Indian J. pure appl. Math., **38**(6): December 2007

LANGLANDS DECOMPOSITION OF AFFINE KAC-MOODY ALGEBRAS

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(*Received* 1 *April* 2005; *after final revision* 31 *October* 2006; *accepted* 9 *March* 2007)

Langlands decompositions of affine Kac-Moody algebras have been obtained by the method of direct determination as introduced by Cornwell for Lie algebras. This method is particularly helpful in the case of lower rank algebras. The involutive automorphisms required for such a study are obtained from the Satake diagrams of the corresponding algebras. This has been well illustrated by taking $A_3^{(1)}$ (untwisted) and $A_4^{(2)}$ (twisted) as representative examples.

Key Words: Kac-Moody algebra; Dynkin diagrams; Satake diagrams

1. INTRODUCTION

Now it is beyond doubt that Kac-Moody algebras [1, 2] more particularly the affine versions have wide physical applications in the context of integrable systems [3], two-dimensional field theories and string theories [4] etc. The representation theory of such Kac-Moody algebras runs almost parallel with that of Lie algebras. Already the highest weight representations of these algebras have been discussed in great detail. Cornwell [5] has introduced the method of direct determination of Iwasawa decomposition of Lie algebras, which plays main role in the construction of unitary irreducible representations. We have already applied this technique to obtain Iwasawa decomposition of various types of Kac-Moody algebras and superalgebras [6-8]. The method of direct determination can also be extended very easily to give the Langlands decomposition [9] of all the parabolic subalgebras, which form an essential part in the construction of various unitary irreducible representations. Keeping this in mind, in this communication we have obtained the Langlands decompositions of affine Kac-Moody algebras taking $A_3^{(1)}$ and $A_4^{(2)}$ as illustrative examples. The involutive automorphisms [10] required for such studies have been obtained from their corresponding Satake diagrams [11, 13]. The Satake diagrams are nothing but modified Dynkin diagrams which are used in the classification of the real forms of Lie algebras as well as the associated symmetric spaces [14]. In recent times, these symmetric spaces have found application in the quantum integrable systems [3] and random matrix models [12] that have been studied in various quantum transport problems.

The organization of the paper is as follows: In section II, we give an introduction to direct determination of Langlands decomposition and briefly outlined the procedures for the construction of Satake diagrams of Kac-Moody algebras. In Section-III, we have applied this method to find out the Langlands decomposition of $A_3^{(1)}$ and $A_4^{(2)}$ respectively. Section-IV contains few concluding remarks.

II(A). Langlands Decomposition of Affine Kac-Moody Algebras

The notion of direct determination of Iwasawa decomposition [6, 8] of Lie algebra is extended to the Langlands decomposition of parabolic subalgebras. Let g_R be a real Kac-Moody algebra generated from its compact real form g_K by an involutive automorphism defined with respect to the Cartan subalgebra h of g, which is the complexification of g_R.

The following commutation relations are satisfied by the elements of g:

$$[h, e_\alpha] = \alpha(h)\ e_\alpha, h \in \mathsf{h}, \alpha \in R$$

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha_\beta} e_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ 0, & \text{otherwise} \end{cases}$$

$$[e_\alpha, e_{-\alpha}] = h_\alpha, h_\alpha \in \mathsf{h}. \tag{2.1}$$

Here R denotes the set of roots of g with respect to h and the Killing form is defined by $B(e_\alpha, e_{-\alpha}) = -1$. Here $\alpha(h) = B(h, h_\alpha)$. The compact real form g_K, which we get using Cartan involution is given by $ih_\alpha, \alpha = \alpha_0, \alpha_1 \ldots \alpha_r$ and $i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha})$ for all $\alpha \in R$.

Let K be the maximal compact subalgebra of g_R defined in such a way that $a \in K$ iff $a \in \mathsf{g}_R$ and $\sigma a = a$ where σ is an involutive automorphism of g_R. Let P be the subspace such that $a \in P$ iff $a \in \mathsf{g}$ and $\sigma a = -a$.

Thus, K and P are given by

$$\begin{aligned} \mathsf{K} &= \{ih_\alpha, \text{for } \alpha = \alpha_0, \alpha_1 \ldots\ldots \alpha_r \text{ and } (e_\alpha + e_{-\alpha}), \\ &\qquad i(e_\alpha - e_{-\alpha}) \text{ for all } \alpha |\exp\ \alpha(h) = +1\} \\ \mathsf{P} &= \{i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for all } \alpha |\exp\ \alpha(h) = -1\} \end{aligned} \tag{2.2}$$

Let 'A' be the maximal abelian subalgebra of P with dimension m and $\mathcal{M}$ be the centralizer of A in K. Thus, A may be taken to have a basis consisting of the elements of the form $i(e_\alpha + e_{-\alpha})$. Let R_{A} denote the set of positive roots α appear in this way in A. Similarly, M may be taken to have a basis consisting of the elements of the form $(e_\alpha + e_{-\alpha})$, with the set of positive root α appearing this way in M, being denoted by R_{M}, together possibly with some elements of $\mathsf{h} \cap \mathsf{g}$. If $h'' \in \mathsf{h} \cap \mathsf{g}$

is an element of M, then $\alpha(h'') = 0$ for all $\alpha \in RA \cup R_M$. Complexification of $A \oplus M$ together with the derivation d' gives a Cartan subalgebra h' of g with basis $\mathsf{h}'_0, \mathsf{h}'_1, \ldots \mathsf{h}'_r$ and d'.

Now there exists an inner automorphism [9] $V : \mathsf{h}' \to \mathsf{h}$ i.e.

$$\begin{aligned} \mathsf{h}_j &= V\mathsf{h}'_j, \text{where } V = \prod_\alpha V_\alpha, \alpha \in R, \\ V_\alpha &= \exp[ad\{ia_\alpha(e_\alpha - e_{-\alpha})\}] \quad \text{and} \quad a_\alpha = \frac{\pi}{\{8(\alpha,\alpha)\}^{1/2}} \end{aligned} \tag{2.3}$$

$$\text{Let}\Delta^+ \text{be the set of positive roots, then} h_\alpha = \sum_{j=0}^{r} b_j(\alpha)\mathsf{h}_j. \tag{2.4}$$

Thus $\alpha \in \Delta^+$ iff $b_j(\alpha) > 0$ where j is the least index such that $b_j(\alpha) \neq 0$. The positive roots can be again divided into the following classes:

$$(i) \quad \Delta^+_+ = \{\alpha | \alpha \in \Delta^+, \alpha(h) \neq \alpha(V\sigma V^{-1}h) \text{ for all } h \in h\}, \tag{2.5}$$

$$(ii) \quad \Delta^+_- = \{\alpha | \alpha \in \Delta^+, \alpha(h) = \alpha(V\sigma V^{-1}h) \text{ for all } h \in h\} \tag{2.6}$$

Let the subalgebra $\tilde{N}$ be spanned by the elements $V^{-1}e_\alpha, \alpha \in \Delta^+_+$ and $N = \tilde{N} \cap \mathsf{g}$, where $\tilde{N}$ and N are the nilpotent subalgebras of g_R and g respectively. Thus the Iwasawa decomposition of g_R is given by

$$\mathsf{g}_R = K \oplus A \oplus N, \tag{2.7}$$

where $\oplus$ indicates the direct sum of vector spaces rather than a sum of mutually commuting Lie subalgebras.

Now, a minimal parabolic subalgebra of g_R is defined to be any subalgebra that is conjugate to

$$P_I = M \oplus A \oplus N. \tag{2.8}$$

A general parabolic subalgebra of g_R may be defined to be any subalgebra of g_R that contains a minimal parabolic subalgebra of g_R. There exist 2^m conjugacy classes of parabolic subalgebras of g_R and in each such class there is a standard parabolic subalgebra P_θ, which can be obtained in the following way:

Let Σ be the set of roots λ for A and Ψ be the set of positive roots in Σ where $\psi = \{\lambda_1, \lambda_2, \ldots\}$. Let θ denote the subset of Ψ and $< \theta >$ the set of roots in Σ which arises as a linear combination of roots in θ. Define $< \theta >_\pm = \Sigma_\pm \cap < \theta >$, where Σ_+ and Σ_- denote the positive and negative roots in Σ. Let $N_+(\theta), N_-(\theta)$ and $N(\theta)$ denote the subspace of A corresponding to $< \theta >_+, < \theta >_-$ and $\Sigma_\pm - < \theta >_\pm$ respectively.

Now define

$$A_\theta = \{h \in A | \lambda(h) = 0 \text{ for all } \lambda \in \theta\}. \tag{2.9}$$

Now for each $\lambda \in \theta$ construct Q_λ such that $Q_\lambda \in VA$ and is a linear combination of all the elements $h_\alpha \in \mathsf{h}$ for which the restriction of α to $V\mathsf{A}$ is λ.

Again, let $\mathsf{A}(\theta)$ be the orthogonal component of A_θ in A with respect to the Cartan Killing form, then the Langlands decomposition of parabolic subalgebra of g_R is

$$\mathsf{P}_\theta = \mathsf{M}_\theta \oplus \mathsf{A}_\theta \oplus \mathsf{N}_\theta \tag{2.10}$$

$$\text{where} \quad \mathsf{M}_\theta = \mathsf{M} \oplus \mathsf{N}_+(\theta) \oplus \mathsf{N}_-(\theta) \oplus \mathsf{A}(\theta). \tag{2.11}$$

A real Cartan subalgebra h'_r is said to be σ invariant if

$$\mathsf{h}'_r = (\mathsf{h}'_r \cap \mathsf{K}) \oplus (\mathsf{h}'_r \cap \mathsf{P}). \tag{2.12}$$

A parabolic subalgebra P_θ is said to be cuspidal, if there exists an σ-invariant real Cartan subalgebra h'_r such that

$$A_\theta = \mathsf{h}'_r \cap P. \tag{2.13}$$

This shows that the minimal parabolic algebra is cuspidal.

II(B). Satake Diagrams of Affine Kac-Moody Algebras

Each Kac-Moody algebra g determines g_R (real form of g) where g is the complexification of g_R i.e. $\mathsf{g} = \mathsf{g}_R \oplus i\mathsf{g}_R$ (direct sum). Such a real form g_R determines a mapping $C : \mathsf{g} \to \mathsf{g}_r$. The mapping C has the following properties:

(i) $[CX, CY] = C[X, Y]$ for $X, Y \in \mathsf{g}$.

(ii) C is an involution i.e. $C^2 = I_g$.

(iii) C is semilinear, i.e. $C(\pi X + \mu Y) = \bar{\pi} C(X) + \bar{\mu} C(Y)$ for $X, Y \in \mathsf{g}$ and $\mu, \pi \in C$.

(2.14)

A bijection $C : \mathsf{g} \to \mathsf{g}$ with the above properties is called conjugation of g. Conversely any conjugation of g determines uniquely a real algebra, $\mathsf{g}_R = \{X \in \mathsf{g} : CX = X\}$ such that $\mathsf{g} = \mathsf{g}_R + i\mathsf{g}_R$. Hence we have a canonical one to one correspondence between conjugation of g and real forms of g. Let C be the conjugation of g defined by g_R, so that $C(X + iY) = (X - iY)$ for $X, Y \in \mathsf{g}_R$. C acts on the root system as follows:

For each root $\alpha \in R$, we define $\sigma(\alpha)$ such that

$$\sigma(\alpha(h)) = \overline{\alpha(C(h))},\ h \in \mathsf{h}. \tag{2.15}$$

Then we have,

$$C(\mathsf{g}_\alpha) = \mathsf{g}_{-\sigma(\alpha)}. \tag{2.16}$$

The mapping $\alpha \to -\sigma(\alpha)$ extends by linearity to an involutary isometry under which $R - R_0$ is stable and R_0 is the set of roots $\alpha \in R$, such that $\sigma(\alpha) = \alpha$. We have $\alpha + \sigma(\alpha) \notin R$ for all

$\alpha \in R$. Therefore, we are led to consider pairs (R, σ), where R is a restricted root system and σ is an involutary isometry such that $\sigma(R) = R$. Each Kac-Moody algebra $\mathfrak{g}$, therefore determines a normal pair (R, σ), which determines $\mathfrak{g}_R$.

The construction of Satake diagrams associated with real Kac-Moody algebras from the Dynkin diagram of the corresponding complex Kac-Moody algebras proceeds as follows:

Let R be the root system of affine Kac-Moody algebra. For $\alpha \in R$, let $\bar{\alpha} = \alpha - \sigma(\alpha)$, where σ is the involutive automorphism of R. Let us introduce $R_- = \{\bar{\alpha}|\bar{\alpha} \neq 0, \alpha \in R\}$. Also let $R_0 = \{\alpha \in R|\bar{\alpha} = 0\}$. Further let B_- (resp. B) denote the basis of R_- (resp. R) and B_0 be a basis of R_0, then $B_0 = B \cap R_0$ is a basis of R_0. Let $B_- = B/B_0 = \{\alpha_i\}$ and $B_0 = \{\beta_i, \}$ then

$$-\sigma(\alpha_i) = \alpha_{\pi(i)} + \sum \eta_{il}\ \beta_l \tag{2.17}$$

where π is the involutive permutation of $\{0, 1, 2 \ldots r\}$ and η_{il} are non-negative integers.

We can now associate with B its Satake diagrams as follows:

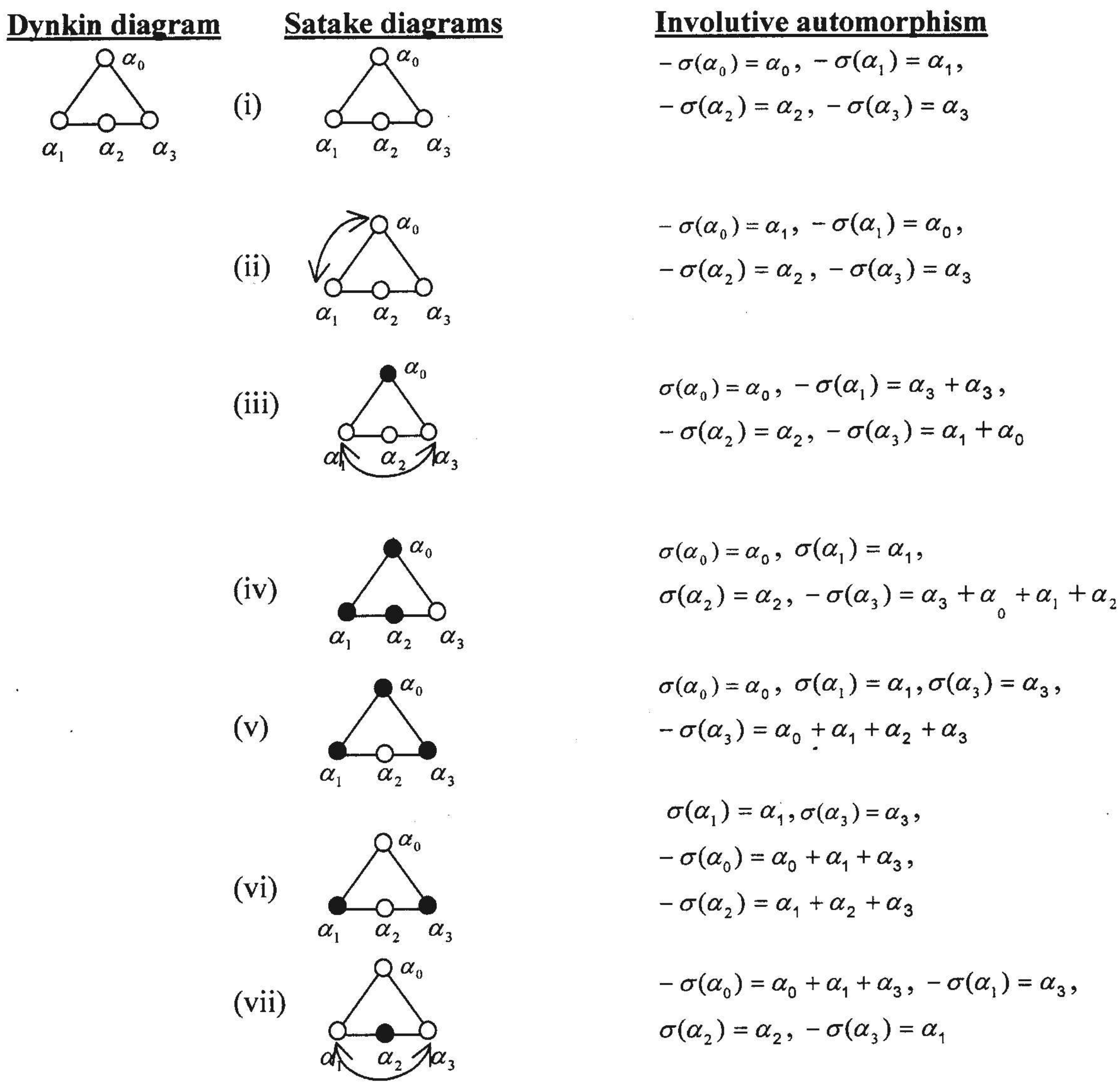
In the Dynkin diagrams of B, denote the roots α_i by white dot ◯ as usual and the roots β_l by black dot ◯. If $\pi(i) = k$, indicate this by ◯ ◯. The Satake diagrams determine the involution σ of R uniquely. We note that $\sigma(\beta_l) = \beta_l$ and if $\alpha \in R$ then $\alpha + \sigma(\alpha) \notin R$.

In finite dimensional cases, the Satake diagrams of simple Lie algebras determines the real forms of these algebras uniquely up to isomorphism and also determine the associated symmetric space. In a similar manner we can also construct Satake diagrams from the Dynkin diagrams of all affine Kac-Moody algebras, which we hope, will provide one way of classification of real forms of these algebras and will also determine the associated symmetric space (if it exists for an infinite setting). Recently symmetric spaces have got wide application in quantum transport problems, quantum integrable systems and random matrix models etc. Thus we hope these new types of symmetric spaces may play important role in such type of studies in future. The same method can also be applied to Lie superalgebra case. In an earlier paper Satake diagrams (super) has been successfully used to determine the real forms of simple Lie superalgebras [16], which have one to one correspondence with real forms determined by Parker [15]. The same method can be used in the case of Kac-Moody superalgebra.

III(A). Langlands Decomposition of $A_3^{(1)}$

The Cartan matrix of is $A_3^{(1)}$ $\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$. The four simple roots of $A_3^{(1)}$ are $\alpha_0, \alpha_1, \alpha_2$ and α_3. The possible Satake diagrams of $A_3^{(1)}$ along with their root automorphisms are depicted in Table I.

Table I: Satake diagram and Involutive automorphism of $A_3^{(1)}$:



Dynkin diagram		Satake diagrams	Involutive automorphism
α_0; α_1 α_2 α_3	(i)		$-\sigma(\alpha_0)=\alpha_0,\ -\sigma(\alpha_1)=\alpha_1,$ $-\sigma(\alpha_2)=\alpha_2,\ -\sigma(\alpha_3)=\alpha_3$
	(ii)		$-\sigma(\alpha_0)=\alpha_1,\ -\sigma(\alpha_1)=\alpha_0,$ $-\sigma(\alpha_2)=\alpha_2,\ -\sigma(\alpha_3)=\alpha_3$
	(iii)		$\sigma(\alpha_0)=\alpha_0,\ -\sigma(\alpha_1)=\alpha_3+\alpha_3,$ $-\sigma(\alpha_2)=\alpha_2,\ -\sigma(\alpha_3)=\alpha_1+\alpha_0$
	(iv)		$\sigma(\alpha_0)=\alpha_0,\ \sigma(\alpha_1)=\alpha_1,$ $\sigma(\alpha_2)=\alpha_2,\ -\sigma(\alpha_3)=\alpha_3+\alpha_0+\alpha_1+\alpha_2$
	(v)		$\sigma(\alpha_0)=\alpha_0,\ \sigma(\alpha_1)=\alpha_1,\sigma(\alpha_3)=\alpha_3,$ $-\sigma(\alpha_3)=\alpha_0+\alpha_1+\alpha_2+\alpha_3$
	(vi)		$\sigma(\alpha_1)=\alpha_1,\sigma(\alpha_3)=\alpha_3,$ $-\sigma(\alpha_0)=\alpha_0+\alpha_1+\alpha_3,$ $-\sigma(\alpha_2)=\alpha_1+\alpha_2+\alpha_3$
	(vii)		$-\sigma(\alpha_0)=\alpha_0+\alpha_1+\alpha_3,\ -\sigma(\alpha_1)=\alpha_3,$ $\sigma(\alpha_2)=\alpha_2,\ -\sigma(\alpha_3)=\alpha_1$

Let us consider the involutive automorphism of $A_3^{(1)}$ determined by any one of the Satake diagrams, say (vi) of table I.

The simple root automorphisms are given by

$$\begin{aligned}\sigma(\alpha_1) &= \alpha_1, -\sigma(\alpha_2) = \alpha_1+\alpha_2+\alpha_3,\\ \sigma(\alpha_3) &= \alpha_3, -\sigma(\alpha_0) = \alpha_0+\alpha_1+\alpha_3.\end{aligned} \tag{3.1}$$

In terms of roots α and purely imaginary root δ, the basic root automorphisms can be written as

$$\sigma(\alpha_1) \;=\; \alpha_1, -\sigma(\alpha_2) = \alpha_1+\alpha_2+\alpha_3,$$

$$\sigma(\alpha_3) = \alpha_3, -\sigma(\delta-\alpha_1-\alpha_2-\alpha_3) = \delta-\alpha_2 \tag{3.2}$$

where

$$\alpha_0 = \delta - (\alpha_1+\alpha_2+\alpha_3).$$

So, the positive roots are given by

$$\Delta = \left\{ \begin{array}{c} \alpha_1, \alpha_2, \alpha_3, \pm\alpha_1+n\delta, \pm\alpha_2+n\delta, \pm\alpha_3+n\delta, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \\ \pm(\alpha_1+\alpha_2)+n\delta, \pm(\alpha_2+\alpha_3)+n\delta, \pm(\alpha_1+\alpha_2+\alpha_3)+n\delta, n\delta, n\in Z^+ \end{array} \right\}. \tag{3.3}$$

We can apply simple root automorphisms to find out the automorphism of other roots and we see that the positive roots can be separated into two categories i.e.

$$\begin{aligned} \exp\alpha(h) &= +1 \text{ for} \\ \alpha &= \left\{ \begin{array}{l} \alpha_1, \alpha_3, \alpha_1+n\delta, \pm\alpha_2+n\delta, -\alpha_3+n\delta, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \\ \pm(\alpha_1+\alpha_2)+n\delta, \end{array} \right\} \quad (3.4) \\ \exp\alpha(h) &= -1 \text{ for} \\ \alpha &= \left\{ \begin{array}{l} \alpha_2, -\alpha_1+n\delta, \alpha_3, +n\delta, \alpha_2+\alpha_3, \pm(\alpha_2+\alpha_3)+n\delta, \\ \pm(\alpha_1+\alpha_2+\alpha_3)+n\delta, n\delta \end{array} \right\}. \quad (3.5) \end{aligned}$$

Thus for $A_3^{(1)}$, K and P are given by

$$K = \{ih_\alpha, \text{for } \alpha = \alpha_0, \alpha_1, \alpha_2, \alpha_3 \text{ and } (e_\alpha + e_{-\alpha}), i(e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by eqn } (3.4)\} \tag{3.6}$$

$$P = \{i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for } \alpha \text{ given by eqn } (3.5)\}. \tag{3.7}$$

We now select a maximal abelian subalgebra, 'A' which is two-dimensional and may be chosen to have basis elements

$$h'_0 = i\{e_{\alpha_2} + e_{-\alpha_2}\}, h'_1 = i\{e_{\alpha_1+\alpha_2+\alpha_3+n\delta} + e_{-(\alpha_1+\alpha_2+\alpha_3)+n\delta}\}. \tag{3.8}$$

So, we have $R_A = \{\alpha_2, \alpha_1+\alpha_2+\alpha_3+n\delta\}$, R_M is empty. 'M' is two-dimensional and its basis elements are given by

$$-ih'_2 = h_{(m+n)\delta}, = ih'_3 = i(h_{\alpha_1} - h_{\alpha_3}) \tag{3.9}$$

Note that h'_0, h'_1, h'_2, h'_3 together with a scaling element d' are the elements of Cartan subalgebra h'. The inner automorphism of $A_3^{(1)}$ is given by

$$V = \prod_\alpha V_\alpha \text{ for all } \alpha \in R_A \cup R_m, \tag{3.10}$$

$$\begin{aligned} V &= V_{\alpha_2} V_{\alpha_1+\alpha_2+\alpha_3+m\delta} \\ &= \exp\{ad[i\, a_{\alpha_2}(e_{\alpha_2} - e_{-\alpha_2})]\} \exp\{ad[ia_{\alpha_1+\alpha_2+\alpha_3+m\delta} \\ &\quad (e_(\alpha_1+\alpha_2+\alpha_3) + m\delta - e_{-(\alpha_1+\alpha_2+\alpha_3)+m\delta})]\} \end{aligned} \tag{3.11}$$

$$\text{where } a_{\alpha_2} = \frac{\pi}{[t^n\, 8(\alpha_2, \alpha_2)]^{(1/2)}}, \tag{3.12}$$

$$a_{\alpha_1+\alpha_2+\alpha_3+m\delta} = \frac{\pi}{[t^n 8(\alpha_1+\alpha_2+\alpha_3+m\delta, \alpha_1+\alpha_2+\alpha_3+m\delta)^{(1/2)}} \quad . \tag{3.13}$$

Here t^n represents the complex variable.

Applying this to Cartan subalgebra of h', we get

$$\begin{aligned} \mathsf{h}_0 &= -h_{\alpha_2} \\ \mathsf{h}_1 &= -(h_{\alpha_1} + h_{\alpha_2} + h_{\alpha_3} + h_{(m+n)\delta}) \\ \mathsf{h}_2 &= -h_{(m+n)}\delta \\ \mathsf{h}_3 &= -(h_{\alpha_1} - h_{\alpha_3}). \end{aligned} \tag{3.14}$$

With respect to this Cartan subalgebra, the set of positive roots is given by

$$\Delta^+ = \left\{ \begin{array}{l} \alpha_1, -\alpha_2, \alpha_3, \alpha_1 + m\delta, -\alpha_2 \pm m\delta, \alpha_3 \pm m\delta, -(\alpha_1+\alpha_2), -(\alpha_2+\alpha_3), -(\alpha_1+\alpha_2+\alpha_3), \\ -(\alpha_1+\alpha_2) + m\delta, -(\alpha_2+\alpha_3) + m\delta, -(\alpha_1+\alpha_2+\alpha_3) + m\delta, m\delta, m \in Z^+ \end{array} \right\} \tag{3.15}$$

Now this can be divided into two categories Δ^+_+ and Δ^+_-, where

$$\Delta^+_- = \{-m\delta\}, \tag{3.16}$$

and

$$\Delta^+_+ = \Delta^+/\Delta^+_-. \tag{3.17}$$

Now choose the fundamental root system,

$$\Psi = \{\lambda_1, \lambda_2\} \tag{3.18}$$

where

$$\lambda_1 = \delta - (\alpha_1 + \alpha_2 + \alpha_3), \alpha_1 \tag{3.19}$$

$$\lambda_2 = \alpha_2, \alpha_3.$$

So that using equation (3.15), we write

$$\Sigma^+ = \{\lambda_1, \lambda_2, (-1 \pm 2m)\lambda_1 + (-1 \pm 2m)\lambda_2, (-2 \pm 2m)\lambda_2 \pm 2m\lambda_1, (-1 \pm 2m)\lambda_1 + (-2 \pm 2m)\lambda_2\}. \tag{3.20}$$

It follows that λ_1 is the restriction of α_0, α_1 to VA, where as λ_2 is the restriction of α_2, α_3.

Now since $dimA = m = 2$, so there are four standard parabolic sub-algebras, namely the minimal parabolic sub-algebra, g itself and two others which will now be determined:

Case (I) — Choose

$$\theta = \{\lambda_2\}. \tag{3.21}$$

Then VA_θ has generator λ_α with

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + n\delta. \tag{3.22}$$

So that A_θ has generator $i(e_\alpha + e_{-\alpha})$, where

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + n\delta. \tag{3.23}$$

As $Q_{\lambda_2} = h_{\lambda_2} = h_{\alpha_2}$, so $A(\theta)$ has generator $i(e_{\alpha_2} + e_{-\alpha_2})$. (3.24)

Now, as $\langle\theta\rangle = \{\lambda_2\}$. So $\langle\theta\rangle_- = -\Sigma_+ \cap \langle\theta\rangle = \{\lambda_2\}$ and $\tilde{N}_-(\theta)$ is generated by the element

$$V^{-1}e_{\alpha_2} = \frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}). \tag{3.25}$$

and as $\langle\theta\rangle_+ = \Sigma_+ \cap \langle\theta\rangle = \{-\lambda_2\}$, So $\tilde{N}_+(\theta)$ is generated by the element

$$V^{-1}e_{\alpha_2} = -\frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}). \tag{3.26}$$

and $\tilde{N}(\theta)$ is generated by elements

$$\begin{aligned} V^{-1}e_{\alpha_1 \pm (m+n)\delta} &= \mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) A\, e_{-(\alpha_2+\alpha_3)\pm(n\pm(m+n)\delta)} + \frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) e_{\alpha_1\pm(m+n)\delta} \\ &\pm\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) B\, e_{-\alpha_3\pm m\delta} - \frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) C\, e_{\alpha_1+\alpha_2\pm(m+n)\delta}, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} A &= Sgn(N_{\alpha_1+\alpha_2+\alpha_3,-\alpha_2-\alpha_3}), \\ B &= Sgn(N_{\alpha_1,\alpha_2}N_{\alpha_1+\alpha_2+\alpha_3,-\alpha_3}), \\ \text{and} \qquad C &= Sgn(N_{\alpha_1,\alpha_2}). \end{aligned} \tag{3.28}$$

$$\begin{aligned} V^{-1}e_{-\alpha_3 \pm (m+n)\delta} &= \mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) D\, e_{-\alpha_1-\alpha_2\pm(n\pm(m+n))\delta} + \frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) e_{\alpha_3\pm(m+n)\delta} \\ &\pm\left(\frac{1}{1+t^{2n}}\right) E e_{-\alpha_1\pm(n\pm(m+n))\delta} - \frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right) F\, e_{\alpha_2+\alpha_3\pm(m+n)\delta}. \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} D &= \mathrm{S}gn(N_{\alpha_1+\alpha_2+\alpha_3,-\alpha_1-\alpha_2}), \\ E &= \mathrm{S}gn(N_{\alpha_1+\alpha_2+\alpha_3,\alpha_2+\alpha_3}N_{\alpha_3,\alpha_2}), \\ \text{and} \qquad F &= \mathrm{S}gn(N_{\alpha_3,\alpha_2}). \end{aligned} \tag{3.30}$$

$$\begin{aligned} V^{-1}e_{-(\alpha_1+\alpha_2)\pm(m+n)\delta} &= \frac{1}{2^{1/2}}\left(\frac{1}{2+t^{2n}}\right)e_{-(\alpha_1+\alpha_2)\pm((m+n))\delta} \\ &\mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)G\, e_{\alpha_3\pm(n\pm(m+n))\delta} \\ &\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)He_{-\alpha_1\pm(m+n)\delta} \\ &\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)Ie_{\alpha_2+\alpha_3\pm(n\pm(m+n))\delta}. \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} G &= Sgn(N_{\alpha_1+\alpha_2+\alpha_3,-(\alpha_1+\alpha_2)}), \\ H &= Sgn(N_{\alpha_2,-(\alpha_1+\alpha_2)}), \\ \text{and} \quad I &= Sgn(N_{\alpha_2,-(\alpha_1+\alpha_2)}N_{\alpha_1+\alpha_2+\alpha_3,-\alpha_1}). \end{aligned} \tag{3.32}$$

$$\begin{aligned} V^{-1}e_{-(\alpha_2+\alpha_3)\pm(m+n)\delta} &= \frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)e_{-(\alpha_2+\alpha_3)\pm(m+n)\delta} \\ &\mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)Je_{\alpha_1\pm(n\pm(m+n))\delta} \\ &\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)Ke_{-\alpha_3\pm(m+n)\delta} \\ &\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)Le_{(\alpha_2+\alpha_1)\pm(n\pm(m+n))\delta}. \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} J &= Sgn(N_{\alpha_1+\alpha_2+\alpha_3,-(\alpha_2+\alpha_3)}), \\ K &= Sgn(N_{\alpha_2,-(\alpha_2+\alpha_3)}), \\ \text{and} \quad L &= Sgn(N_{\alpha_2,-(\alpha_2+\alpha_3)}N_{\alpha_1+\alpha_2+\alpha_3,-\alpha_3}). \end{aligned} \tag{3.34}$$

$$\begin{aligned} V^{-1}e_{-(\alpha_1+\alpha_2+\alpha_3)\pm(m+n)\delta} &= -\frac{i}{2}(h_{(\alpha_1+\alpha_2+\alpha_3)\pm m\delta)}) \\ &-\frac{1}{2}(e_{(\alpha_1+\alpha_2+\alpha_3)\pm(m+n)\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)\pm(m+n)\delta}). \end{aligned} \tag{3.35}$$

Consequently the basis elements of $N(\theta)$ are given by

$$\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)A\ (e_{-(\alpha_2+\alpha_3)\pm(n\pm(m+n)\delta} - e_{(\alpha_2+\alpha_3)\pm(n\pm(m+n)\delta})$$

$$\pm\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)(e_{\alpha_1\pm(m+n)\delta} - e_{\alpha_1\pm(m+n)\delta})$$

$$\pm\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)B(e_{-\alpha_3\pm m\delta} - e_{\alpha_3\pm m\delta})$$

$$-\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)C(e_{\alpha_1+\alpha_2\pm(m+n)\delta} - e_{-\alpha_1-\alpha_2\pm(m+n)\delta}),$$

$$\mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)D\ (e_{-\alpha_1-\alpha_2\pm(n\pm(m+n))\delta} - e_{\alpha_1+\alpha_2\pm(n\pm(m+n))\delta})$$

$$+\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)(e_{\alpha_3\pm(m+n)\delta} - e_{-\alpha_3\pm(m+n)\delta})$$

$$\pm\left(\frac{1}{1+t^{2n}}\right)E\ (e_{-\alpha_1\pm(n\pm(m+n))\delta} - e_{\alpha_1\pm(n\pm(m+n))\delta})$$

$$-\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)F(e_{\alpha_2+\alpha_3\pm(m+n)\delta} - e_{-\alpha_2-\alpha_3\pm(m+n)\delta}),$$

$$\frac{1}{2^{1/2}}\left(\frac{1}{2+t^{2n}}\right)(e_{-(\alpha_1+\alpha_2)\pm(m+n)\delta} - e_{(\alpha_1+\alpha_2)\pm(m+n)\delta})$$

$$\mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)G(e_{\alpha_3\pm(n\pm(m+n))\delta} - e_{-\alpha_3\pm(n\pm(m+n)\delta)})$$

$$\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)H(e_{-\alpha_1\pm(m+n)\delta} - e_{\alpha_1\pm(m+n)\delta})$$

$$\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)I(e_{\alpha_2+\alpha_3\pm(n\pm(m+n))\delta} - e_{-\alpha_2-\alpha_3\pm(n\pm(m+n))\delta}),$$

$$\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)(e_{-(\alpha_2+\alpha_3)\pm(m+n)\delta} - e_{(\alpha_2+\alpha_3)\pm(m+n)\delta})$$

$$\mp\frac{1}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)J(e_{\alpha_1\pm(n\pm(m+n))\delta} - e_{-\alpha_1\pm(n\pm(m+n))\delta})$$

$$\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)K(e_{-\alpha_3\pm(m+n)\delta} - e_{\alpha_3\pm(m+n)\delta})$$

$$\mp\frac{i}{2^{1/2}}\left(\frac{1}{1+t^{2n}}\right)L(e_{(\alpha_2+\alpha_1)\pm(n\pm(m+n))\delta} - e_{-(\alpha_2+\alpha_1)\pm(n\pm(m+n))\delta}),$$

$$-\frac{i}{2}(h_{(\alpha_1+\alpha_2+\alpha_3)\pm m\delta}) - \frac{1}{2}(e_{(\alpha_1+\alpha_2+\alpha_3)\pm(m+n)\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)\pm(m+n)\delta}). \quad (3.36)$$

and the basis elements of M_θ may be taken to be

$$\frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}),$$

$$-\frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}), i(e_{\alpha_2} + e_{-\alpha_2}), i(h_{\alpha_1} - h_{\alpha_3}). \quad (3.37)$$

Clearly $ih_{\alpha_2}, i(e_{(\alpha_1+\alpha_2+\alpha_3)+n\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)+n\delta}), i(h_{\alpha_1} - h_{\alpha_3})$ are the generators of a real Z-invariant Cartan subalgebra $\mathfrak{h}'_r$ for which $\mathfrak{h}'_r \cap P = A_\theta$. So the parabolic subalgebra is cuspidal and $P_\theta = M_\theta \oplus A_\theta \oplus N_\theta$.

Case (II) — Here, choose

$$\theta = \{\lambda_1\} \quad (3.38)$$

In this case, VA_θ has generator

$$(h_{\alpha_2} + h_{\alpha_1+\alpha_2+\alpha_3+n\delta}). \quad (3.39)$$

So that A_θ has generator

$$i(e_{\alpha_2} + e_{-\alpha_2}) + i(e_{\alpha_1+\alpha_2+\alpha_3+n\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)+n\delta}). \quad (3.40)$$

As Q_{λ_1} is a linear combination of α_2 and $\alpha_1 + \alpha_2 + \alpha_3 + n\delta$, so $A(\theta)$ has generator

$$\lfloor i(e_{\alpha_1+\alpha_2+\alpha_3+n\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)+n\delta}) - i(e_{\alpha_2} + e_{-\alpha_2})\rfloor. \quad (3.41)$$

Moreover as, $\langle\theta\rangle_- = \{\lambda_1\}$, so elements of $\tilde{N}_-(\theta)$ is generated by

$$V^{-1}e_{\alpha_1} = \frac{1}{2}e_{\alpha_1} - \frac{i}{2}A_1 e_{-\alpha_2-\alpha_3-n\delta} - \frac{i}{2}B_1 e_{\alpha_1+\alpha_2} - \frac{1}{2}C_1 e_{-\alpha_3-n\delta} \quad (3.42)$$

where

$$\begin{aligned} A_1 &= Sgn(N_{\alpha_1+\alpha_2+\alpha_3,-(\alpha_2+\alpha_3)}), \\ B_1 &= Sgn(N_{\alpha_1,\alpha_2}), \\ \text{and} \quad C_1 &= Sgn(N_{\alpha_1,\alpha_2} N_{-\alpha_3,(\alpha_1+\alpha_2+\alpha_3)}). \end{aligned} \quad (3.43)$$

and as $\langle\theta\rangle_+ = \{-\lambda_1\}$, so elements of $\tilde{N}_+(\theta)$ is generated by

$$V^{-1}e_{-\alpha_1} = \frac{1}{2}e_{-\alpha_1} - \frac{i}{2}A_2 e_{\alpha_2+\alpha_3+n\delta} - \frac{i}{2}B_2 e_{-\alpha_1-\alpha_2} - \frac{1}{2}C_2 e_{\alpha_3+n\delta} \quad (3.44)$$

where

$$\begin{aligned} A_2 &= Sgn(N_{\alpha_1+\alpha_2+\alpha_3,-\alpha_1}), \\ B_2 &= Sgn(N_{-\alpha_1,-\alpha_1-\alpha_2}), \\ \text{and} \quad C_2 &= Sgn(N_{-\alpha_1,-\alpha_2} N_{\alpha_1+\alpha_2+\alpha_3,\,-\alpha_1-\alpha_2}). \end{aligned} \quad (3.45)$$

and elements of $\tilde{N}(\theta)$ are given by

$$V^{-1}e_{-\alpha_2\pm(m+n)\delta} = -\frac{1}{2}(e_{-\alpha_2\pm(m+n)\delta}$$

$$-e_{\alpha_2\pm(m+n)\delta}) - i(e_{\alpha_2\pm(m+n)\delta} - e_{-\alpha_2\pm(m+n)\delta}) \text{ and } V^{-1}e_{-\alpha_3\pm(m+n)\delta},$$

$$V^{-1}e_{-(\alpha_1+\alpha_2)\pm(m+n)\delta}, V^{-1}e_{-(\alpha_2+\alpha_3)\pm(m+n)\delta} \text{ and } V^{-1}e_{-(\alpha_1+\alpha_2+\alpha_3)\pm(m+n)\delta},$$

all of which are mentioned in Case I. (3.46)

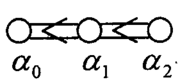
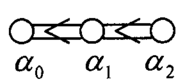
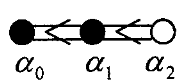
The elements of $N(\theta)$ can be known by considering the elements $\tilde{N}\cap\mathfrak{g}$ and basis elements of M_θ may be chosen in a similar way as mentioned in case-I.

As there does not exist two mutually commuting linearly independent generators in $M_\theta \cap K$, so this parabolic subalgebra is not cuspidal and $P_\theta = M_\theta \oplus A_\theta \oplus N_\theta$. (3.47)

III(B). Langlands Decomposition of $A_4^{(2)}$

The Cartan matrix of $A_4^{(2)}$ is $\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$. The three simple roots are α_0, α_1 and α_2. The possible Satake diagrams of $A_4^{(2)}$ along with their root automorphism are given in table II.

Table II: Satake diagram and Involutive automorphism of $A_4^{(2)}$.

Dynkin diagram		Satake diagrams	Involutive automorphism
	(i)		$-\sigma(\alpha_0)=\alpha_0$, $-\sigma(\alpha_1)=\alpha_1$, $-\sigma(\alpha_2)=\alpha_2$
	(ii)		$\sigma(\alpha_0)=\alpha_0$, $\sigma(\alpha_1)=\alpha_1$, $-\sigma(\alpha_2)=\alpha_2+4\alpha_0+4\alpha_1$.

Let us consider the involutive automorphism of $A_4^{(2)}$ determined by any one of the Satake diagrams, say (ii) of table II. The simple root automorphisms are given by

$$\begin{aligned} -\sigma(\alpha_2) &= \alpha_2 + 4\alpha_1 + 4\alpha_0 \\ \sigma(\alpha_1) &= \alpha_1, \\ \sigma(\alpha_0) &= \alpha_0. \end{aligned} \tag{4.1}$$

In terms of roots α and purely imaginary root δ the basic root automorphisms can be written as

$$-\sigma(\alpha_2) = \alpha_2 + 4\alpha_1 + 4\alpha_0, \sigma(\alpha_1) = \alpha_1,$$

$$\sigma\left(\frac{1}{2}(\delta - 2\alpha_1 - \alpha_2)\right) = \frac{1}{2}(\delta - 2\alpha_1 - \alpha_2),$$

where

$$\alpha_0 = \frac{1}{2}(\delta - 2\alpha_1 - \alpha_2). \quad (4.2)$$

So, the positive roots of $A_4^{(2)}$ are given by

$$\Delta = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \pm\alpha_1 + n\delta, \pm\alpha_2 + 2n\delta, \pm(2\alpha_1 + \alpha_2) + 2n\delta,$$
$$\pm(\alpha_1 + \alpha_2) + n\delta, \frac{1}{2}(\pm\alpha_2 + (2n-1)\delta),$$
$$\frac{1}{2}(\pm(2\alpha_1 + \alpha_2) + (2n-1)\delta), n\delta, n \in Z\}. \quad (4.3)$$

We can apply the simple root automorphism to find out automorphism of other roots and we see that the positive roots can be separated into two categories, i.e.

$$\exp\ \alpha(h) = +1 \text{ for } \alpha = \{\alpha_1, -\alpha_1 + n\delta, -\alpha_2 + 2n\delta, -(2\alpha_1 + \alpha_2) + 2n\delta,$$
$$-(\alpha_1 + \alpha_2) + n\delta, \frac{1}{2}(-\alpha_2 + (2n-1)\delta), \frac{1}{2}(-(2\alpha_1 + \alpha_2) + (2n-1)\delta)\}, \quad (4.4)$$
$$\exp\ \alpha(h) = -1 \text{ for } \alpha = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + n\delta, \alpha_2 + 2n\delta, (2\alpha_1 + \alpha_2) + 2n\delta,$$
$$(\alpha_1 + \alpha_2) + n\delta, \frac{1}{2}(\alpha_2 + (2n-1)\delta), \frac{1}{2}((2\alpha_1 + \alpha_2) + (2n-1)\delta), n\delta\}. \quad (4.5)$$

Thus for $A_4^{(2)}, \mathsf{K}$ and P are given by

$$\mathsf{K} = \{ih_\alpha \text{ for } \alpha = \alpha_0, \alpha_1, \alpha_2, (e_\alpha + e_{-\alpha}), i(e_{\alpha - e_{-\alpha}}), \text{ for } \alpha \text{ given by eq}(4.4)\}, \quad (4.6)$$

$$\text{and } \mathsf{P} = \{i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}), \text{ for } \alpha \text{ given by eq}(4.5)\}. \quad (4.7)$$

We now select a maximal abelian subalgebra, A which is two-dimensional and may be chosen to have basis elements,

$$\mathsf{h}'_0 = i(e_{\alpha_2} + e_{-\alpha_2}) \quad (4.8)$$
$$\mathsf{h}'_1 = i[e_{(2\alpha_1+\alpha_2)} + e_{-(2\alpha_1+\alpha_2)}]. \quad (4.9)$$

So, we have $R_{\mathsf{A}} = \{\alpha_2, 2\alpha_1 + \alpha_2\}$ and R_{M} is empty and M is one-dimensional and its basis element is given by

$$-i\mathsf{h}'_2 = n\delta. \quad (4.10)$$

Note that $\mathsf{h}'_0, \mathsf{h}'_1, \mathsf{h}'_2$ together with the scaling element d' forms the Cartan subalgebra h'.

The required inner automorphism of $A_4^{(2)}$ is

$$V = V_{\alpha_2} \cdot V_{2\alpha_1+\alpha_2} \tag{4.11}$$

$$\text{where } V_\alpha = \exp[adj\{ia_{\alpha_2}(e_{\alpha_2} - e_{-\alpha_2})\}] \exp[adj\{ia_{2\alpha_1+\alpha_2}(e_{2\alpha_1+\alpha_2} - e_{-2\alpha_1-\alpha_2})\}] \tag{4.12}$$

$$\text{and} \quad a_{\alpha_2} = \frac{\pi}{\{8(\alpha_2,\alpha_2)\}^{(1/2)}}, a_{2\alpha_1+\alpha_2} = \frac{\pi}{\{8(2\alpha_1+\alpha_2, 2\alpha_1+\alpha_2)\}^{(1/2)}}. \tag{4.13}$$

Applying this to the Cartan subalgebra h' of $A_4^{(2)}$, we obtain

$$\begin{aligned} \mathsf{h}_0 &= -h_{\alpha_2}, \\ \mathsf{h}_1 &= -(2h_{\alpha_1} + h_{\alpha_2}), \\ \text{and} \quad \mathsf{h}_2 &= -h_{n\delta}. \end{aligned} \tag{4.14}$$

With respect to this Cartan- subalgebra, the set of positive roots is given by

$$\Delta^+ = \left\{ \begin{array}{l} \alpha_1, -\alpha_2, -(\alpha_1+\alpha_2), -(2\alpha_1+\alpha_2), \alpha_1 \pm n\delta, -\alpha_2 \pm 2n\delta, \\ -(2\alpha_1+\alpha_2) \pm 2n\delta, -(\alpha_1+\alpha_2) \pm n\delta, \\ \frac{1}{2}(-\alpha_2 \pm (2n-1)\delta), \frac{1}{2}(-(2\alpha_1+\alpha_2) \pm (2n-1)\delta) \end{array} \right\}. \tag{4.15}$$

The sets Δ_+^+ and Δ_-^+ can be written as

$$\Delta_+^+ = \Delta^+/\Delta_-^+ \tag{4.16}$$

$$\text{and} \quad \Delta_-^+ = \{-n\delta\}. \tag{4.17}$$

Now Choose the fundamental root system

$$\Psi = \{\lambda_1, \lambda_2\}, \tag{4.18}$$

where

$$\lambda_1 = \frac{1}{2}(\delta - (2\alpha_1+\alpha_2)), \alpha_1 \tag{4.19}$$

$$\text{and} \quad \lambda_2 = \alpha_2. \tag{4.20}$$

It follows that λ_1 is the restriction of $\frac{1}{2}(\delta - (2\alpha_1+\alpha_2)), \alpha_1$ and λ_2 is the restriction of α_2. So that using these in eqn. (4.15), we can write

$$\Sigma_+ = \left\{ \begin{array}{l} (1 \pm 4n)\lambda_1 \pm n\lambda_2, (-1 \pm 2n)\lambda_2 \pm 8n\lambda_1, (-2 \pm 8n)\lambda_1 \\ \quad +(-1 \pm 2n)\lambda_2, (-1 \pm 4n)\lambda_1 + (-1 \pm n)\lambda_2, \\ \frac{1}{2}[(-1 \pm (2n-1))\lambda_2 \pm 4(2n-1)\lambda_1], \frac{1}{2}[(-2 \pm 4((2n-1))\lambda_1 + (-1 \pm (2n-1))\lambda_2] \end{array} \right\} \tag{4.21}$$

Since dim $\mathsf{A} = m = 2$, there are four conjugacy classes of parabolic subalgebras, namely the minimal parabolic subalgebra, g itself and two others which will now be constructed.

Case (I) — Now, Choose the non-empty proper subset θ of Ψ_1 as

$$\theta = \{\lambda_2\}. \tag{4.22}$$

Then VA_θ has generator h_α with

$$\alpha = 2\alpha_1 + \alpha_2. \tag{4.23}$$

So A_θ has generator $(e_\alpha + e_{-\alpha})$ with

$$\alpha = 2\alpha_1 + \alpha_2. \tag{4.24}$$

Now as $\theta_{\lambda_2} = h_{\lambda_2} = h_{\alpha_2}$, so $A(\theta)$ has generator

$$i(e_{\alpha_2} + e_{-\alpha_2}). \tag{4.25}$$

Moreover, as $\langle\theta\rangle_- = \{\lambda_2\}$ $\tilde{N}_-(\theta)$ is generated by

$$V^{-1}e_{\alpha_2} = \frac{1}{2}(e_{\alpha_2} + e_{-\alpha_2}) - \frac{i}{2}h_{\alpha_2}. \tag{4.26}$$

Then, as $\langle\theta\rangle_+ = \{-\lambda_2\}$. So $\tilde{N}_+(\theta)$ is generated by

$$V^{-1}e_{-\alpha_2} = -\frac{1}{2}(e_{\alpha_2} + e_{-\alpha_2}) - \frac{i}{2}h_{\alpha_2}. \tag{4.27}$$

and $\tilde{N}_\theta$ is generated by

$$V^{-1}e_{\alpha_1} = -\frac{i}{2}A_1\ e_{-\alpha_1-\alpha_2} + \frac{1}{2}e_{\alpha_1} - \frac{i}{2}B_1\ e_{-\alpha_1} - \frac{i}{2}C_1\ e_{\alpha_1+\alpha_2}, \tag{4.28}$$

where

$$\begin{aligned} A_1 &= Sgn(N_{2\alpha_1+\alpha_2,-\alpha_1-\alpha_2}),\\ B_1 &= Sgn(N_{\alpha_1,\alpha_2}\ N_{2\alpha_1+\alpha_2,-\alpha_1}),\\ \text{and}\qquad C_1 &= Sgn(N_{\alpha_1,\alpha_2}) \end{aligned} \tag{4.29}$$

$$V^{-1}e_{-(\alpha_1+\alpha_2)} = \frac{1}{2}e_{-\alpha_1-\alpha_2} - \frac{i}{2}\ A_2\ e_\alpha - \frac{i}{2}B_2\ e_{-\alpha_2} - \frac{1}{2}\ C_2\ e_{\alpha_1+\alpha_2}, \tag{4.30}$$

where

$$\begin{aligned} A_2 &= Sgn(N_{2\alpha_1+\alpha_2,-\alpha_1-\alpha_2}),\\ B_2 &= Sgn(N_{\alpha_2,-\alpha_1-\alpha_2}),\\ \text{and}\qquad C_2 &= Sgn(N_{\alpha_2,-\alpha_1-\alpha_2}\ N_{2\alpha_1+\alpha_2,-\alpha_1}). \end{aligned} \tag{4.31}$$

$$V^{-1}e_{-2\alpha_1-\alpha_2} = e_{-2\alpha_1-\alpha_2}. \tag{4.32}$$

$$\begin{aligned} V^{-1}e_{\alpha_1\pm n\delta} &= \frac{1}{2}e_{\alpha_1\pm n\delta} - \frac{i}{2}A_3\; e_{-\alpha_1-\alpha_2\pm n\delta} \\ &\quad -\frac{i}{2}B_3\; e_{\alpha_1+\alpha_2\pm n\delta} - \frac{1}{2}C_3\; e_{-\alpha_1\pm n\delta}, \end{aligned} \tag{4.33}$$

where

$$\begin{aligned} A_3 &= Sgn(N_{-\alpha_1-\alpha_2\pm n\delta,\, 2\alpha_1+\alpha_2}), \\ B_3 &= Sgn(N_{\alpha_2,\, \alpha_2\pm n\delta}), \\ \text{and} \qquad C_3 &= Sgn(N_{\alpha_2,\, \alpha_1\pm n\delta}\; N_{2\alpha_1+\alpha_2,-\alpha_1\pm n\delta}). \end{aligned} \tag{4.34}$$

$$V^{-1}e_{-\alpha_2\pm 2n\delta} = \frac{1}{2^{1/2}}e_{-\alpha_2\pm 2n\delta} - \frac{i}{2^{1/2}}A_4\; e_{\pm 2n\delta,} \tag{4.35}$$

where

$$A_4 = Sgn(N_{\alpha_2,-\alpha_2\pm 2n\delta}). \tag{4.36}$$

$$V^{-1}e_{-(2\alpha_1+\alpha_2)\pm 2n\delta} = \frac{1}{2^{1/2}}e_{-(2\alpha_1+\alpha_2)\pm 2n\delta} - \frac{i}{2^{1/2}}A_5\; e_{\pm 2n\delta}, \tag{4.37}$$

where

$$A_5 = Sgn(N_{2\alpha_1+\alpha_2,-(2\alpha_1+\alpha_2)\pm 2n\delta}). \tag{4.38}$$

$$\begin{aligned} V^{-1}e_{-(\alpha_1+\alpha_2)\pm n\delta} &= \frac{1}{2}e_{-\alpha_1-\alpha_2\pm n\delta} - \frac{i}{2}A_6\; e_{\alpha_1\pm n\delta} \\ &\quad -\frac{i}{2}B_6\; e_{-\alpha_1\pm n\delta} - \frac{1}{2}C_6 e_{\alpha_1+\alpha_2\pm n\delta}, \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} A_6 &= Sgn(N_{2\alpha_1+\alpha_2,-\alpha_1-\alpha_2\pm n\delta}), \\ B_6 &= Sgn(N_{\alpha_2,-\alpha_1-\alpha_2\pm n\delta}), \\ \text{and} \qquad C_6 &= Sgn(N_{\alpha_2,-\alpha_1-\alpha_2\pm n\delta}\; N_{2\alpha_1+\alpha_2,-\alpha_1-\alpha_2\pm n\delta}). \end{aligned} \tag{4.40}$$

$$\begin{aligned} V^{-1}e_{\frac{1}{2}\{-(\alpha_2)\pm(2n-1)\delta\}} &= \frac{1}{2^{1/2}}e_{\frac{1}{2}\{-(\alpha_2)\pm(2n-1)\delta\}} \\ &\quad -\frac{i}{2^{1/2}}A_7\; e_{\frac{1}{2}(\alpha_2)\pm\frac{1}{2}(2n-1)\delta}, \end{aligned} \tag{4.41}$$

where

$$A_7 = Sgn\left(N_{\alpha_2,\frac{1}{2}(-\alpha_2)\pm\frac{1}{2}(2n-1)\delta}\right). \tag{4.42}$$

$$V^{-1}e_{\frac{1}{2}\{-(2\alpha_1+\alpha_2)\pm(2n-1)\delta\}} = \frac{1}{2^{1/2}}e_{\frac{1}{2}\{-(2\alpha_1+\alpha_2)\pm(2n-1)\delta\}} - \frac{i}{2^{1/2}}A_8\ e_{-\frac{1}{2}(2\alpha_1+\alpha_2)\pm\frac{1}{2}(2n-1)\delta}, \qquad (4.43)$$

where

$$A_8 = Sgn\left(N_{2\alpha_1+\alpha_2,-\frac{1}{2}(2\alpha_1+\alpha_2)\pm\frac{1}{2}(2n-1)\delta}\right). \qquad (4.44)$$

It follows that the basis elements of N_θ may be taken to be

$$-\frac{i}{2}A_1(e_{-\alpha_1-\alpha_2} - e_{\alpha_1+\alpha_2}) + \frac{1}{2}(e_{\alpha_1} - e_{-\alpha_1})$$

$$-\frac{1}{2}B_1(e_{-\alpha_1} - e_{\alpha_1}) - \frac{i}{2}C_1(e_{\alpha_1+\alpha_2} - e_{-\alpha_1-\alpha_2}),$$

$$\frac{1}{2}(e_{-\alpha_1-\alpha_2} - e_{\alpha_1+\alpha_2}) - \frac{i}{2}A_2(e_{\alpha_1} - e_{-\alpha_1}) - \frac{i}{2}B_2(e_{-\alpha_1} - e_{\alpha_1})$$

$$-\frac{i}{2}C_2(e_{\alpha_1+\alpha_2} - e_{-\alpha_1-\alpha_2}), (e_{-2\alpha_1-\alpha_2} - e_{2\alpha_1+\alpha_2}),$$

$$\frac{1}{2}(e_{\alpha_1\pm n\delta} - e_{-\alpha_1\pm n\delta}) - \frac{i}{2}A_3(e_{-\alpha_1-\alpha_2\pm n\delta} - e_{\alpha_1+\alpha_2\pm n\delta})$$

$$-\frac{i}{2}B_3(e_{\alpha_1+\alpha_2\pm n\delta} - e_{-\alpha_1-\alpha_2\pm n\delta})$$

$$\frac{1}{2}C_3(e_{-\alpha_2\pm n\delta} - e_{\alpha_2\pm n\delta}), \frac{1}{2^{1/2}}(e_{-\alpha_1\pm n\delta} - e_{\alpha_2\pm n\delta})$$

$$-\frac{i}{2^{1/2}}A_4(e_{\pm 2n\delta} - e_{\mp 2n\delta}),$$

$$\frac{1}{2^{1/2}}(e_{-(2\alpha_1+\alpha_2)\pm 2n\delta} - e_{(2\alpha_1+\alpha_2)\pm 2n\delta})$$

$$-\frac{i}{2^{1/2}}A_5(e_{\pm 2n\delta} - e_{\mp 2n\delta}), \frac{1}{2}(e_{-\alpha_1-\alpha_2\pm n\delta} - e_{\alpha_1+\alpha_2\pm n\delta}) -$$

$$\frac{i}{2}A_6(e_{\alpha_1\pm n\delta} - e_{-\alpha_1\pm n\delta}) - \frac{i}{2}B_6(e_{-\alpha_1\pm n\delta} - e_{\alpha_1\pm n\delta})$$

$$-\frac{1}{2}C_6(e_{\alpha_1+\alpha_2\pm n\delta} - e_{-\alpha_1-\alpha_2\pm n\delta}),$$

$$\frac{1}{2^{1/2}}\left(e_{\frac{1}{2}(-\alpha_2\pm(2n-1)\delta)} - e_{-\frac{1}{2}(-\alpha_2\pm(2n-1)\delta)}\right)$$

$$-\frac{i}{2^{1/2}}A_7\left(e_{\frac{1}{2}(\alpha_2\pm(2n-1)\delta)}-e_{-\frac{1}{2}(\alpha_2\pm(2n-1)\delta)}\right),$$

$$\frac{1}{2^{1/2}}\left(e_{\frac{1}{2}(-(2\alpha_1+\alpha_2)\pm(2n-1)\delta)}-e_{-\frac{1}{2}(-(2\alpha_1+\alpha_2)\pm(2n-1)\delta)}\right)$$

$$-\frac{i}{2^{1/2}}A_8\left(e_{-\frac{1}{2}\left((2\alpha_1+\alpha_2)\pm\frac{1}{2}(2n-1)\delta\right)}-e_{\frac{1}{2}\left((2\alpha_1+\alpha_2)\pm\frac{1}{2}(2n-1)\delta\right)}\right), \tag{4.45}$$

and the basis element of M_θ may be taken to be

$$n\delta,\frac{1}{2}(e_{\alpha_2}-e_{-\alpha_2})-\frac{i}{2}h_{\alpha_2},-\frac{1}{2}(e_{\alpha_2}-e_{-\alpha_2})-\frac{i}{2}h_{-\alpha_2},i(e_{\alpha_2}+e_{-\alpha_2}). \tag{4.46}$$

Since $\mathsf{h}'_r\cap P=A_\theta$, so this parabolic subalgebra P_θ is cuspidal and

$$P_\theta=M_\theta\oplus A_\theta\oplus N_\theta \tag{4.47}$$

Case (II) — Here choose the non-empty proper subset θ of Ψ_1 as

$$\theta=\{\lambda_1\}. \tag{4.48}$$

Then, VA_θ has generator

$$(h_{\alpha_2}+h_{2\alpha_1+\alpha_2}). \tag{4.49}$$

So that A_θ has generator

$$i(e_{\alpha_2}+e_{-\alpha_2})+i(e_{2\alpha_1+\alpha_2}+e_{-(2\alpha_1+\alpha_2)}) \tag{4.50}$$

As $Q_{\lambda_1}=h_{\alpha_1}=\frac{1}{2}h_{2\alpha_1+\alpha_2}-\frac{1}{2}h_{\alpha_2}$, so that the generator of $A(\theta)$ may be taken to be,

$$A(\theta)=\{i(e_{(2\alpha_1+\alpha_2)}+e_{-(2\alpha_1+\alpha_2)})-i(e_{\alpha_2}-e_{-\alpha_2})\}. \tag{4.51}$$

Moreover, as $\langle\theta\rangle_-=\{-\lambda_1\}$, So $\tilde{N}_-(\theta)$ is generated by

$$V^{-1}e_{-\alpha_1}=\frac{1}{2}e_{-\alpha_1}-\frac{1}{2}A_9\ e_{\alpha_1+\alpha_2}-\frac{i}{2}B_9\ e_{-\alpha_1-\alpha_2}-\frac{1}{2}C_9e_{\alpha_1}, \tag{4.52}$$

where

$$\begin{aligned} A_9 &= Sgn(N_{-\alpha_1,2\alpha_1+\alpha_2}),\\ B_9 &= Sgn(N_{\alpha_2,-\alpha_1-\alpha_2}),\\ \text{and}\qquad C_9 &= Sgn(N_{\alpha_2,-\alpha_1-\alpha_2}\ N_{2\alpha_1+\alpha_2,-\alpha_1-\alpha_2}). \end{aligned} \tag{4.53}$$

and as $\langle\theta\rangle_+=\{\lambda_1\}$, so $\tilde{N}_+(\theta)$ is generated by $V^{-1}e_{\alpha_1}$ which is as mentioned in eq. (4.28) and $\tilde{N}_\theta$ is generated by elements, $V^{-1}e_{-\alpha_2},V^{-1}e_{-(\alpha_1+\alpha_2)},V^{-1}e_{-(2\alpha_1+\alpha_2)},V^{-1}e_{\alpha_1\pm n\delta},V^{-1}e_{-\alpha_2\pm2n\delta}$,

$$V^{-1}e_{-(\alpha_1+\alpha_2)\pm n\delta},V^{-1}e_{\frac{1}{2}(-\alpha_2\pm(2n-1)\delta)},V^{-1}e_{\frac{1}{2}[-(2\alpha_1+\alpha_2)\pm(2n-1)\delta]}. \tag{4.54}$$

All of which are already mentioned in eq. (4.27), eq. (4.29–4.44) as in case (a).

Preceding in the same way as in Case-I the elements of N_θ and M_θ may be generated.

Clearly $n\delta, i(e_{2\alpha_1+\alpha_2} - e_{-(2\alpha_1+\alpha_2)}) + i(e_{\alpha_2} - e_{-\alpha_2}), (e_{-\alpha_1} - e_{\alpha_1}) - iA_9(e_{\alpha_1+\alpha_2} - e_{-(\alpha_1+\alpha_2})$

$-iB_9(e_{-(\alpha_1+\alpha_2)} - e_{\alpha_1+\alpha_2}) - C_9(e_{\alpha_1} - e_{-\alpha_1}), (e_{\alpha_1} - e_{-\alpha_1}) - iA_1(e_{-(\alpha_1+\alpha_2)} - e_{(\alpha_1+\alpha_2)})$

$iB_1(e_{-\alpha_1} - e_{\alpha_1}) - C_1(e_{\alpha_1+\alpha_2} - e_{-\alpha_1-\alpha_2})$ are generators of a real Z-invariant Cartan-subalgebra h'_r for which $\mathsf{h}'_r \cap P = A_\theta$. So this parabolic subalgebra is also cuspidal and

$$P_\theta = M_\theta \oplus A_\theta \oplus N_\theta. \tag{4.55}$$

IV. Conclusion

We have presented the Langlands decomposition of lower rank affine Kac-Moody algebras $A_3^{(1)}$ (untwisted) and $A_4^{(2)}$ (twisted) in detail. Use of Satake diagrams facilitates Iwasawa decomposition, which in turn leads to Langland decomposition. The minimal parabolic subalgebra being cuspidal leads to an induced representation. The Satake diagrams can be studied in a different angle altogether. They can be used to classify the real forms of affine Kac-Moody algebras and automatically give the Dynkin diagram of reduced root system, which are nothing but Dynkin diagrams of the associated symmetric space (if it can be defined for a infinite dimensional algebra). Such studies are currently in progress.

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