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THE EFFECT OF MICROSTRUCTURE ON THE THERMAL CONVECTION IN A RECTANGULAR BOX OF FLUID HEATED FROM BELOW

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Abstract—The effect of microstructure on the thermal convection in a rectangular box of fluid heated from below has been investigated by applying the micropolar fluid theory. The influence of lateral walls on the convection process in a rectangular box has been determined. The Galerkin method has been employed to get an approximate solution for the eigenvalue problem. The beam functions which satisfy two boundary conditions on each rigid surface have been used to construct the finite roll (cells with two nonzero velocity components depend on all three spatial variables) trial functions for the Galerkin method. The effect of variations of material parameters at the onset of convection has been presented graphically. It is observed that as the distance between the lateral walls increases the effect of one of the material parameters, characterizing the spin-gradient viscosity, at the onset of stability diminishes. A comparison has been made with the corresponding results for a Newtonian fluid.

1. INTRODUCTION

THE STUDY of buoyancy driven convection in a fluid layer heated from below is always important from the technological point of view, for it can reveal hitherto unknown properties of fluids of practical interest. Using a classical linear stability analysis, it is possible to predict the conditions for the onset of this convective motion, as well as the characteristic scale of the resulting flow pattern [1]. At the onset of convection, the shape of the convective motion depends on the nature and geometry of the boundaries.

The problem of convection in a rectangular box is considered as fundamental in the studies of thermal convection in enclosures. A wealth of articles concerning theoretical and experimental study of this problem has appeared in the pertinent literature. Davis [2] and Segel [3] have investigated the thermal convection in a rectangular box of fluid heated from below. The finite rolls with their axes parallel to the shorter side are predicted solely on the basis of a linear treatment in [2] and [3]. This is consistent with Koschmieder's experimental observation [4] who found that, near the critical point, the cell pattern that emerges is strongly influenced by the geometry of the lateral boundaries. Davis [2] considered the lateral walls to be perfectly conducting. He found upper bounds of the critical Rayleigh number by using the Galerkin method. However, while using the Galerkin method, he violated the Weierstrass theorem, and his set of trial functions were not complete within the region of interest. Incorporating the Weierstrass theorem, Catton [5] has constructed a set of trial functions for the Galerkin method. Using the modified trial functions, Catton [5] has found meaningful results even for small aspect ratio where Davis [2] was unable to get any such results. Recently, McDonough and Catton [6] have studied two-dimensional convection in a square box using a mixed finite-difference Galerkin method. In this method, Galerkin procedure was applied in the horizontal direction and a finite-difference scheme was used in the vertical direction. Kihm *et al.* [7] have investigated the thermal instability in a fluid layer, subjected to a sudden change in surface temperature, by applying the marginal state method of modified frozen time analysis. This analysis also predicts, like the amplification theory, the proper dependence of wave number on Rayleigh number. Luijkx and Platten [8] have experimentally obtained the wave number at the onset of thermal convection in a long rectangular duct. The experimental data presented in [8] do not fit closely with the finite rolls approximation. Thermal convection in an inclined rectangular enclosure of low aspect ratio with heating from below has been studied by Ozoe *et al.* [9] using a finite-difference technique. However, the applicability of calculations in [9] to enclosures depends on the advance postulate of a known cell width for each Rayleigh number, Prandtl number and pair of aspect ratios. An experimental and analytical study of the phenomenon of natural convection in a partially divided rectangular enclosure has been made very recently by Lin and Bejan [10].

Although a number of studies on the thermal convection in enclosures have been reported in literature, these do not give satisfactory results if the fluid is a mixture of heterogeneous means such as liquid crystals, ferro liquid, liquid with polymer additives, which is more realistic and important from the technological point of view. For the realistic description of the flow of such rheologically complex fluids there exist several theories, e.g. polar fluids, dipolar fluids, couple stress fluids, anisotropic fluids, etc. However, it has been demonstrated by Ariman *et al.* [11] that for linear, viscous and isotropic fluids all these theories can be considered as equivalent to micropolar fluid theory [12]. This theory deals with viscous fluids in which the microconstituents are rigid and spherical or randomly oriented. Polymeric fluids, colloidal fluids, liquid crystals, fluid suspensions, animal blood, etc. can be characterized by this fluid model.

The effect of microstructure in the Benard problem of thermal instability has been studied by Ahmadi [13], Datta and Sastry [14] and Bhattacharyya and Jena [15]. But, to the best of the authors' knowledge, the thermal convection in fluid with microstructures, in enclosures, has not yet been investigated.

In this article, we have studied the effect of microstructure on the thermal convection in a rectangular box of fluid heated from below by applying the theory of micropolar fluids. The Galerkin method with trial functions of the type suggested by Catton [5] has been employed to get an approximate solution for the eigenvalue problem. The influence of lateral walls and material parameters on the convection process in a rectangular box has been determined. The results have been compared with the results of the corresponding problem of a Newtonian fluid.

2. FORMULATION OF THE PROBLEM

We consider an incompressible fluid with microstructures confined in a rectangular box of height d, having no body couple and no heat source. The domain of flow is $|x| \leq h_1/2$, $|y| \leq h_2/2$, and $|z| \leq 1/2$, where x, y, z are the dimensionless coordinates of a point in the flow field referred to the centre of the box as origin. The characteristic length used for nondimensionalising the coordinates is d. The base of the box is kept at a higher temperature than the top so that an adverse temperature gradient ϕ along the vertical direction (along the z axis) is maintained. We assume that instability sets in via a convective marginal state so that the terms with time derivatives will not appear in the equations governing the disturbances [14]. The linearized equations governing the disturbances [14], in vector notation, now can be written in nondimensional form as

$$\operatorname{div} \mathbf{V} = 0, \tag{2.1}$$

$$(1 + N_1)\nabla^2\mathbf{V} + N_1\nabla \times \boldsymbol{\nu} + \mathrm{Ra}\theta\mathbf{k} - \operatorname{grad}p = 0, \tag{2.2}$$

$$N_7\nabla(\nabla \cdot \boldsymbol{\nu}) + N_3\nabla^2\boldsymbol{\nu} + N_1(\nabla \times \mathbf{V} - 2\boldsymbol{\nu}) = 0, \tag{2.3}$$

$$\nabla^2\theta + w - N_5(\nabla \times \boldsymbol{\nu})\cdot\mathbf{k} = \mathbf{0}. \tag{2.4}$$

In the above equations, $\mathbf{V}$, $\boldsymbol{\nu}$, θ and p are respectively the velocity, microrotation, temperature and pressure disturbances measured in units of κ/d, κ/d^2, ϕd and $\kappa\mu/d^2$, where κ is the thermal diffusivity and μ is the coefficient of viscosity. $\mathbf{k}$ is the unit vector in z direction and w is the third component of $\mathbf{V}$. Ra is the Rayleigh number and N_1, N_3, N_5 and N_7 are the material parameters defined as

$$\mathrm{Ra} = \frac{\rho_0 g\phi\lambda d^4}{\kappa\mu}, \quad N_1 = \frac{K}{\mu}, \quad N_3 = \frac{\gamma}{\mu d^2}, \quad N_5 = \frac{\delta}{\rho_0 c_v d^2}, \quad N_7 = \frac{(\alpha + \beta)}{\mu d^2}, \tag{2.5}$$

where c_v is the specific heat of the fluid at constant volume, g is the acceleration due to gravity, ρ_0 is the mass density, and λ is the coefficient of thermal expansion in the Boussinesq equation of state. The dimensionless parameters N_1, N_3, N_5 and N_7, respectively, characterize the vortex viscosity K, spin-gradient viscosity γ, the micropolar heat conduction δ, and gradient viscosities α and β.

The boundaries are all considered to be rigid and perfect heat conductors so that

$$\mathbf{V} = 0, \quad \boldsymbol{\nu} = 0, \quad \theta = 0 \quad \text{on} \quad |x| = \tfrac{1}{2}h_1, \quad |y| = \tfrac{1}{2}h_2, \quad |z| = \tfrac{1}{2}. \tag{2.6}$$

Equations (2.1)–(2.4) together with the boundary conditions (2.6) constitute an eigenvalue problem for the Rayleigh number. It may be remarked here that eqns (2.1)–(2.4) reduce to the corresponding equations for Newtonian fluids [2, 5] by setting $N_1 = N_3 = N_5 = N_7 = 0$.

3. METHOD OF SOLUTION

In order to get an approximate solution for the eigenvalue problem we have employed the Galerkin method. The advantage of this method is that we can use the original full system of equations rather than a higher-order equation obtained by cross-differentiation and elimination as in the case of the variational method. The details of the Galerkin method are available in [16].

In accordance with the Galerkin method we can represent $\mathbf{V}$, $\boldsymbol{\nu}$, θ and p as follows:

$$\mathbf{V} = \sum_{i=1}^{\infty} a_i\mathbf{V}_i, \quad \boldsymbol{\nu} = \sum_{i=1}^{\infty} b_i\boldsymbol{\nu}_i, \quad \theta = \sum_{i=1}^{\infty} c_i\theta_i, \quad p = \sum_{i=1}^{\infty} d_i p_i, \tag{3.1}$$

where $\mathbf{V}_i$, $\boldsymbol{\nu}_i$, θ_i, p_i are all functions of x, y and z. Substituting the first N terms of eqns (3.1) into eqns (2.1)–(2.4) and applying the interior orthogonality relations [16] for the resulting system of equations we get

$$(1 + N_1)\sum_{i=1}^{N} a_i \int_V (\nabla^2\mathbf{V}_i)\cdot\mathbf{V}_j\mathrm{d}V + N_1\sum_{i=1}^{N} b_i\int_V (\nabla\times\boldsymbol{\nu}_i)\cdot\mathbf{V}_j\mathrm{d}V + \mathrm{Ra}\sum_{i=1}^{N} c_i\int_V \theta_i\mathbf{k}\cdot\mathbf{V}_j\mathrm{d}V = 0, \tag{3.2}$$

$$N_1\sum_{i=1}^{N} a_i\int_V (\nabla\times\mathbf{V}_i)\cdot\boldsymbol{\nu}_j\mathrm{d}V + \sum_{i=1}^{N} b_i\int_V [N_7\nabla(\nabla\cdot\boldsymbol{\nu}_i) + N_3\nabla^2\boldsymbol{\nu}_i - 2N_1\boldsymbol{\nu}_i]\cdot\boldsymbol{\nu}_j\mathrm{d}V = 0, \tag{3.3}$$

$$\sum_{i=1}^{N} a_i\int_V w_i\theta_j\mathrm{d}V - N_5\sum_{i=1}^{N} b_i\int_V (\nabla\times\boldsymbol{\nu}_i)\cdot\mathbf{k}\theta_j\mathrm{d}V + \sum_{i=1}^{N} c_i\int_V (\nabla^2\theta_i)\theta_j\mathrm{d}V = 0, \tag{3.4}$$

where $j = 1, 2, \ldots, N$.

It may be noted that

$$\begin{aligned}\sum_{i=1}^{N} d_i\int_V \operatorname{grad} p_i\cdot\mathbf{V}_j\mathrm{d}V &= \sum_{i=1}^{N} d_i\int_V [\nabla\cdot(p_i\mathbf{V}_j) - p_i\nabla\cdot\mathbf{V}_j]\mathrm{d}V \\ &= \sum_{i=1}^{N} d_i\int_S p_i\mathbf{V}_j\cdot\mathrm{d}\mathbf{S} \qquad (\text{since div } \mathbf{V}_j = 0) \\ &= 0 \qquad (\text{since } \mathbf{V}_j = 0 \text{ on the boundary}).\end{aligned}$$

Thus, the pressure term in eqn (3.2) vanishes.

Equations (3.2)–(3.4) have nontrivial solutions if and only if the secular determinant is zero, that is, if

$$\begin{vmatrix} M_{11} & M_{12} & \mathrm{Ra}\, M_{13} \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & M_{33} \end{vmatrix} \equiv \det M = 0, \tag{3.5}$$

where M_{11}, M_{12}, M_{13}, M_{21}, M_{22}, M_{31}, M_{32} and M_{33} are $N \times N$ matrices defined as

$$
\begin{aligned}
(M_{11})_{ij} &= (1 + N_1) \int_V (\nabla^2 \mathbf{V}_i) \cdot \mathbf{V}_j \mathrm{d}V, \\
(M_{12})_{ij} &= N_1 \int_V (\nabla \times \boldsymbol{\nu}_i) \cdot \mathbf{V}_j \mathrm{d}V, \\
(M_{13})_{ij} &= \int_V \theta_i \mathbf{k} \cdot \mathbf{V}_j \mathrm{d}V, \\
(M_{21})_{ij} &= N_1 \int_V (\nabla \times \mathbf{V}_i) \cdot \boldsymbol{\nu}_j \mathrm{d}V, \\
(M_{22})_{ij} &= \int_V [N_7 \nabla(\nabla \cdot \boldsymbol{\nu}_i) + N_3 \nabla^2 \boldsymbol{\nu}_i - 2N_1 \boldsymbol{\nu}_i] \cdot \boldsymbol{\nu}_j \mathrm{d}V, \\
(M_{31})_{ij} &= \int_V w_i \theta_j \mathrm{d}V, \\
(M_{32})_{ij} &= -N_5 \int_V (\nabla \times \boldsymbol{\nu}_i) \cdot \mathbf{k} \theta_j \mathrm{d}V, \\
(M_{33})_{ij} &= \int_V (\nabla^2 \theta_i) \theta_j \mathrm{d}V.
\end{aligned}
\tag{3.6}
$$

Selection of trial functions

In order to solve (3.5), we have considered trial functions which have properties of a finite roll, that is, cells with two nonzero velocity components dependent on all three spatial variables [2], and satisfy the boundary conditions and also the continuity equation exactly.

The trial functions for the velocity components (u_i, v_i, w_i) are constructed by expanding them in terms of beam functions. The trial functions for the components of microrotation and temperature are constructed from a linear combination of a complete set of orthogonal functions. The trial functions are selected to allow for the possibility of fully three-dimensional flow configurations. The construction of trial functions for velocity and temperature has been done in a similar way by Catton [5]. The trial functions used in the present analysis have been presented in Appendix A.

Substituting the trial functions (A.1) of Appendix A into eqns (3.6) and carrying out the integration as indicated, we obtain the roots of det $M = 0$, i.e., the values of Ra for the set (A.1). Similarly, making use of other sets of trial functions (A.2)–(A.4), we obtain the corresponding values of Ra. The minimum among all these Ra's gives the critical Rayleigh number Ra_c for this problem. The integrals (3.6) of various combinations of trial functions (A.1)–(A.4) are available in [17]. In the calculation of Ra, we have truncated the element of the secular determinant (3.5) at finite N. We increased N until five significant figures of accuracy were obtained in Ra, for each of the sets of trial functions (A.1)–(A.4). The above procedure was carried out for various horizontal dimensions h_1 and h_2. For numerical computations, we have chosen the following values for the parameters:

$$
N_1 = (0.1, 1.0, 1.5), \qquad N_3 = (2.0, 4.0, 6.0),
$$
$$
N_5 = (0.5, 1.0, 1.5), \qquad N_7 = (0.0, 1.0, 2.5).
$$

These values of N_1, N_3, N_5 and N_7 satisfy the nondimensional equivalent of the thermodynamic restrictions given by Eringen [18]. In accordance with this restriction N_1, N_3, N_5 and N_7 are all non-negative.

Ahmadi [19] and Tözeren and Skalak [20] have stated that the parameter N_1 depends on the shape and concentration of the microelements. For a given shape of the microelements N_1 directly gives a measure of concentration of the microelements. The parameters N_3 and N_7 can be thought of as fluid properties depending on the relative size of the microstructure in relation to a geometrical length. The parameter N_5 depends on the heat conduction of the microelements.

4. RESULTS AND DISCUSSION

It is worth mentioning that convection in a box gives rise to two types of dependencies of the critical Rayleigh number upon horizontal dimensions for finite rolls. The first type occurs when a finite roll has its width varied which is the same as in the problem of an infinite layer [13–15]. The second type of dependence is obtained when the length of a finite roll is varied. This type of dependence is not present in the infinite layer where rolls have infinite lengths. In the present discussion we confine ourselves to the second type of dependence by studying the variation of aspect ratios at the onset of convection. The results of the present investigation are summarized in Table 1 and Figs. 1–5.

We have recorded in Table 1 the effect of lateral walls (i.e. the variation of aspect ratios) on the onset of convection. Some representative stability curves for different values of aspect ratios h_1 and h_2 have been plotted in Fig. 1. We see from Fig. 1 that there exist kinks for small values of h_1/h_2 or h_2/h_1. This happens because flow goes from single roll to multiple rolls configuration. A similar phenomenon has been observed by Davis [2] and Catton [5] for a Newtonian fluid. For the sake of direct comparison, the results for a Newtonian fluid have been included in Table 1 and Fig. 1. It is observed from Table 1 and Fig. 1 that the critical Rayleigh number of the present problem is always higher than that of a Newtonian fluid. From this we conclude that the fluid with microstructure heated from below is more stable in comparison to a Newtonian fluid. This is in agreement with the results obtained in [13]. Further, we notice from Fig. 1 that the critical Rayleigh number of the present problem at $h_1 = 1.8$, and $h_2 = 1.8$ is quite close to the critical Rayleigh number obtained for a micropolar fluid in an infinite channel for $N_1 = 1.5$, $N_3 = 2.0$ and $N_5 = 1.0$, and with no heat source [17, 21]. It is, in fact, expected because increase in h_1 and h_2 leads to the case of infinite layer. However, for a Newtonian fluid the corresponding values of the aspect ratios are $h_1 = h_2 = 6.0$ [5].

The effects of variation of N_1, N_3, N_5 and N_7 at the onset of convection have been

Table 1. Critical Rayleigh number for the onset of convection for various aspect ratios

h_1	$\log_{10} Ra_c$						
	h_2						
	0.05	0.1	0.8	1.0	1.2	1.4	1.6
			$N_1 = 1.5$, $N_3 = 2.0$, $N_5 = 1.0$, $N_7 = 2.5$				
0.05	9.06239	8.90648	8.81362	8.81199	8.81093	8.81020	8.80967
0.1	8.90648	7.86183	7.62142	7.61752	7.61507	7.61340	7.61219
0.8	8.81362	7.62142	4.61230	4.54259	4.49892	4.46964	4.44896
1.0	8.81199	7.61752	4.54259	4.41054	4.35459	4.31675	4.28991
1.2	8.81093	7.61507	4.49892	4.35459	4.29650	4.25134	4.21914
1.4	8.81020	7.61340	4.46964	4.31675	4.25134	4.23683	4.19999
1.6	8.80967	7.61219	4.44896	4.28991	4.21914	4.19999	4.17960
			$N_1 = 0.0$, $N_3 = 0.0$, $N_5 = 0.0$, $N_7 = 0.0$				
0.05	8.84731	7.98156	7.58113	7.57006	7.56342	7.55711	7.54862
0.1	7.98156	7.76704	6.78243	6.77312	6.76412	6.75623	7.75223
0.8	7.58113	6.78243	4.08941	4.01972	3.97603	3.94475	3.92603
1.0	7.57006	6.77312	4.01972	3,84725	3.82168	3.79386	3,76701
1.2	7.56342	6.76412	3.97603	3.87168	3.77357	3.72843	3,69618
1.4	7.55711	6.75623	3.94675	3.79386	3.72843	3.68391	3.64706
1.6	7.54862	6.75223	3.92603	3,76701	3.69618	3.64706	3.62677

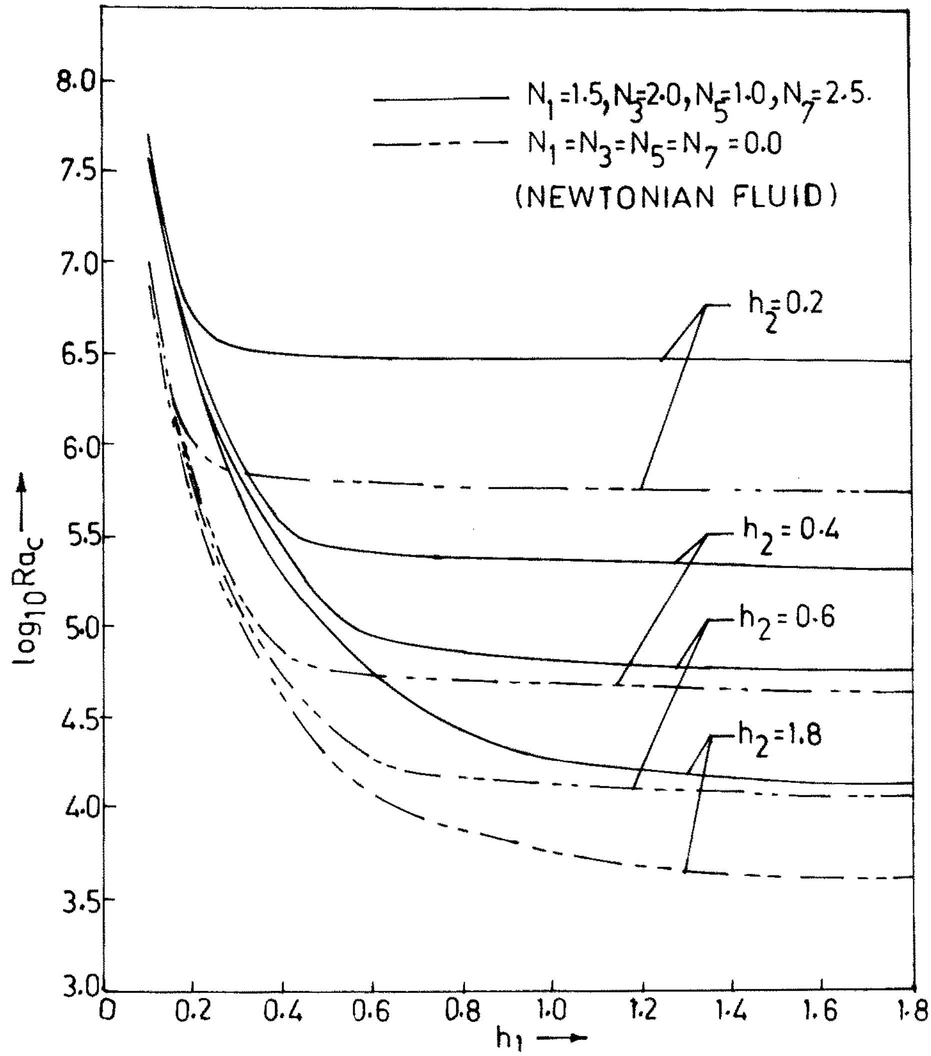

Fig. 1. Critical Rayleigh number for various aspect ratios.

shown in Figs. 2–5, respectively. It is found that the increase of N_1 or N_5 increases the critical Rayleigh number whereas the increase of N_3 or N_7 decreases the Rayleigh number. This means that increase in N_1 or N_5 delays the onset of instability while the increase in N_3 or N_7 hastens.

The concentration of microelements increases with the increase in N_1. Therefore, a greater part of the kinetic energy of the system is consumed in developing gyrational velocities of the fluid, and as a result, the onset of instability is delayed. When N_5 increases, the heat induced into the fluid is also increased, thus reducing the heat transfer from the bottom to the top. The decrease in heat transfer is responsible for delaying the onset of instability due to any increase in N_5. However, increase in N_3 or N_7 increases the couple stress of the fluid, which causes a decrease in microrotation and hence makes the system more unstable.

An interesting phenomenon can be observed from Fig. 5. When h_1 increases, the effect of N_7 seems to reduce gradually. The same is true when h_2 increases because of the symmetric nature of this problem. This means that as the distance between the lateral walls increases, the effect of N_7 on the onset of stability diminishes.

We notice from Figs. 2–5 that N_1 and N_5 play a significant roll at the onset of stability in comparison to N_3 or N_7. Therefore, we conclude that the effect of variation of concentration and heat conduction of the microelements is more important in the study of stability of flow under consideration as compared to the effect of the variation of the size of the microelements.

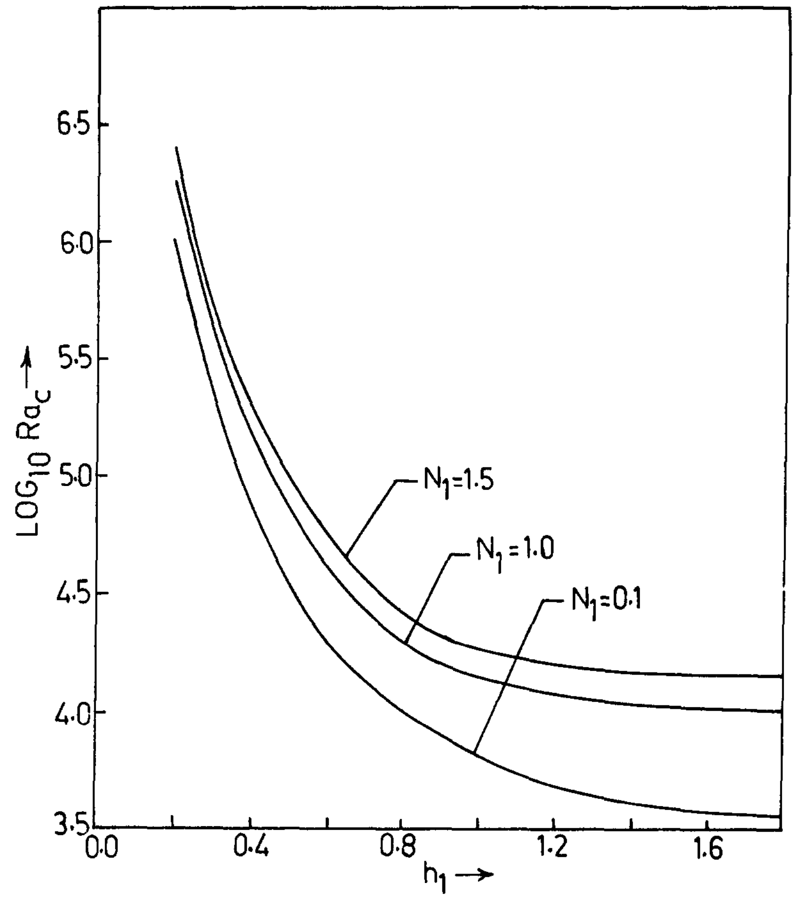

Fig. 2. Critical Rayleigh number for various values of N_1 (N_3 = 2.0, N_5 = 1.0, N_7 = 2.5, h_2 = 1.8).

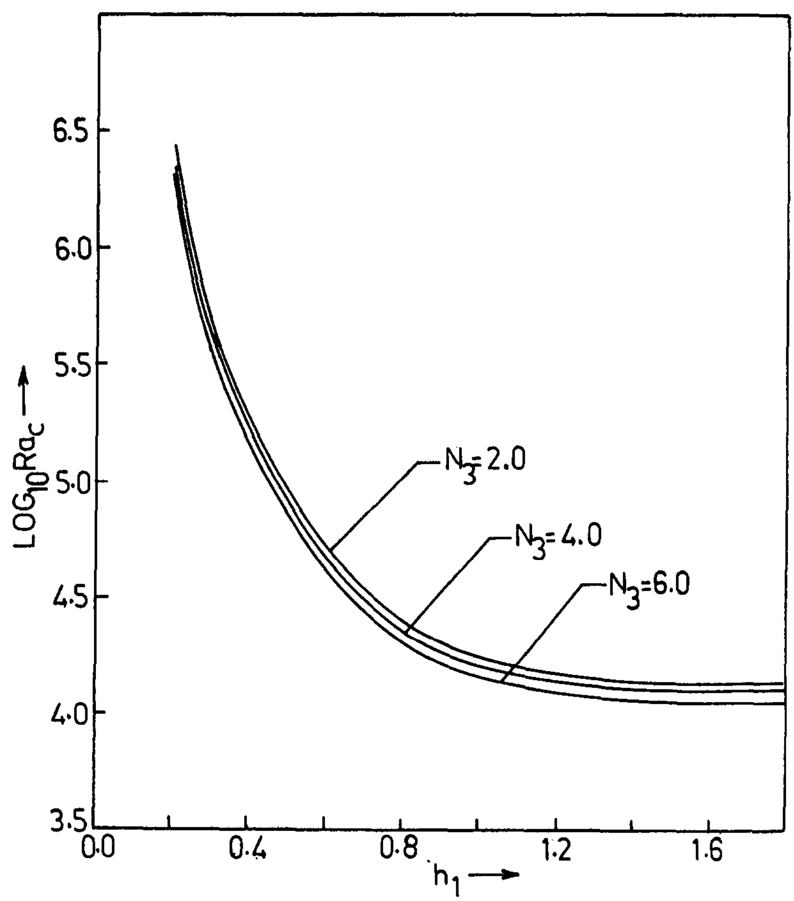

Fig. 3. Critical Rayleigh number for various values of N_3 (N_1 = 1.5, N_5 = 1.0, N_7 = 2.5, h_2 = 1.8).

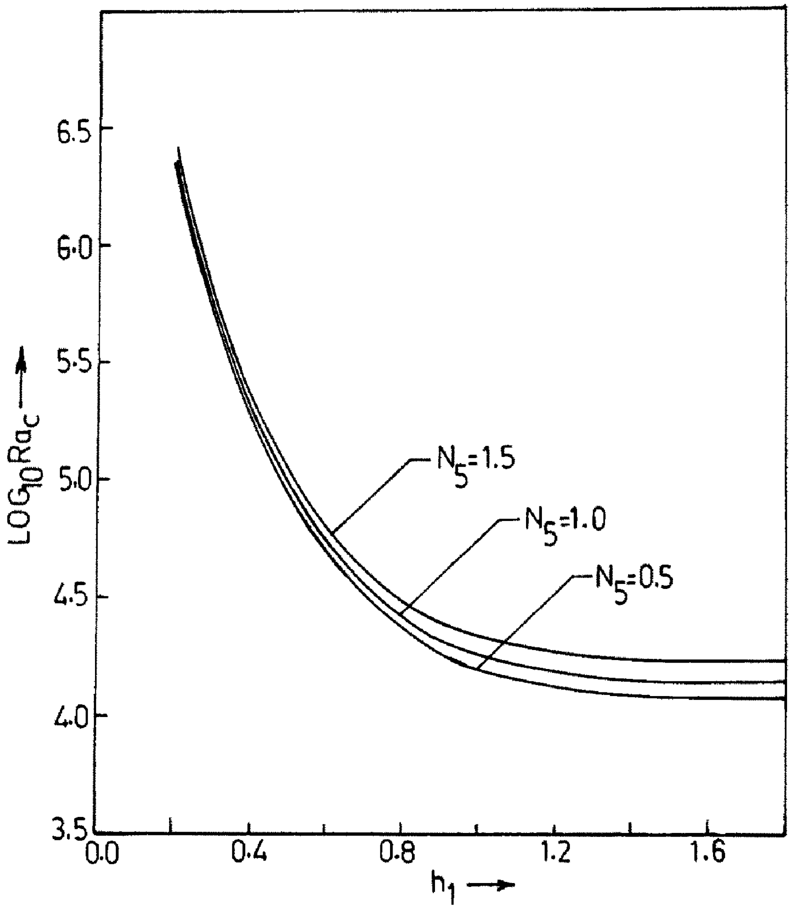

Fig. 4. Critical Rayleigh number for various values of N_5 (N_1 = 1.5, N_3 = 2.0, N_7 = 2.5, h_2 = 1.8).

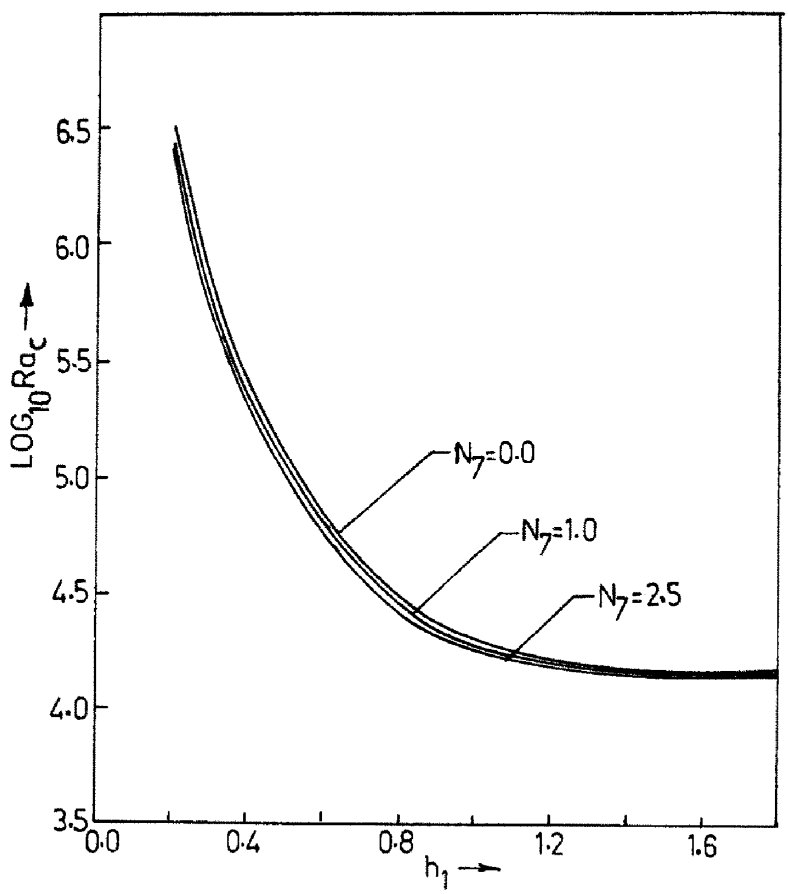

Fig. 5. Critical Rayleigh number for various values of N_7 (N_1 = 1.5, N_3 = 2.0, N_5 = 1.0, h_2 = 1.8).

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APPENDIX A

Finite roll trial functions

1. Odd number of finite x rolls:

$$
\begin{aligned}
u_{pqr} &= -\frac{h_1}{\lambda_p} C_p(\bar{x}) \cos[(2q-1)\pi\bar{y}]C_r'(z),\\
v_{pqr} &= 0,\\
w_{pqr} &= \frac{1}{\lambda_p} C_p'(\bar{x}) \cos[(2q-1)\pi\bar{y}]C_r(z),\\
f_{pqr} &= \sin(2\pi p\bar{x}) \sin(2q\pi\bar{y}) \cos[(2r-1)\pi z],\\
g_{pqr} &= \cos[(2p-1)\pi\bar{x}] \cos[(2q-1)\pi\bar{y}] \cos[(2r-1)\pi z],\\
h_{pqr} &= \cos[(2p-1)\pi\bar{x}] \sin(2q\pi\bar{y}) \sin(2r\pi z),\\
\theta_{pqr} &= \sin(2p\pi\bar{x}) \cos[(2q-1)\pi\bar{y}] \cos[(2r-1)\pi z].
\end{aligned}
\tag{A.1}
$$

2. Even number of finite x rolls:

$$
\begin{aligned}
u_{pqr} &= -\frac{h_1}{\mu_p} S_p(\bar{x}) \cos[(2q-1)\pi\bar{y}]C_r'(z),\\
v_{pqr} &= 0,\\
w_{pqr} &= \frac{1}{\mu_p} S_p'(\bar{x}) \cos[(2q-1)\pi\bar{y}]C_r(z),\\
f_{pqr} &= \cos[(2p-1)\pi\bar{x}] \sin(2q\pi\bar{y}) \cos[(2r-1)\pi z],\\
g_{pqr} &= \sin(2p\pi\bar{x}) \cos[(2q-1)\pi\bar{y}] \cos[(2r-1)\pi z],\\
h_{pqr} &= \sin(2p\pi\bar{x}) \sin(2q\pi\bar{y}) \sin(2r\pi z),\\
\theta_{pqr} &= \cos[(2p-1)\pi\bar{x}] \cos[(2q-1)\pi\bar{y}] \cos[(2r-1)\pi z].
\end{aligned}
\tag{A.2}
$$

3. Odd number of finite y rolls:

$$u_{pqr} = 0,$$

$$v_{pqr} = -\frac{h_2}{\lambda_q}\cos[(2p-1)\pi\bar{x}]C_q(\bar{y})C'_r(z),$$

$$w_{pqr} = \frac{1}{\lambda_q}\cos[(2p-1)\pi\bar{x}]C'_q(\bar{y})C_r(z),$$

$$f_{pqr} = \cos[(2p-1)\pi\bar{x}]\cos[(2q-1)\pi\bar{y}]\cos[(2r-1)\pi z], \tag{A.3}$$

$$g_{pqr} = \sin(2p\pi\bar{x})\sin(2q\pi\bar{y})\cos[(2r-1)\pi z],$$

$$h_{pqr} = \sin(2p\pi\bar{x})\cos[(2q-1)\pi\bar{y}]\sin(2r\pi z),$$

$$\theta_{pqr} = \cos[(2p-1)\pi\bar{x}]\sin(2q\pi\bar{y})\cos[(2r-1)\pi z].$$

4. Even number of finite y rolls:

$$u_{pqr} = 0,$$

$$v_{pqr} = -\frac{h_2}{\mu_q}\cos[(2p-1)\pi\bar{x}]S_q(\bar{y})C'_r(z),$$

$$w_{pqr} = \frac{1}{\mu_p}\cos[(2p-1)\pi\bar{x}]S'_q(\bar{y})C_r(z),$$

$$f_{pqr} = \cos[(2p-1)\pi\bar{x}]\sin(2q\pi\bar{y})\cos[(2r-1)\pi z], \tag{A.4}$$

$$g_{pqr} = \sin(2p\pi\bar{x})\cos[(2q-1)\pi\bar{y}]\cos[(2r-1)\pi z],$$

$$h_{pqr} = \sin(2p\pi\bar{x})\sin(2q\pi\bar{y})\sin(2r\pi z),$$

$$\theta_{pqr} = \cos[(2p-1)\pi\bar{x}]\cos[(2q-1)\pi\bar{y}]\cos[(2r-1)\pi z].$$

In the above expressions, $\bar{x} = x/h_1$ and $\bar{y} = y/h_1$. $C_p(\bar{x})$ and $S_p(\bar{x})$ are, respectively, the even- and the odd-beam functions, defined as

$$C_p(\bar{x}) = \frac{\cosh\lambda_p\bar{x}}{\cosh\lambda_p/2} - \frac{\cos\lambda_p\bar{x}}{\cos\lambda_p/2}, \qquad S_p(\bar{x}) = \frac{\sinh\mu_p\bar{x}}{\sinh\mu_p/2} - \frac{\sin\mu_p\bar{x}}{\sin\mu_p/2}$$

and satisfy the conditions $C_p(1/2) = C'_p(1/2) = S_p(1/2) = S'_p(1/2) = 0$, where the prime denotes differentiation with respect to the independent variable. The method for the construction of the beam functions has been described in [22].