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Exploding Julia sets in the dynamics of $ f_{\lambda}(z)=\lambda J_{1}(iz)/iz

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Exploding Julia sets in the Dynamics of $f_\lambda(z) = \lambda J_1(iz)/iz$

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Abstract— **In the present paper, we study the dynamics of the one parameter family of entire functions $\{f_\lambda(z) = \lambda f(z) \;:\; f(z) = J_1(iz)/iz$ for $z \in \mathbb{C}$ and λ is a non-zero real number$\}$ where $J_1(z)$ is the Bessel function of the first kind of order one. We have found a critical parameter $\lambda^* \approx 2.598$ and show that the Julia set of f_λ is a nowhere dense subset of the complex plane $\mathbb{C}$ for $0 < |\lambda| \leq \lambda^*$ and is equal to extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for $|\lambda| > \lambda^*$. This sudden change in the Julia sets is known as *explosion* in the Julia sets or *chaotic burst* in the dynamics.**

Keywords—**Complex dynamics, Julia set, Chaotic burst.**

I. Introduction

A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next or from one stage to the next. A basic goal of the mathematical theory of dynamical systems is to determine or characterize the long term behavior of the system. The simplest model of a dynamical process supposes that $(n+1)$-th state, z_{n+1} can be determined solely from the knowledge of the previous state z_n, that is $z_{n+1} = f(z_n)$ where f is a function. These systems are called Discrete Dynamical Systems. We shall deal with one such systems, namely *Complex Dynamical System* where the the function f is a complex valued function of one complex variable.

In the study of *Complex Dynamical Systems*, the evolution of the system is realized by the iterations of entire complex functions $f : \mathbb{C} \to \mathbb{C}$. Entire functions are functions that are analytic everywhere in $\mathbb{C}$. For a point $z_0 \in \widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$, the sequence of iterates of z_0 (or orbit of z_0) is given by $z_0 = f^0(z_0)$, $z_1 = f(z_0)$, $z_2 = f(z_1) = f(f(z_0))$ and $z_n = f(z_{n-1}) = f^n(z_0)$ for $n \geq 3$ where f^n is the n-th iterate of f. The complex dynamics problem is to study the long term behavior of the sequence of iterates of z_0 for any given initial point z_0 in $\widehat{\mathbb{C}}$. The set of all points in $\widehat{\mathbb{C}}$ whose sequences of iterates exhibit stable behavior is called the Fatou set and the set of all points in $\widehat{\mathbb{C}}$ whose sequences of iterates exhibit unstable or chaotic behavior is called the Julia set. The following two definitions give a precise mathematical meaning to this idea.

Definition I.1: A family $\mathcal{T}$ of analytic functions defined in a domain $D \subseteq \mathbb{C}$ is said to be normal at a point $z_0 \in D$ if every sequence extracted from $\mathcal{T}$ has a subsequence which converges uniformly either to a bounded function or to ∞ on each compact subset of some neighborhood of z_0.

Definition I.2: The *Fatou set* of an entire function $f(z)$, is denoted by $\mathcal{F}(f)$, is defined as
$\mathcal{F}(f) = \{z \in \widehat{\mathbb{C}} :$ the sequence of iterates $\{f^n\}$ is normal at $z\}$
The complement of the Fatou set $\mathcal{F}(f)$ in the extended complex plane $\widehat{\mathbb{C}}$ is known as the *Julia set* of f and is denoted by $\mathcal{J}(f)$. The point at ∞ is always in the Julia set since it is an essential singularity for which f can not be defined there.
The Fatou set of a function is open by definition. The Julia set is always a non empty and perfect set. Also the interior of the Julia set is empty, unless it is whole of $\hat{\mathbb{C}}$ [2].

The dynamics of a function is effectuated basically by the periodic points of the function. The definition and the nature of the periodic points are given below.

Definition I.3: A point z is called a p-periodic point of f if p is the smallest natural number such that $f^p(z) = z$. If $p = 1$, z is called a fixed point. A p-periodic point z is said to be attracting, indifferent or repelling if $|(f^p)'(z)| < 1, = 1$ or > 1 respectively. Further, an indifferent p-periodic point is called rationally (irrationally) indifferent if $(f^p)'(z) = e^{i2\pi t}$ where t is rational (irrational). A rationally indifferent periodic point is also called parabolic periodic point.

A Fatou component is a maximal connected open subset of $\mathcal{F}(f)$. A component U_0 of $\mathcal{F}(f)$ is p-periodic if p is the smallest natural number such that $f^p(U_0) \subseteq U_0$. The set $\{U_0,\ U_1 = f(U_0),\ U_2 = f^2(U_0),\ \cdots,\ U_{p-1} = f^{p-1}(U)\}$ is called a p-periodic cycle of Fatou components. If U is a Fatou component such that $f^p(U) \bigcap f^q(U) = \emptyset$ for all natural numbers p and q, then U is called a wandering domain.

The classification of periodic Fatou components for transcendental entire functions is given below (See also: [2]).
Suppose that U is a p-periodic Fatou component. Then exactly one of the following possibilities occur.
1. *Attracting Basin*: If for all points z in U, $\lim_{n\to\infty} f^{np}(z) = z^*$ where z^* is an attracting p-periodic point lying in U, then the component U is called an attracting basin.
2. *Parabolic domain*: In this case ∂U (the boundary of U) contains a rationally indifferent p-periodic point z^*. Further $\lim_{n\to\infty} f^{np}(z) = z^*$ for all $z \in U$.
3. *Baker Domain*: If for all points $z \in U$, $\lim_{n\to\infty} f^{np}(z) = \infty$ then the Fatou component U is called a Baker domain.
4. *Rotational Domain*: A Fatou component U is said to be a rotational domain if there exists an analytic homeomorphism $\phi : U \to D$ such that $\phi(f^p(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some irrational number α where D is either the unit disc or an annulus $\{z : 0 < r < |z| < 1\}$. In the first case, U is called Siegel disk and in the second case Herman ring. Entire functions do not have Herman rings [2]. Siegel disk is simply connected.

Besides periodic points, the singular values and its forward orbits play an important role in determining the dynamics of a function.

Definition I.4: A point z is a critical point of f if $f'(z) = 0$. The value of the function f at z, $w = f(z)$ is called the critical value of f. A point w is called an asymptotic value of f if there exists a continuous curve $\gamma(t) :\ (0, \infty) \to \mathbb{C}$ such that

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$\lim_{t\to\infty}\gamma(t)=\infty$ and $\lim_{t\to\infty} f(\gamma(t))=w$. All the critical and asymptotic values of a function are known as singular values. The set of all singular values of a function f is denoted by S_f.
The set of all forward orbits of all singular values is denoted by $O^+(S_f)$ and is given by $\{f^n(w) \,:\, w\in S_f$ and $n=0,\ 1,\ 2,\ \cdots\}$. The relation between the set $O^+(S_f)$ and the periodic Fatou components of f is summarized in the following theorem.

Theorem I.1: [2] Let f be an entire function and $C=\{U_0,\ U_1,\ \cdots,\ U_{p-1}\}$ be a p-periodic cycle of components of $\mathcal{F}(f)$.
1. If C is a cycle of attracting basins or parabolic domains, then there exists a natural number j with $j\in\{0,\ 1,\ \cdots,\ p-1\}$ such that $U_j\bigcap S_f\neq\emptyset$.
2. If C is a cycle of rotational domains then $\partial U_j\subset\overline{O^+(S_f)}$ for all $j\in\{0,\ 1,\ \cdots,\ p-1\}$.

In recent years, the dynamics of transcendental entire functions has been studied by many researchers. While studying the dynamics of one parameter family $\mathcal{E}\equiv\{\lambda e^z \,:\, \lambda>0\}$, Devaney and coworkers [5] observed that the Julia set of λe^z is a nowhere dense subset of extended complex plane for $0<\lambda<1/e$, where as it becomes the whole of the extended complex plane for $\lambda>1/e$. This sudden change in the Julia sets is known as *explosion* in the Julia sets or *chaotic burst* in the dynamics of one parameter family $\mathcal{E}$. Similar chaotic bursts are exhibited for the family $\{f_\lambda(z)=\lambda\frac{e^z-1}{z} \,:\, \lambda>0\}$ by Kapoor and Prasad [9] and for the family $\{f_\lambda(z)=\lambda\frac{\sinh(z)}{z} \,:\, \lambda$ is a non zero real parameter$\}$ by Prasad [7].

In the present paper, we study the dynamics of the one parameter family of entire functions $\{f_\lambda(z)=\lambda f(z) \,:\, f(z)=J_1(iz)/iz$ for $z\in\mathbb{C}$ and λ is a non-zero real number$\}$ where $J_1(z)$ is the Bessel function of the first kind of order one given by $J_1(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+1)!}\left(\frac{z}{2}\right)^{2k+1}$ for $z\in\mathbb{C}$. We remark that $f(z)=\frac{J_1(iz)}{iz}=z^{-1}I_1(z)$ where $I_1(z)$ denotes the modified Bessel function of first kind and order one. Clearly

$$f(z)=\frac{J_1(iz)}{iz}=\sum_{k=0}^{\infty}\frac{z^{2k}}{2^{2k+1}\,k!\,(k+1)!}\ \text{for } z\in\mathbb{C}$$

is an entire function.

II. Dynamics of f_λ

Let $\mathcal{B}\equiv\{f_\lambda(z)=\lambda f(z) \,:\, f(z)=\dfrac{J_1(iz)}{iz}$ for $z\in\mathbb{C}$ and λ is a non-zero real number$\}$.

For $f_\lambda\in\mathcal{B}$, observe that $f_\lambda(-z)=f_\lambda(z)$. So, $f_{-\lambda}(z)=-f_\lambda(z)=-f_\lambda(-z)$ for all $z\in\mathbb{C}$. Consequently, $f^n_{-\lambda}(z)=-f^n_\lambda(-z)$ for all $z\in\mathbb{C}$ and $n\in\mathbb{N}$, and dynamics of f_λ and $f_{-\lambda}$ are essentially same. The functions f_λ and $f_{-\lambda}$ are called conformally conjugate. So, it is sufficient to study the dynamics of the one parameter family $\mathcal{B}^+\equiv\{f_\lambda(z)=\lambda f(z) \,:\, f(z)=\dfrac{J_1(iz)}{iz}$ for $z\in\mathbb{C}$ and $\lambda>0\}$.

We first prove that the function f_λ has infinitely many singular values in Proposition II.1. The existence and nature of the fixed points for f_λ is proved in Theorem II.1. Non-existence of certain type of periodic components in Fatou set of f_λ is established in Propositions II.2 and II.3. Finally, a complete picture of the dynamics of the functions f_λ is presented.

A. Singular values of f_λ

The following proposition locates all singular values of $f_\lambda\in\mathcal{B}^+$.

Proposition II.1: Let $f_\lambda\in\mathcal{B}^+$. Then, f_λ has infinitely many singular values all lying in a bounded set of $\mathbb{R}$.

Proof: We first observe that $f'_\lambda(z)=\frac{-\lambda i\,J_2(iz)}{(iz)}$ where $J_2(z)$ is the Bessel function of the first kind of order two [4]. The critical points of $f_\lambda(z)$ are the solutions of $J_2(iz)=0$ and these are infinitely many purely imaginary numbers [4]. They form an unbounded sequence as $J_2(iz)$ is entire. Let these be arranged in an increasing sequence in magnitude, namely, $\{z_k=ix_k\}$ where $x_k\in\mathbb{R}$ for all $k\in\mathbb{N}$. Now, the critical value corresponding to the critical point z_k is given by $f_\lambda(z_k)=f_\lambda(ix_k)=\lambda\frac{J_1(-x_k)}{(-x_k)}$ which is a real number. Since $\lim_{k\to\infty}f_\lambda(z_k)=0$ [4] and $f_\lambda(z_k)\neq 0$ for all k, there are infinitely many critical values of f_λ. Since $\frac{J_1(x)}{x}$ is bounded on $\mathbb{R}$, all the critical values lie in an bounded interval in $\mathbb{R}$.

It is easy to show that the order (which measures the growth of maximum modulus) [8] of the entire function $f_\lambda(z)$ is one. By Ahlfors-Denjoy theorem [1], it follows that f_λ has at most two finite asymptotic values. The function f_λ tends to 0 when z tends to ∞ along the positive and the negative imaginary axis. So, 0 is an asymptotic value for f_λ. If $a\neq 0$ is an asymptotic value of f, then $-a$ and $\bar{a}$ will be also asymptotic values since $f_\lambda(z)=f_\lambda(-z)$ and $f_\lambda(\bar{z})=\overline{f_\lambda(z)}$. This is not possible by Ahlfors-Denjoy theorem [1]. Therefore, f_λ has only one finite asymptotic value, namely, 0. This completes the proof. ∎

B. Real Periodic Points of f_λ

In this subsection, the existence and nature of real periodic points of f_λ is studied. The function $f(x)=J_1(ix)/(ix)$ takes the positive values for all $x\in\mathbb{R}$. It gives that, all the real periodic points of $f_\lambda(x)$ lie on the positive real axis. Suppose x_0 is a real periodic point such that $f^p_\lambda(x_0)=x_0$ for some $p\geq 1$. Since $f'_\lambda(x)>0$ for $x>0$, $f^p_\lambda(x_0)=x_0$ is not possible for $p>1$. Therefore, any real periodic point of f_λ is a fixed point.

Consider the function $\phi(x)=f(x)-xf'(x)$ for $x>0$. As $\phi'(x)=-xf''(x)<0$ for all $x>0$, $\phi(x)$ is decreasing for $x>0$. Using the intermediate value theorem and the facts that $\phi(0)=f(0)>0$ and $\lim_{x\to\infty}\phi(x)=-\infty$, we get a unique point $x^*>0$ such that

$$\phi(x)\begin{cases} >0 & \text{for } 0\leq x<x^* \\ =0 & \text{for } x=x^* \\ <0 & \text{for } x>x^* \end{cases}$$

Throughout this paper, we denote λ^* by $\frac{1}{f'(x^*)}$ where x^* is the unique positive real root of $\phi(x)=f(x)-xf'(x)=0$. Note that $0<\lambda^*<\frac{1}{f'(0)}$, since $x^*>0$ and $\frac{1}{f'(x)}$ is decreasing in $\mathbb{R}^+$. Numerically it is found that $\lambda^*\approx 2.598$.

The following theorem describes the existence and nature of the real fixed points of f_λ for $\lambda>0$.

Theorem II.1: Let $f_\lambda(x)=\lambda J_1(ix)/(ix)$ for $x\in\mathbb{R}$ where $\lambda>0$. Then,

1. For $0 < \lambda < \lambda^*$, the function f_λ has only two real fixed points a_λ and r_λ (say) with $0 < a_\lambda < r_\lambda$ where a_λ is attracting and r_λ is repelling.
2. For $\lambda = \lambda^*$, the function f_λ has only one real fixed point at x^*, and it is rationally indifferent.
3. For $\lambda > \lambda^*$, the function f_λ has no real fixed point.

Proof:

Let $g_\lambda(x) = f_\lambda(x) - x$ for $x \in \mathbb{R}$. Since all the coefficients of the Taylor series of f_λ about the point $z = 0$ are non-negative, the functions $f_\lambda(x)$, $f'_\lambda(x)$ and $f''_\lambda(x)$ are positive for $x > 0$. It gives that $f_\lambda(x)$ and $f'_\lambda(x)$ are increasing in $\mathbb{R}^+$, the positive real axis.

Suppose that $\lambda < \frac{1}{f'(0)}$. Then $g'_\lambda(0) < 0$. Since the function $g'_\lambda(x) = f'_\lambda(x) - 1$ is increasing in $\mathbb{R}^+$ and tends to $+\infty$ as x approaches to $+\infty$, there exists a unique $x_\lambda > 0$ such that $g'_\lambda(x) < 0$ for $x \in (0,\ x_\lambda)$, $g'_\lambda(x_\lambda) = 0$ and $g'_\lambda(x) > 0$ for $x \in (x_\lambda,\ \infty)$. Therefore, g_λ decreases in the interval $[0,\ x_\lambda]$, attains its minimum value at x_λ and then increases in the interval $(x_\lambda,\ \infty)$. For each $\lambda < \frac{1}{f'(0)}$ there exists a unique positive real number x_λ such that $\lambda = \frac{1}{f'(x_\lambda)}$.

1. If $\lambda < \lambda^*$, then $\frac{1}{f'(x_\lambda)} < \frac{1}{f'(x^*)}$. Since $\frac{1}{f'(x)}$ is strictly decreasing in $\mathbb{R}^+$, it follows that $x_\lambda > x^*$ and $\phi(x_\lambda) < \phi(x^*) = 0$ as $\phi(x)$ is decreasing. Since $f'(x_\lambda) > 0$ and for that, $\frac{\phi(x_\lambda)}{f'(x_\lambda)} =$

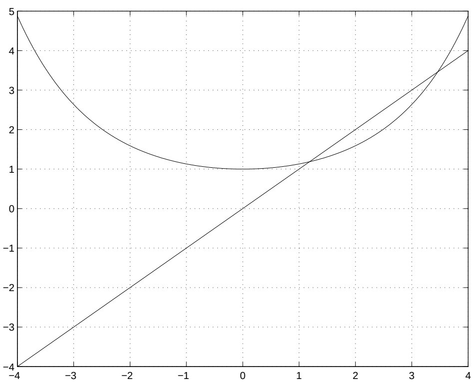

Fig. 1. Graph of f_λ for $\lambda < \lambda^*$ and the line $y = x$, ($\lambda = 2$).

$g_\lambda(x_\lambda) < 0$, the minimum value $g_\lambda(x_\lambda) = f_\lambda(x_\lambda) - x_\lambda$ of $g_\lambda(x)$ is negative. Therefore, there exist two points a_λ and r_λ (say) with $a_\lambda < x_\lambda < r_\lambda$ such that $g_\lambda(a_\lambda) = 0 = g_\lambda(r_\lambda)$. That means, the points a_λ and r_λ are the fixed points of $f_\lambda(x)$ (See Fig. 1). Observe that $f'_\lambda(a_\lambda) < f'_\lambda(x_\lambda) = 1$ and $f'_\lambda(x_\lambda) > f'_\lambda(r_\lambda) = 1$. Therefore, a_λ is attracting and r_λ is repelling.
2. By similar arguments as in proof of (1), we conclude that $g_\lambda(x_\lambda) = 0$ for $\lambda = \lambda^*$ and $x_\lambda = x^*$. As $g_\lambda(x_\lambda)$ is the minimum value of $g_\lambda(x)$, x_λ is the only zero of $g_\lambda(x)$. Hence $f_\lambda(x)$ has only one real fixed point x^* (See Fig. 2) and it is rationally indifferent.
3. For $\lambda^* < \lambda < \frac{1}{f'(0)}$, $\frac{1}{f'(x^*)} < \frac{1}{f'(x_\lambda)}$. It implies that $x_\lambda < x^*$ and, consequently, $\phi(x_\lambda) > 0$. Further, $g_\lambda(x) > g_\lambda(x_\lambda) = 0$ for all $x > 0$. Therefore, there is no real fixed point of $f_\lambda(x)$ for $\frac{1}{f'(0)} > \lambda > \lambda^*$. For $\lambda \geq \frac{1}{f'(0)}$, $g'_\lambda(0) \geq 0$ and $g_\lambda(x) > g_\lambda(0) \geq 0$ for all $x > 0$ as g_λ is increasing in positive real axis. So f_λ has no fixed point in $\mathbb{R}$ (See Fig. 3).

■

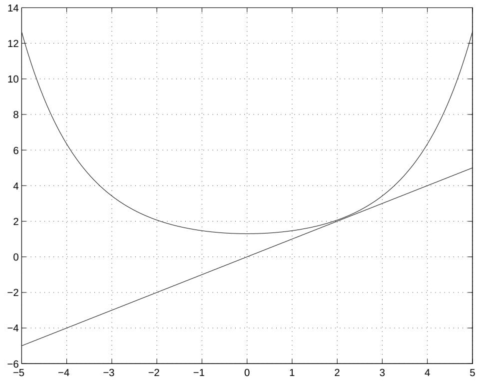

Fig. 2. Graph of f_λ for $\lambda = \lambda^* \approx 2.598$ and the line $y = x$.

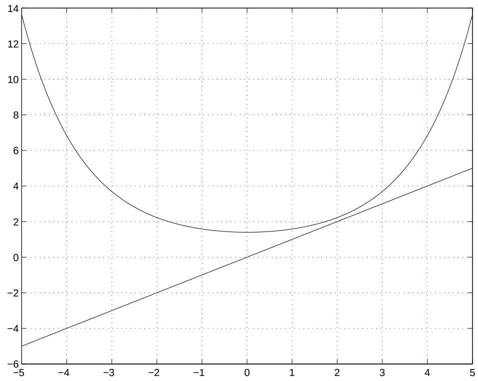

Fig. 3. Graph of f_λ for $\lambda > \lambda^*$ and the line $y = x$, ($\lambda = 2.8$).

C. Fatou Components of f_λ

We show in this subsection that the Fatou set of f_λ does not contain certain kinds of Fatou components.

Proposition II.2: Let $f_\lambda \in \mathcal{B}^+$. Then, the Fatou set $\mathcal{F}(f_\lambda)$ does not contain any Siegel disk.

Proof: As $\lambda > 0$, $f_\lambda(x) > 0$ for all $x \in \mathbb{R}$. By Proposition II.1, the set of all singular values of f_λ is contained in $\mathbb{R}^+$. Consequently, the forward orbits of all singular values $O^+(S_{f_\lambda})$ is also contained in $\mathbb{R}^+$.

Let U be a Siegel disk in the Fatou set of f_λ. Then f_λ is a bijection on U by definition. It follows from Picard's theorem that, there are infinitely many pre-periodic components in $\mathcal{F}(f_\lambda)$ each of which is equal to $f_\lambda^{-k}(U)$ for some $k \in \mathbb{N}$. It is known from Theorem I.1 that $O^+(S_{f_\lambda})$ is dense in ∂U, the boundary of U. So ∂U is contained in $\mathbb{R}^+$. But this is not possible since U is simply connected. Therefore, the Fatou set of f_λ for $\lambda > 0$ does not contain a Siegel disk. ■

Now, we prove non-existence of Baker domains and wandering domains for $f_\lambda \in \mathcal{B}^+$.

Proposition II.3: Let $f_\lambda \in \mathcal{B}^+$. Then, $\mathcal{F}(f_\lambda)$ contains neither wandering domain nor Baker domain.

Proof: Let on contrary, W be a wandering domain of f_λ. It is already shown that any real number tends either to a non repelling (attracting or parabolic) real fixed point or to ∞ under iteration of f_λ. It is known that for a wandering domain W of an entire function f, every limit function of $\{f^n(z)\}_{n>0}$ for $z \in W$ is in Julia set and is a limit point of $O^+(S_f)$ (∞ can well be such a limit point) [3]. Let $z \in W$ and $\{f_\lambda^{n_k}(z)\}_{k>0}$

converges to z_0. Then, z_0 is a real number and in Julia set. Being a limit point of the set $\{f_\lambda^n(z)\}_{n>0},\ z \in W \subset \mathcal{F}(f_\lambda)$, z_0 can not be a repelling fixed point. Therefore, z_0 is not a fixed point of f_λ and, hence $\lim_{m\to\infty} f^m(z_0) = \infty$ as f_λ is increasing in the positive real axis. By continuity of f_λ, it follows that $\lim_{k\to\infty} f^{n_k+m}(z) = f^m(z_0)$. Since $\lim_{k\to\infty} f^{n_k+m}(z) = f^m(z_0)$ and $\lim_{m\to\infty} f^m(z_0) = \infty$, we can find a subsequence $\{f_\lambda^{m_k}(z)\}_{k>0}$ of $\{f_\lambda^n(z)\}_{n>0}$ such that $\lim_{k\to\infty} f_\lambda^{m_k}(z) = \infty$.

Using logarithmic change of variable, it has been proved that, if f is an entire function and the set of all its singular values is bounded then the sequence of iterates $\{f^n(z)\}_{n>0}$ can not converge to ∞ for any z in the Fatou set of f [6]. From the proof of this result, we can show that the above result is true for any subsequence $\{f_\lambda^{m_k}(z)\}_{k>0}$. It contradicts the conclusion made in the previous paragraph. Therefore, f_λ has no wandering domain.

Non-existence of Baker domains of any period follows from the fact that no subsequence of $\{f_\lambda^n(z)\}_{n>0}$ can converge to ∞ for any $z \in \mathcal{F}(f_\lambda)$ [6]. ∎

D. Fatou and Julia sets of f_λ

In this subsection, the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ is described for each non zero real number λ.

Theorem II.2: Let $f_\lambda \in \mathcal{B}^+$. Then,
1. For $0 < \lambda < \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the attracting basin $A(a_\lambda)$ of the real attracting fixed point a_λ.
2. For $\lambda = \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the parabolic domain $P(x^*)$ corresponding to the real rationally indifferent fixed point x^*.
3. For $\lambda > \lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is empty.

Proof: The Fatou set of f_λ does not contain any Siegel disk, Baker domain or wandering domain by Propositions II.2 and II.3. So, every periodic Fatou component is an attracting basin or parabolic domain. All the singular values of f_λ and their forward orbits lie in the real axis. If there is a periodic attracting basin or parabolic domain U, say corresponding to a non-real periodic point then there is a singular value w of f_λ in U by Theorem I.1. In that case $f_\lambda^n(w)$ must be non-real for sufficiently large n which is not possible. So, any periodic Fatou component corresponds only to a real attracting or a parabolic periodic point. Further, it is shown that any real periodic point of f_λ is a fixed point.
1. By Theorem II.1(1), f_λ has only two real fixed points namely a_λ and r_λ. Let $A(a_\lambda) = \{z \in \mathbb{C} : f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$ be the attracting basin of the real attracting fixed point a_λ. Since there are no other attracting or parabolic real fixed points, $\mathcal{F}(f_\lambda) = A(a_\lambda)$.
2. By Theorem II.1(2), f_λ has only one real rationally indifferent fixed point x^*. Let $P(x^*) = \{z \in \mathbb{C} : f_\lambda^n(z) \to x^* \text{ as } n \to \infty\}$ be the parabolic domain corresponding to the real rationally indifferent fixed point. Since there are no other attracting or parabolic real fixed points, $\mathcal{F}(f_\lambda) = P(x^*)$ for $\lambda = \lambda^*$.
3. By Theorem II.1(3), f_λ has no real fixed point. So $\mathcal{F}(f_\lambda) = \emptyset$.

∎

The following result gives an algorithm to computationally generate the pictures of the Julia sets of f_λ.

Corollary II.1: Let $f_\lambda \in \mathcal{B}^+$. Then,
1. For $0 < \lambda < \lambda^*$, the Julia set $\mathcal{J}(f_\lambda)$ is equal to the complement of the attracting basin $A(a_\lambda)$ of the real attracting fixed point a_λ.
2. For $\lambda = \lambda^*$, the Julia set $\mathcal{J}(f_\lambda)$ is equal to the complement of the parabolic basin $P(x^*)$ of the real rationally indifferent fixed point x^*.
3. For $\lambda > \lambda^*$, the Julia set $\mathcal{J}(f_\lambda) = \widehat{\mathbb{C}}$.

The following theorem is an easy consequence of Theorem II.2 and the conformal conjugacy between f_λ and $f_{-\lambda}$.

Theorem II.3: Let $\lambda < 0$. Then,
1. For $-\lambda^* < \lambda < 0$, the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the attracting basin $A(-a_\lambda)$ of the real attracting fixed point $-a_\lambda$.
2. For $\lambda = -\lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is equal to the parabolic basin $P(-x^*)$ of the real rationally indifferent fixed point $-x^*$.
3. For $\lambda < -\lambda^*$, the Fatou set $\mathcal{F}(f_\lambda)$ is empty.

Corollary II.2: Let $\lambda < 0$. Then,
1. For $-\lambda^* < \lambda < 0$, the Julia set $\mathcal{J}(f_\lambda)$ is equal to the complement of the attracting basin $A(-a_\lambda)$ of the real attracting fixed point $-a_\lambda$.
2. For $\lambda = -\lambda^*$, the Julia set $\mathcal{J}(f_\lambda)$ is equal to the complement of the parabolic basin $P(-x^*)$ of the real rationally indifferent fixed point $-x^*$.
3. For $\lambda < -\lambda^*$, the Julia set $\mathcal{J}(f_\lambda) = \widehat{\mathbb{C}}$.

III. CONCLUSION

In the previous section, it is proved that the Julia set $\mathcal{J}(f_\lambda)$ is equal to the complement of either the attracting basin or the parabolic domain for $0 < |\lambda| \le \lambda^*$. Since the Fatou set of f_λ is non-empty for $0 < |\lambda| \le \lambda^*$, it follows that the Julia set of f_λ has empty interior. That is, the Julia set of f_λ is a nowhere dense subset of the complex plane for $0 < |\lambda| \le \lambda^*$. If $|\lambda|$ crosses the value λ^*, the Julia set suddenly explodes and equals to the extended complex plane. Thus, an *explosion* in the Julia sets or *chaotic burst* in the dynamics of the one parameter family $\mathcal{B} \equiv \{f_\lambda(z) = \lambda f(z) : f(z) = \frac{J_1(iz)}{iz}$ for $z \in \mathbb{C}$ and λ is a non-zero real number$\}$ of entire transcendental functions occurs at $|\lambda| = \lambda^*$.

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