractive spatial soliton, and since then it has been extensively studied. PR solitons exhibit several interesting properties. First, these PR solitons are found to be stable in both transverse dimensions and can be created even at low power [7]. In addition to these properties, PR screening soliton beams can also induce waveguiding and can guide intense light beams, which is photorefractively less sensitive [8].

An optical beam propagating through a PR crystal biased with a DC electric field excites charge carriers, which drift and become retrapped on impurity centers. This results in the build-up of a space charge field, which screens out the externally applied field. The spatially varying electric field can increase or decrease the index of refraction depending on the polarity of the applied field, i.e. the refractive index change resulting from the nonuniformly screened electric field can have either a self-focusing or a defocusing effect thereby producing so-called bright and dark screening spatial solitons. The aforementioned interesting properties make the screening solitons (both bright and dark) attractive for practical applications. Also the screening solitons provide a useful tool in experimental verification of theoretical predictions regarding generic properties of individual solitons as well as their interactions. Zhigang Chen *et al.* [9] investigated the interaction of these photorefractive screening solitons and have experimentally confirmed these results.

In this paper we provide a theoretical study of spatial solitons in coupled photorefractive media. The paper is arranged as follows. In section 2, we present the necessary theoretical model related to this work. We discuss the construction of an integro-differential equation in section 3. Anharmonic and undamped Duffing oscillators are discussed in section 4. We derive the necessary condition for the formation of both bright and dark solitons in section 5, and then finally we give results and conclusions.

2. Theoretical model

The theoretical model for one-dimensional PR screening solitons in biased photorefractive media is well known [1,10]. For convenience of the analysis, a light beam is assumed to propagate in a PR crystal in the x-direction. The electric field component E of the light beam satisfies the following equation [11]:

$$\nabla^2 E + (k\hat{n}_b)^2 E = 0, \tag{1}$$

where the term k is equal to $2\pi n_b/\lambda$, λ is the free-space wavelength, $(\hat{n}_b)^2 = n_b^2 - n_b^4 r_{33} E_{sc}$ is the perturbed extraordinary refractive index, r_{33} is the electro-optic coefficient, n_b is the unperturbed refractive index, and $E_{sc} = E_{sc}\hat{x}$ is the induced space-charge field. The equation (1) is considered to be the fundamental equation for the slowly varying spatial amplitude of the optical field and a set of charge-transport equations that describe the photorefractive effect in a nonlinear medium. These equations reduce to a single nonlinear equation as follows:

$$\left[\frac{\partial}{\partial z} - \frac{i}{2k}\frac{\partial^2}{\partial x^2} - \frac{ik}{n_b}\Delta n(E_{sc})\right]\Phi(x, z) = 0, \tag{2}$$

where $\Delta n(E_{sc})$ is the change in refractive index, which is driven by the space-charge field $E_{sc}(x, z)$ through the electro-optic effect:

$$\Delta n(E_{sc}) = -\frac{1}{2} n_b^3 r_{33} E_{sc}. \tag{3}$$

With a PR crystal of strontium barium niobate (SBN), the most favorable configuration for soliton formation is when the crystalline c axis is parallel to the x direction, which is also the direction of beam polarization, and has a large electro-optic coefficient r_{33}. Under appropriate conditions, the steady-state space-charge field is given (approximately) by

$$E_{sc} = E_0 \frac{I_\infty + I_d}{I(x, z) + I_d}, \tag{4}$$

where E_0 is the space charge field at $x \to \pm \propto$, and it is approximately equal to $\pm V/l$, i.e., the voltage V applied across the crystal of width l. $I(x, z)$ is the total intensity of the two optical beams. $I_\propto = I(x \to \pm \propto)$ represents the total intensity far away from the centre of the beams, and I_d is the dark irradiance.

As our prime concern is to analyse spatial solitons in coupled photorefractive media, the above equation can be extended to the case of two optical beams co-propagating in a photorefractive medium. Therefore in steady state, the coupled wave equations can be written as

$$\left[\frac{\partial}{\partial z} - \frac{i}{2k}\frac{\partial^2}{\partial x^2} - \frac{ik}{n_b}\Delta n(E_{sc})\right]\Phi(x, z) = 0,$$
$$\left[\frac{\partial}{\partial z} - \frac{i}{2k}\frac{\partial^2}{\partial x^2} - \frac{ik}{n_b}\Delta n(E_{sc})\right]\Psi(x, z) = 0, \tag{5}$$

where $\Phi(x, z)$ and $\Psi(x, z)$ are the slowly varying amplitudes of the two optical fields, and x and z are the transverse and longitudinal coordinates, respectively.

For convenience, we transform this envelope equation into a normalized equation by the substitutions $\zeta = z/(kx_0^2)$, $\xi = x/x_0$, $\Phi = (2\eta_0 I_d/n_b)u$ and $\Psi = (2\eta_0 I_d/n_b)v$. Now equation (5) can be written as two coupled nonlinear equations in dimensionless variables [8]:

$$i\frac{\partial u}{\partial \varsigma} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} - \beta(1+\rho)\frac{u}{1+|u|^2+|v|^2} = 0,$$
$$i\frac{\partial v}{\partial \varsigma} + \frac{1}{2}\frac{\partial^2 v}{\partial \xi^2} - \beta(1+\rho)\frac{v}{1+|u|^2+|v|^2} = 0, \tag{6}$$

where $\rho = I_\propto/I_d$, $\beta = 1/2(kx_0)^2 {n_b}^2 r_{33}E_0$, and x_0 is an arbitrary spatial width for scaling. These are the basic equations we use in our theoretical model.

For further investigation we also include cubic nonlinearity and coupling between the two beams in the above saturable coupled nonlinear Schrödinger (SCNLS) equation. Therefore the beam propagation in a coupled photorefractive media is governed by the following equation [12]:

$$\left[i\frac{\partial u}{\partial \varsigma} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + (\beta+\alpha)(\rho+l)\frac{u}{1+\rho\left(|u|^2+|v|^2\right)} + k\frac{v}{1+\rho\left(|u|^2+|v|^2\right)} - \frac{2\delta\rho\left[1-\left(|u|^2+A|v|^2\right)\right]u}{1+\rho\left(|u|^2+|v|^2\right)}\right] = 0,$$
$$\left[i\frac{\partial u}{\partial \varsigma} + \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} - (\beta+\alpha)(\rho+l)\frac{v}{1+\rho\left(|u|^2+|v|^2\right)} + k\frac{u}{1+\rho\left(|u|^2+|v|^2\right)} - \frac{2\delta\rho\left[1-\left(A|u|^2+|v|^2\right)\right]v}{1+\rho\left(|u|^2+|v|^2\right)}\right] = 0. \tag{7}$$

From the above equation it is obvious that the saturable nonlinearity depends on the total intensity of the light beam and, in general, does not satisfy the additional properties of solitons such as elastic collision and integrability of the system under consideration. That is, the total intensity ratio in the denominator of the above equation breaks the integrability criterion and in general they are non-integrable. Therefore, to the best of our knowledge, no analytical solitary wave solutions have been reported with saturable nonlinearity, but numerical methods have been applied [4, 19]. However, in this paper, we present analytical stationary solutions (with saturable nonlinearity) of both bright and dark screening solitons through the undamped and unforced Duffing oscillator equation.

As we are mainly interested in analysing bright and dark screening solitons in photorefractive media, we reduce the above SCNLS equations to anharmonic oscillator type equations, which contain all the information about the SCNLS equations. Therefore, solution of the anharmonic oscillator equations is equivalent to solution of the SCNLS equation. We can achieve the above goal using the Stokes parameters, defined by [13, 14]

$$\begin{aligned} s_0 &= |u|^2+|v|^2, \\ s_1 &= |u|^2-|v|^2, \\ s_2 &= u^*v + uv^*, \\ s_3 &= -i(u^*v - uv^*). \end{aligned} \tag{8}$$

By using the above four Stokes parameters, we derive a set of so-called integro-differential equations which appear to be complicated when compared to the beam-governing equation (7). Therefore, to proceed further, we assume that the amplitude functions are separable ($u = X_{(\varsigma)}f_{(\xi)}$ and $v = Y_{(\varsigma)}f_{(\xi)}$), as used by Akhmediev *et al*. [15], and the resulting equations are

$$\begin{aligned} &\frac{d}{d\varsigma}S_0 = 0, \\ &\frac{d}{d\varsigma}S_1 = -2kg_1\frac{S_3}{\rho S_0}, \\ &\frac{d}{d\varsigma}S_2 = 2g_2\delta\frac{S_1S_3}{\rho S_0} + 2(\beta+\alpha)(\rho+l)g_1\frac{S_3}{\rho S_0}, \\ &\frac{d}{d\varsigma}S_3 = -2g_2\delta\frac{S_1S_2}{\rho S_0} - 2(\beta+\alpha)(\rho+l)g_1\frac{S_2}{\rho S_0} + 2kg_1\frac{S_1}{\rho S_0}, \end{aligned} \tag{9}$$

where g_1 and g_2 are nonlinear birefringent coefficients and are related to the nonlinear beat length by

$$g_1 = \frac{\int_{-\infty}^{\infty} f^2 d\xi}{\left(\int_{-\infty}^{\infty} 1+f^2 d\xi\right)\int_{-\infty}^{\infty} f^2 d\xi}, \quad g_2 = (1-A)\frac{\int_{-\infty}^{\infty} f^4 d\xi}{\left(\int_{-\infty}^{\infty} 1+f^2 d\xi\right)\int_{-\infty}^{\infty} f^2 d\xi},$$

where f is a real function defining the common profiles.

3. Construction of anharmonic and Duffing oscillator equation

In this section, we construct the anharmonic oscillator equation from the integro-differential equations. After finding the bright spatial solution for the above-mentioned oscillator equation, we show that for a particular special case it also leads to the bright spatial soliton solution for quadratic nonlinearity. As has been discussed in the introduction, we reduce the anharmonic oscillator equation to the well-known undamped Duffing oscillator equation. After constructing the Hamiltonian from the Duffing oscillator equation, we impose the appropriate necessary conditions for the formation of both bright and dark solitons. In order to derive the anharmonic oscillator equation, it is necessary to find the conserved quantities. We have the following conserved quantities from the equations (8) and (9):

$$\begin{aligned} &S_0^2 = S_1^2 + S_2^2 + S_3^2, \\ &\delta \frac{g_2}{g_1} \frac{S_1^2}{2} + (\beta + \alpha)(\rho + l)S_1 + kS_2 = \Gamma. \end{aligned} \tag{10}$$

Now using equation (10) in (9), we obtain the following anharmonic oscillator equation:

$$\frac{d^2 S_1}{d\varsigma^2} - aS_1 + bS_1^2 + cS_1^3 = \frac{4(\beta + \alpha)(\rho + l)g_1^2 \Gamma}{(\rho S_0)^2}, \tag{11}$$

where the parameters a, b, and c are given by

$$a = \frac{4g_1\left[\delta g_2 \Gamma - g_1\left(k^2 + B^2\right)\right]}{(\rho S_0)^2}, \quad b = \frac{6\delta g_1 g_2 B}{(\rho S_0)^2}, \quad c = \frac{2g_2^2 \delta^2}{(\rho S_0)^2},$$

where $B = (\beta + \alpha)(\rho + l)$. Equation (11) contains both quadratic and cubic nonlinearity. First let us find the solution of equation (11) with both quadratic and cubic nonlinearity, which contains all the information of the saturable CNLS equation. This is equivalent to finding the solution of the SCNLS equation. We choose the integration constant $\Gamma = 0$ and then integrating equation (11), we get the following equation:

$$\left(\frac{dS_1}{d\varsigma}\right)^2 - aS_1^2 + bS_1^3 + cS_1^4 = C.$$

We choose the constant of integration $C = 0$ and then obtain the bright soliton solution in the form

$$S_1 = \frac{3a}{\sqrt{b^2 + 9ca}\cosh\left(\sqrt{a}\right)\varsigma + 1}. \tag{12}$$

S_1 represents the bright soliton solution in the presence of both quadratic and cubic nonlinearity. When we set the cubic nonlinearity to zero, we have only quadratic nonlinearity. Therefore, under this criterion, the solution of equation (11) reduces to the solution of quadratic nonlinearity in the form

$$S_1 = \frac{3a}{2b\cosh^2\left(\sqrt{a}\right)\varsigma/2}. \tag{13}$$

Our results for the quadratic nonlinearity can easily be checked with earlier available results [16–18]. The above results simply resemble the family of stationary localized solutions of two coupled nonlinear equations for the envelopes of the fundamental and second-harmonic field components.

Similarly, we also analyse equation (11) in the absence of quadratic nonlinearity. Under this criterion, equation (11) reduces to the following unforced and undamped Duffing oscillator equation [19]:

$$\frac{d^2 S_1}{d\varsigma^2} + aS_1 + cS_1^3 = 0. \tag{14}$$

Further, equation (14) is similar to the equations obtained in [20], and therefore bright and dark spatial screening solitons are easily possible in the above Duffing oscillator-type equation. For equation (14) the Hamiltonian is found to be

$$H = \frac{1}{2}\left(\frac{dS_1}{d\varsigma}\right)^2 + c\frac{S_1^4}{4} + a\frac{S_1^2}{2}. \tag{15}$$

The phase plane analysis of the Hamiltonian system described by equation (15) is equivalent to investigating the nature of the stationary waves (both bright and dark) of equation (7).

4. Necessary conditions for the formation of screening solitons

From the Hamiltonian structure, we present the necessary condition for the formation of both bright and dark screening solitons:
For bright screening soliton formation:

$$a < 0,\ c > 0 \text{ and } H = 0. \tag{16}$$

For dark screening soliton formation:

$$a > 0,\ b < 0 \text{ and } H = a^2/4b. \tag{17}$$

Now we apply these conditions to the Hamiltonian equation (15), and we obtain both the bright and dark screening soliton solutions. For equation (16), we obtain the bright solitary wave solution as

$$S_1 = \pm\sqrt{\frac{2a}{c}}\,\text{sech}\,\sqrt{a}\,\varsigma. \tag{18}$$

Similarly by applying the condition (17) we obtain the dark screening soliton solution as

$$S_1 = \pm\sqrt{\frac{a}{c}}\tanh\sqrt{\frac{a}{2}}\,\varsigma. \tag{19}$$

Thus we have constructed both bright and dark screening soliton solutions in a coupled photorefractive media. The terms a and c in the above solutions are the physical parameters of the system under consideration and they are given in equation (11). While constructing the screening soliton solution, we do not perform any series truncation of the nonlinear term in the denominator of equation (7). Therefore the screening solitons in our model form a new set of solitary wave solutions.

5. Conclusions

We have considered coupled beam propagation in photorefractive media. Using the Stokes parameters, we have obtained an anharmonic oscillator equation (without change in the physical parameters) from the saturable CNLS equation and obtained the general bright screening soliton solution in the presence of both quadratic and cubic nonlinearity. As a special case, we have also obtained the solution for a quadratic nonlinearity, and comparison has been made with known results. Moreover, in the absence of quadratic nonlinearity, we have reduced our dynamical system to an unforced and undamped Duffing oscillator.

After constructing the Hamiltonian, we have provided the necessary conditions for the formation of both bright and dark screening solitons in coupled photorefractive media, and finally obtained screening soliton solutions for both of them.

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