Int. J. Engng Sci. Vol. 21, No. 9, pp. 1019 1024, 1983
Printed in Great Britain.

0020 7225/83 $3.00 + .00

ON THE STABILITY OF A HOT LAYER OF MICROPOLAR FLUID

S. P. BHATTACHARYYA and S. K. JENA
Department of Mathematics, Indian Institute of Technology, Powai, Bombay 400 076, India

Abstract—A critical study on the stability of a hot layer of micropolar fluid heated from below with free boundaries has been investigated. The analysis shows that the method by which the previous investigators (Datta and Sastry, and Pérez-García and Rubí) obtained the critical Rayleigh number is not justified and the final result obtained thereby is erroneous. The correct solution to the problem has been presented. Moreover, it is found that the possibility of having an overstable marginal state which was shown by one of the previous investigators (Pérez-García and Rubí) is not justified. The correct approach proves the validity of the principle of exchange of stabilities for this problem. The results show that the criteria of micropolar stability have some interesting features having no classical analogue.

1. INTRODUCTION

THE THEORY of micropolar fluids has been developed by Eringen[1] to explain the behaviour of fluids which exhibit certain microscopic effects arising from the local structure and micromotion of the fluid elements. Such fluids can be subjected to surface and body couples only and material points in a volume element can undergo only rigid rotational motions about their centers of mass. Eringen[2] further extended the theory of micropolar fluids to take into account thermal effects. Physically, this theory may be considered as a satisfactory model for description of the flow behaviour of liquid crystals, real fluids with suspensions, polymeric fluids and animal blood. On the basis of this theory, experimentally observed phenomenon of skin-friction reduction (up to 30–35%) near a rigid body[3, 4] can also be explained satisfactorily.

The stability criteria for a micropolar fluid have aroused a great amount of interest in recent years[5–8]. In stability analysis, coupling between thermal and microinertial effects has been ignored by Ahmadi[5]. In [5], it has been found that the principle of exchange of stabilities is valid and further, there is no region of subcritical instability for a micropolar fluid heated from below. When the coupling between thermal and microinertial effects is included, the results obtained by Ahmadi were found to be erroneous[6]. Incorporating the coupling between thermal and microinertial effects, Datta and Sastry[7] have presented the plot $Ra(a)$, the critical Rayleigh number corresponding to the wave number a. Recently, Pérez-García and Rubí[8] showed the possibility of having an overstable marginal state when the coupling between thermal and micropolar effects is included. In the present paper, we observe that even after taking into consideration the coupling between thermal and microinertial effects, the principle of exchange of stabilities is valid (Section 3). We also find that the method employed by Dutta and Sastry, and Pérez-García and Rubí is not justified and the critical Rayleigh number obtained thereby is erroneous. A correct solution to the problem of finding the critical Rayleigh number is presented in Section 4.

2. LINEAR THEORY AND THE CHARACTERISTIC EQUATION

We consider an infinite horizontal layer of an incompressible micropolar fluid of finite depth, heated from below, having no body couple and no heat source. The normal mode analysis based on the perturbation equations for this case has been given in great details in [6–8]. For the sake of brevity, this analysis is not presented here. The final form of linearized nondimensional equations governing the disturbances as given by Pérez-García and Rubí[8] is

$$(D^2-k^2)[(1+K)(D^2-k^2)-\sigma]W+K(D^2-k^2)G-Rak^2\Theta=0 \tag{2.1}$$

$$[C_0(D^2-k^2)-2K-\bar{j}\sigma]G-K(D^2-k^2)W=0 \tag{2.2}$$

$$[(D^2-k^2)-Pr\sigma]\Theta+W-\bar{\delta}G=0 \tag{2.3}$$

with the boundary conditions

$$W = D^2W = G = \Theta = 0 \text{ at } z = 0, 1. \tag{2.4}$$

In the above equations, Pr is the Prandtl number, Ra is Rayleigh number, k is the wave number, σ is the stability parameter, K, $\bar{j}$, C_0 and $\bar{\delta}$ are the micropolar fluid parameters. Notations used in eqns (2.1)–(2.3) and the boundary conditions (2.4) are same as given by Pérez-García and Rubí[8], only with the exception of U_z replaced here by W for our convenience. The reader is referred to Ref. [8] for a detailed account of these notations and parameters. For a direct comparison of our results with that of Datta and Sastry[7], it may be remarked here that Datta and Sastry used the parameters $\bar{\delta}$, R, A and n_1 which have the following relations with the parameters used in this paper:

$$K = R, \quad \bar{j} = n_1, \quad C_0 = R/A, \quad \bar{\delta} = \bar{\delta}.$$

Further, it may be noted here that under the thermodynamic restriction given by Eringen[2], Pr, K, $\bar{j}$, C_0 and $\bar{\delta}$ are all non-negative.

In view of eqns (2.1)–(2.3) and the boundary conditions (2.4), we find that $D^{(2m)}W = D^{(2m)}\Theta = D^{(2m)}G = 0$ for $z = 0, 1$ and $m = 1, 2, \ldots$. To consider a general case, we take

$$[W, G, \Theta] = [W_n, G_n, \Theta_n] \sin n\pi z \tag{2.5}$$

and substitute into eqns (2.1)–(2.3) to get the following relationships between W_n, G_n and Θ_n

$$[(1 + K)k_n^2 + \sigma_n k_n]W_n - Kk_n G_n - Rak^2\Theta_n = 0 \tag{2.6}$$

$$Kk_n W_n - (C_0 k_n + 2K + \bar{j}\sigma_n)G_n = 0 \tag{2.7}$$

$$W_n - \bar{\delta}G_n - (k_n + Pr\sigma_n)\Theta_n = 0, \tag{2.8}$$

where $k_n = n^2\pi^2 + k^2$ and σ is replaced by σ_n. Eliminating W_n, G_n and Θ_n from eqns (2.6)–(2.8), we get the following characteristic equation in σ_n

$$A_n\sigma_n^3 + B_n\sigma_n^2 + C_n\sigma_n + D_n = 0, \tag{2.9}$$

where

$$A_n = \bar{j}Prk_n$$

$$B_n = (C_0k_n + 2K)Prk_n + [Pr(1 + K) + 1]\bar{j}k_n^2$$

$$C_n = [(1 + K)(C_0k_n + 2K) - K^2]Prk_n^2 + (C_0k_n + 2K)k_n^2 + \bar{j}(1 + K)k_n^3 - \bar{j}Rak^2$$

$$D_n = Rak^2[(K\bar{\delta} - C_0)k_n - 2K] + [C_0(1 + K)k_n + K(2 + K)]k_n^3. \tag{2.10}$$

3. THE PRINCIPLE OF EXCHANGE OF STABILITIES

Substituting $\sigma_n = \sigma_n^{(r)} + i\sigma_n^{(i)}$ in eqn (2.9), where $\sigma_n^{(r)}$ and $\sigma_n^{(i)}$ are the real and the imaginary parts of σ_n, respectively, and separating out the real and imaginary parts, we get

$$A_n[\sigma_n^{(r)3} - 3\sigma_n^{(r)2}\sigma_n^{(i)}] + B_n[\sigma_n^{(r)2} - \sigma_n^{(i)2}] + C_n[\sigma_n^{(r)}] + D_n = 0 \tag{3.1}$$

and

$$\sigma_n^{(i)}[A_n\{3\sigma_n^{(r)2} - \sigma_n^{(i)2}\} + 2B_n\sigma_n^{(r)} + C_n] = 0. \tag{3.2}$$

To investigate the overstability, we assume $\sigma_n^{(i)} \neq 0$ and finally, after eliminating $\sigma_n^{(i)}$ between eqns (3.1) and (3.2), we get the following equations in $\sigma_n^{(r)}$

$$
\begin{aligned}
&26A_n^3\sigma_n^{(r)6} + 22A_n^3B_n\sigma_n^{(r)5} + 7A_n^3C_n\sigma_n^{(r)4} + [2A_n^2(3B_nC_n - A_nD_n) - 8A_nB_n^3]\sigma_n^{(r)3} \\
&\quad + [4B_n(A_n^2D_n - B_n^3) - A_n^2C_n^2]\sigma_n^{(r)2} - 2(A_nC_n - 2B_n^2)(A_nD_n - B_nC_n)\sigma_n^{(r)} \\
&\quad - (A_nD_n - B_nC_n)^2 = 0.
\end{aligned}
$$

For critical state, at least one root of $\sigma_n^{(r)}$ must be zero whereas all other roots must be negative. Thus, taking $A_nD_n - B_nC_n = 0$ to give the critical Rayleigh number for overstability, the above equation reduces to the form

$$26A_n^3\sigma_n^{(r)6} + 22A_n^3B_n\sigma_n^{(r)5} + 7A_n^3C_n\sigma_n^{(r)4} + 4A_nB_n(A_nC_n - 2B_n^2)\sigma_n^{(r)3} - (A_nC_n - 2B_n^2)^2\sigma_n^{(r)2} = 0. \tag{3.3}$$

It is clear from eqn (3.3) that though two roots of $\sigma_n^{(r)}$ are zero, the product of the other four roots is negative showing that either three or one root is positive. Since this result is true for any n ($n = 1, 2, \ldots$), we do not get an oscillatory marginal state when $\sigma_n^{(i)} \neq 0$. We therefore conclude that the marginal state must necessarily be convective and hence the principle of exchange of stabilities is valid for this problem.

It may be mentioned here that the conclusion that the marginal state has got the possibility of being overstable which was arrived at by Pérez-García and Rubí[8] is not justified since only setting real part of σ_n to be zero, one does not get the marginal state unless all other values of σ_n are found to have non-positive real parts. This criterion was not applied by Pérez-García and Rubí which led to an erroneous result.

4. THE CRITICAL RAYLEIGH NUMBER

Let the disturbance given by

$$[W, G, \Theta] = [W_n, G_n, \Theta_n] \sin n\pi z \cdot \exp(ik_x x + ik_y y + \sigma_n t), \tag{4.1}$$

be marginally stable. This means that at least one root of σ_n in eqn (2.9) must be zero whereas none of the other roots should be positive. Thus, from (2.9) and (2.10), we have

$$Ra = \frac{C_0(1 + K)k_n + K(2 + K)}{2K + (C_0 - K\bar{\delta})k_n} \cdot \frac{k_n^3}{k^2} = (Ra)_n \,(\text{say}). \tag{4.2}$$

With this value of Ra, the value of C_n is obtained from (2.10) as

$$C_n = \frac{(S_1k_n^2 + S_2k_n + S_3)k_n^2}{2K + (C_0 - K\bar{\delta})k_n} = \frac{t_1k_n^2}{t_2} \,(\text{say}), \tag{4.3}$$

where

$$
\begin{aligned}
t_1 &= S_1k_n^2 + S_2k_n + S_3, \\
t_2 &= 2K + (C_0 - K\bar{\delta})k_n, \\
S_1 &= (C_0 - K\bar{\delta})[(1 + K)(PrC_0 + \bar{j}) + C_0] - \bar{j}C_0(1 + K), \\
S_2 &= K(C_0 - K\bar{\delta})[(2 + K)Pr + 2] + 2KC_0[(1 + K)Pr + 1] + K^2\bar{j}, \\
S_3 &= 2K^2[(2 + K)Pr + 2],
\end{aligned} \tag{4.4}
$$

whereas A_n and B_n remain the same as given in (2.10). Since none of the remaining roots of eqn (2.9) is positive, A_n, B_n and C_n must have same sign. Due to the restrictions given by

Eringen[2], signs of A_n as well as B_n are positive. It is, therefore, required to show that C_n is positive in order that the disturbance given in eqn (4.1) is marginally stable.

In subsequent analysis, we consider the following cases separately.

Case I. $C_0 - K\bar{\delta} \geq 0$

In this case $(Ra)_n$'s for various values of n are positive and moreover, it can be easily seen from eqn (4.2) that

$$(Ra)_1 < (Ra)_2 < (Ra)_3 < \cdots . \tag{4.5}$$

Further, we observe from (4.4) that S_2 and S_3 are positive. Therefore, if S_1 is positive, that is, if

$$C_0 - K\bar{\delta} \geq \frac{\bar{j}C_0(1+K)}{(1+K)(PrC_0+\bar{j})+C_0},$$

C_n is positive and $(Ra)_c$, the critical Rayleigh number, is obtained from eqn (4.2) as $(Ra)_1$. Thus, we have

$$(Ra)_c = \frac{C_0(1+K)(\pi^2+k^2)+K(2+K)}{2K+(C_0-K\bar{\delta})(\pi^2+k^2)} \cdot \frac{(\pi^2+k^2)^3}{k^2}. \tag{4.6}$$

If

$$0 \leq C_0 - K\bar{\delta} < \frac{\bar{j}C_0(1+K)}{(1+K)(PrC_0+\bar{j})+C_0},$$

we find that though $(Ra)_1 < (Ra)_2 < (Ra)_3 < \ldots$, S_1 is negative, and, therefore, we have to consider only those n for which $C_n \geq 0$. We see that $t_2 > 0$ for all k_n. Moreover, $t_1 > 0$ only for those values of k_n lying between the two roots of $t_1 = 0$, one of which is positive and the other negative. If α^* be the positive root of $t_1 = 0$, then we have

$$\alpha^* = \frac{1}{2}\left[\left\{\left(\frac{S_2}{S_1}\right)^2 - 4\left(\frac{S_3}{S_1}\right)\right\}^{1/2} - \frac{S_2}{S_1}\right]. \tag{4.7}$$

Obviously k_n is greater than the negative root. Therefore, C_n is positive only for those n corresponding to the particular wave number k, for which $k_n \leq \alpha^*$. It may be mentioned here that if $\alpha^* < \pi^2$, no such n exists and hence convective marginal state does not exist. In case $\alpha^* > \pi^2$, the marginal convective state exists only for those values of k for which $k^2 < (\alpha^* - \pi^2)$, and $(Ra)_c$ is obtained as $(Ra)_1$ given by eqn (4.6).

Case II. $C_0 - K\bar{\delta} < 0$

In this case also, we observe that $t_1 > 0$ for those values of k_n lying between the two roots of $t_1 = 0$, one being positive (α^*, say) and the other negative. Further, we observe that $t_2 > 0$ for $k_n < M$ and $t_2 < 0$ for $k_n > M$, where M is given by

$$M = 2K/(K\bar{\delta} - C_0).$$

Thus, if

$$\gamma_1 = \min(\alpha^*, M) \tag{4.8}$$

and

$$\gamma_2 = \max(\alpha^*, M), \tag{4.9}$$

C_n is positive only when either $n^2\pi^2 \leq k_n < \gamma_1$ or $k_n > \gamma_2$. It is noted that $(Ra)_n > 0$ in the case $n^2\pi^2 \leq k_n < \gamma_1$ and $(Ra)_n < 0$ in the case $k_n > \gamma_2$. From (2.9), we find that if $Ra = (Ra)_n + \delta R_n$, where δR_n is a small change in Ra, then to the first order of the small quantity σ_n, which was zero when $Ra = (Ra)_n$, assumes the value $\delta\sigma_n$

$$\delta\sigma_n = \frac{\delta R_n \cdot k^2}{C_n}[2K - (K\bar{\delta} - C_0)k_n].$$

Hence for $k_n < M$ any increase in the value of Ra from $(Ra)_n$ leads to instability and for $k_n > M$ any decrease in the value of Ra from $(Ra)_n$ leads to instability. Since $C_0 - K\bar{\delta} < 0$, it is noted after some calculations that the general order relation $(Ra)_1 < (Ra)_2 < (Ra)_3 < \ldots$ is not valid for all k's. We, therefore, conclude that to obtain $(Ra)_c$, the critical Rayleigh number corresponding to the particular wave number k, all n's are to be considered for which $k_n < \gamma_1$. In case such n's exist, we find that the minimum of the corresponding $(Ra)_n$'s is positive and gives one value of the critical Rayleigh number lying on the positive branch of the stability curve. Similarly, for the same value of k, we consider all n's for which $k_n > \gamma_2$. We find that the maximum of the corresponding $(Ra)_n$'s is negative and gives another value of the critical Rayleigh number lying on the negative branch of the stability curve. We further observe that for any value of $k > 0$, there exists n such that $k_n > \gamma_2$ and hence the negative branch of the stability curve exists for all k. If $\gamma_1 < \pi^2$, we find that no positive branch of the stability curve exists and if $\gamma_1 \geq \pi^2$, then for all those values of k for which $k^2 \leq (\gamma_1 - \pi^2)$ the positive branch exists and ceases to exist for those values of k for which $k^2 > (\gamma_1 - \pi^2)$.

5. CONCLUSIONS

We have shown in Section 3 that the principle of exchange of stabilities is valid even when the coupling between thermal and microrotational effects is taken into account, which was not done by Ahmadi[5] or any other investigators.

We find after combining the results of the various case studies in Section 4 that if

$$C_0 - K\bar{\delta} \geq \frac{\bar{j}C_0(1+K)}{(1+K)(PrC_0+\bar{j})+C_0},$$

the convective marginal state exists and the critical Rayleigh number is the same as given by Datta and Sastry[7]. If

$$0 \leq C_0 - K\bar{\delta} < \frac{\bar{j}C_0(1+K)}{(1+K)(PrC_0+\bar{j})+C_0},$$

we find that the convective marginal state exists only if α^*, given by eqn (4.7), is greater than π^2 and even in that case, convective stability exists for only those disturbances with wave number k such that $k^2 < (\alpha^* - \pi^2)$. Moreover, when such a disturbance exists in the marginal state, the critical Rayleigh number is found to be of the same form as in the earlier case.

If $C_0 - K\bar{\delta} < 0$, we find that though a negative branch of the convective stability curve always exists, the positive branch can exist only if (i) $\gamma_1 \geq \pi^2$, where γ_1 is given by (4.8) and (ii) $k^2 \leq (\gamma_1 - \pi^2)$. Therefore, the asymptote separating the positive and the negative branches as presented by Datta and Sastry (Ref. [7], Figs. 1–3) does not exist. Considering all the possibilities, we find that the assumption $(Ra)_c = (Ra)_1$ in all cases, as was made by previous investigators[7, 8], is not justified.

Acknowledgement—One of the authors, (S. K. Jena), is grateful to the Council of Scientific and Industrial Research, Government of India, for the award of a Research Fellowship for undertaking this research work.

REFERENCES

[1] A. C. ERINGEN, *J. Math. Mech.* **16**, 1 (1966).
[2] A. C. ERINGEN, *J. Math. Anal. Appl.* **38**, 480 (1972).
[3] J. W. HOYT and A. G. FABULA, U.S. Naval Ordnance Test Report (1964).
[4] W. M. VOGEL and A. M. PATTERSON, Pacific Naval Lab. of the Defence Research Board of Canada Report, 64-2 (1964).

[5] G. AHMADI, *Int. J. Engng Sci.* **14**, 81 (1976).
[6] G. LEBON and C. PÉREZ-GARCÍA, *Int. J. Engng Sci.* **19**, 1321 (1981).
[7] A. B. DATTA and V. U. K. SASTRY, *Int. J. Engng Sci.* **14**, 631 (1976).
[8] C. PÉREZ-GARCÍA and J. M. RUBÍ, *Int. J. Engng Sci.* **20**, 873 (1982).

(*Received* 13 *October* 1982)