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SUPER KAC-MOODY ALGEBRA : CLASSIFICATION THEORY (I)

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The explicit construction and classification of super Kac-Moody algebras associated with a Z_2-graded Lie algebra is presented. The canonical realisation of such algebras and the central extension problem is also briefly discussed.

1. INTRODUCTION

In recent years the study of infinite-dimensional Lie algebras popularly known as Kac-Moody algebras[1] and loop algebras[2] has got wide acceptance in the context of 2-dimensional conformal field theory[3], string theory[4] and soluble statistical models[5]. The existence of such Lie algebras is also felt while analysing the symmetries associated with non-linear systems[6,7] (equations). Even in linear equations, the symmetry possesses a loop algebra structure[8,9]. From all these angles, the study of classification theory and representations of infinite-dimensional Lie algebras is highly promising. We do have some systematic analysis of generating Kac-Moody and loop algebras from finite dimensional Lie algebras. Kac[1] has thrown a lot of light on the construction of Verma modules (the highest weight representation) for such Lie algebras. The reducibility properties and complete classification of all infinite dimensional Lie algebras are yet to be completely analysed. Here, following the work of Goddard-Olive[10] and Kac[1], we present a systematic construction of infinite dimensional Lie algebras (super) starting from a Z_2-graded Lie algebra. The classification of Z_2-graded Lie algebras (from cohomological analysis) have been dealt with in detail elsewhere[11-14]. We thus start from a Z_2-graded Lie algebra and from the knowledge of their automorphic structures construct (super) Kac-Moody Lie algebras. We make no pretention that in the present analysis, we have only addressed to the construction and classification of (super) Kac-Moody algebras. The representations of such Lie algebras will be presented in future communication. Our material is arranged as follows.

In section 2, we present a brief resume of Z_2-graded Lie algebras and the Dynkin diagrams. We have introduced the notion of extended Dynkin diagrams for such super Lie algebras from the structure of the extended cartan matrices. These diagrams correspond to the identity automorphism of Z_2-graded Lie algebras and correspondingly yield untwisted infinite dimensional Kac-Moody Lie algebras. From the outer automorphisms of Z_2-graded Lie algebras, we obtain exhaustive the twisted infinite dimensional Kac-Moody algebras.

In Section 3, we introduce the Chevally basis and from the vanishing properties of the Cartan matrix generate the (super) Kac-Moody algebras.

In section 4, we intıoduce the canonical basis for realizing the (super) Kac-Moody algebras and analyse the central extension properties. As expected, if the Z_2-graded Lie algebıa is strongly semi-simple[14], then the central extensions of Kac-Moody algebras are trivial. We also discuss the role of 2-cocycles in the central extension problem. The computation of 2-cocycles have been discussed elsewhere using spectral analysis[14]. The study of central extension problem is intimately connected with the deformation of Z_2-graded Lie algebras and the super quantization problem.

In an appendix, we discuss the degeneracy properties of the roots introducing the Cartan-Weyl basis for the super Kac-Moody algebras.

2. Resume of Super Lie Algebras

A super Lie algebra is a Z_2-graded Lie algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ (i.e if $a \in L_\alpha$; $b \in L_\beta$; $|\alpha|, |\beta| = \{0, 1\}$ then $[a, b] \in L_{\alpha+\beta}$) with a bracket [,] satisfying the following properties

$$[a, b] = -(-1)^{|\alpha||\beta|}[b, a] \qquad \ldots(2.1)$$

$$[a, [b, c]] = [[a, b], c] + (-1)^{|\alpha||\beta|}[b, [a, c]]. \qquad \ldots(2.2)$$

Here we briefly review some important theorems and lemmas (without proof) regarding the classification of simple super Lie algebra.

Definition 1—A super Lie algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is called simple if it satisfies the following conditions.

(i) The representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is faithful and $[L_{\bar{1}}, L_{\bar{1}}] = L_{\bar{0}}$.

(ii) The representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is irreducible.

Definition 2—A finite dimensional super Lie algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is called classical if it is simple and the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is completely reducible.

Now if L is a simple super Lie algebra,, then an invariant form on it is either non-degenerate or identically zero and any two invariant forms on L are proportional. Throughout this article we discuss about simple super Lie algebras with non-degenerate Killing forms. A simple super Lie algebra with non-degenerate Killing form is classical[15].

Theorem 1—A simple finite dimensional super Lie algebra with non-degenerate Killing form is isomorphic to one of the simple Lie algebra or to one of the following classical super Lie algebras $A(m, n)$, $m \neq n$; $B(m, n)$, $C(n)$, $D(m, n)$, $m - n \neq 1$, $F(4)$, $G(3)$: We have

$$A(m, n) = SL(m + 1, n + 1), m \neq n$$

$$B(m, n) = OSP(2m + 1, 2n)$$

$$C(n) = OSP(2, 2n - 2)$$

$$D(m, n) = OSP(2m, 2n).$$

$F(4)$, $G(3)$ are exceptional super Lie algebras of 40 and 31 dimension respectively.

Root Space Decomposition of Super Lie Algebras

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Z_2-graded Lie algebra and h be Cartan subalgebra. h is also called Cartan subalgebra of L. A Cartan sub algebra of a classical super Lie algebra is diagonalizable. If h^* be the dual space of h we can now write L as

$$L = \bigoplus_{\lambda \in h^*} L^{\lambda}(h) \qquad \ldots(2.3)$$

and

$$[L^{\lambda}(h), L^{\mu}(h)] \subseteq L^{\lambda+\mu}(h) \ \forall \ \lambda, \mu \in h^*. \qquad \ldots(2.4)$$

Let us define

$$\Delta_{\bar{0}} = \{\lambda \in h^* \mid \lambda \neq 0, L^{\lambda}_{\bar{0}}(h) \neq \{0\}\}$$

$$\Delta_{\bar{1}} = \{\lambda \in h^* \mid L^{\lambda}_{\bar{1}}(h) \neq \{0\}\}$$

$$\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}.$$

The elements of Δ are called the roots of L w.r.t. h, $\Delta_{\bar{0}}$ is called the system of even roots and $\Delta_{\bar{1}}$, is called the system of odd roots which are nothing but system of weights of the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$. Since h is a Cartan subalgebra of $L_{\bar{0}}$, we have $L^{0}_{\bar{0}}(h) = h$.

Thus we can write

$$L = h \bigoplus_{\lambda \in \Delta_{\bar{0}}} (L^{\lambda}_{\bar{0}} \bigoplus_{\lambda \in \Delta_{\bar{1}}} L^{\lambda}_{\bar{1}}(h) \qquad \ldots(2.5)$$

Like ordinary Lie algebra, here also we can define a system of simple roots $\pi = \{\alpha_1 \ldots \alpha_r\} \subseteq \Delta$, if every other root of the super Lie algebra can be obtained as a result of linear combination of these system of simple roots. r is called the rank of the super Lie algebra. Now we can define an $r \times r$ square matrix called Cartan matrix with elements $a_{ij} = \alpha_j(h_i)$ which can be read of from the Dynkin diagram. The rules are same as ordinary Lie algebras with a little modification. A super Lie algebra of rank r can be represented by a Dynkin diagram consisting of r white, grey or black circles. The white circles imply even roots whereas the black and grey circles denote odd roots. The roots are expressed in terms of linear functions of $e_1, e_2 \ldots e_m$ and $\delta_1, \delta_2 \ldots \delta_n$ which forms a unit basis of h^*, with inner product $(e_i, e_j) = \delta_{ij}$ $1 \leqq i, j \leqq m$; $(\delta_k, \delta_l) = -\delta_{kl}$ $1 \leqq k, l \leqq n$; $(e_i, \delta_k) = 0$. The white and black circles correspond to 2 in the diagonal of the Cartan matrix while grey circles correspond to zero in the diagonal. If the ith and jth aircles are not joined then $a_{ij} = a_{ji} = 0$.

(a) *Root System*

We now list all the simplest root systems of super Lie algebras.

(i) $A(m, n)$: The roots are expressed in terms of $e_1 \ldots e_{m+1}$, $\delta_1 = e_{m+2} \ldots$, $\delta_{n+1} = e_{m+n+2}$

$$\Delta_{\bar{0}} = \{e_i - e_j,\ \delta_i - \delta_j\}\ i \neq j;\ \Delta_{\bar{1}} = \{\pm\, e_i - \delta_j\}.$$

The simplest root system is

$$\{e_1 - e_2,\ e_2 - e_3 \ldots e_{m+1} - \delta_1,\ \delta_1 - \delta_2 \ldots \delta_n - \delta_{n+1}\}$$

(ii) $B(m, n)$: The roots are expressed in terms of $e_1 \ldots e_m$, $\delta_1 = e_{2m+1} \ldots$, $\delta_n = e_{2m+n}$

$$\Delta_{\bar{0}} = \{\pm e_i \pm e_j,\ \pm 2\delta_i,\ \pm e_i,\ \pm\delta_i \pm \delta_j\},\ i \neq j$$

$$\Delta_{\bar{1}} = \{\pm\delta_i,\ \pm e_i \pm \delta_j\}$$

The simplest root system is

$$\{\delta_1 - \delta_2, \ldots \delta_n - e_1,\ e_1 - e_2 \ldots e_{m-1} - e_m,\ e_m\} \quad \text{for } m > 0.$$

$$\{\delta_1 - \delta_2,\ \delta_{n-1} - \delta_n,\ \delta_n\} \quad \text{for } m = 0.$$

(iii) $C(n)$: The roots are expressed in terms of e_1, $\delta_1 = e_3, \ldots, \delta_{n-1} = e_{n+1}$

$$\Delta_{\bar{0}} = \{\pm 2\delta_i,\ \pm\delta_i \pm \delta_j\},\ \Delta_{\bar{1}} = \{\pm e_1 \pm \delta_i\}$$

One of the simplest root system is

$$\pm\{e_i - \delta_1,\ \delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1},\ 2\delta_{n-1}\}$$

(iv) $D(m, n)$: The roots are expressed in terms of linear function of $e_1 \ldots, e_m$, $\delta_1 = e_{2m+1}, \ldots, \delta_n = e_{2m+n}$.

$$\Delta_{\bar{0}} = \{\pm\, e_i \pm e_i,\ \pm 2\delta_i,\ \pm\delta_i \pm \delta_j\}$$

$$\Delta_{\bar{1}} = \{\pm e_i \pm \delta_j\}$$

One of the simplest root system is

$$\{\delta_1 - \delta_2, \ldots \delta_n - e_1,\ e_1 - e_2 \ldots e_{m-1} - e_m,\ e_{m-1} + e_m\}$$

(v) $F(4)$: The roots are expressed in terms of e_1, e_2, e_3 corresponding to B_3 and δ corresponding to A_1.

$$\Delta_{\bar{0}} = \{\pm e_i \pm e_j,\ \pm e_i \pm \delta\}\ i \neq j$$

$$\Delta_{\bar{1}} = \{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm \delta)\}.$$

One of the simplest such system is

$$\{\tfrac{1}{2}(e_1 + e_2 + e_3 + \delta),\ -e_1,\ e_1 - e_2,\ e_2 - e_3\}.$$

(vi) $G(3)$: The roots are expressed in terms of linear functions of e_1, e_2, e_3 corresponding to G_2 and $e_1 + e_2 + e_3 = 0$ and δ corresponding to A_1.

$$\Delta_{\bar{0}} = \{e_i - e_j,\ e_i,\ \pm 2\delta\},\ \Delta_{\bar{1}} = \{\pm e_i \pm \delta,\ \pm \delta\}.$$

The unique system of simple roots is

$$\{\delta + e_1,\ e_2,\ e_3 - e_2\}.$$

(b) *Extended Root System*

We know that the $r \times r$ square Cartan matrices constructed from the system of simple roots of a super Lie algebra of rank r are indecomposable and the determinants do not vanish. We now extend the simple root system by adding one more extra root which is the negative of the highest root. This new $(r+1) \times (r+1)$

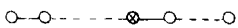
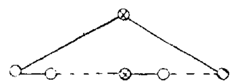
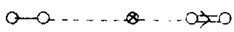
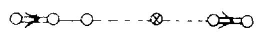
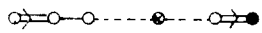
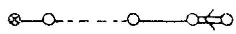
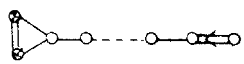
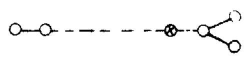
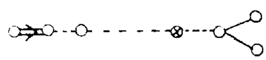
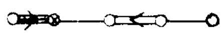
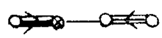
Super Lie algebras	Dynkin diagrams	Super Kac-Moody algebras	Extended Dynkin diagrams
$A(m,n)$		$A^{(1)}(m,n)$	
$B(m,n)$		$B^{(1)}(m,n)$	
$B(0,n)$		$B^{(1)}(0,n)$	
$C(n)$		$C^{(1)}(n)$	
$D(m,n)$		$D^{(1)}(m,n)$	
$F(4)$		$F^{(1)}(4)$	
$G(3)$		$G^{(1)}(3)$	

Table I

square matrix is called the exteneded Cartan matrix and the corresponding Dynkin diagram is called extended Dynkin diagram. It is easily seen that the determinant of the extended cartan matrix is null. These extended root systems and extended Dynkin diagrams yield the super Kac-Moody algebras (infinite dimensional). In Table I, we have shown the Dynkin diagrams for the super Kac-Moody algebras while in Table II we have examined some special case of super Kac-Moody algebras and displayed the corresponding cartan matrices.

However, this list does not exhaust all possible Kac-Moody algebras. The above extended Dynkin diagrams correspond to identity automorphism of the super Lie algebras. Such algebras are called untwisted Kac-Moody algebras.

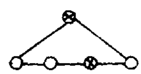
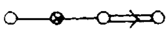
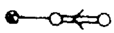
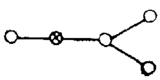
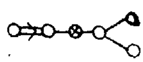
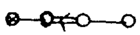
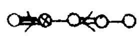
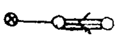
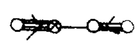
Super Lie algebra	Dynkin diagram	Cartan matrix	Extended Cartan matrix	Extended Dynkin diagram
A(2,1)		$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$	
B(2,2)		$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}$	
C (3)		$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	
D(4,2)		$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$	
F (4)		$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	
G (3)		$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	

Table II

(c) *Outer Automorphisms of Super Lie Algebras and the Twisted Kac-Moody Algebras*

The outer automophism of a super Lie algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is defined as Out L = Aut L/α_0, where α_0 is the connected component of the identity of a Lie group with Lie algebra L_0. α_0 is embedded in the automorphism group of the super Lie algebra.

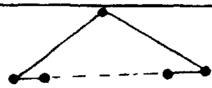
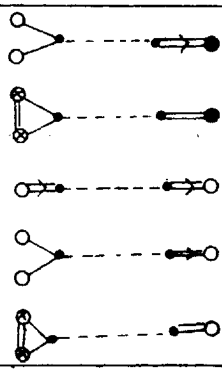
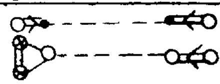
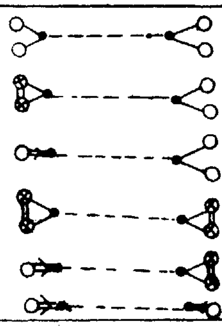
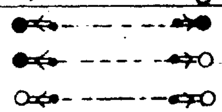
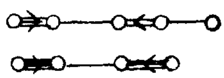
Super Lie algebras and their automorpnisms (the superscript indicates the order of automorphisms)	Extended Dynkin diagrams
$SL^{(1)}(m, n)$	
$OSP^{(1)}(2m+1, 2n)$ $SL^{(2)}(2m+1\ \ 2n)$	
$OSP^{(1)}(2, 2n)$	
$OSP^{(1)}(2m, 2n)$ $SL^{(2)}(2m, 2n)$	
$OSP^{(2)}(2m, 2n)$ $SL^{(2)}(2m+1\ \ 2n+1)$	
$F^{(1)}(4)$ $C^{(1)}(3)$	

Table III

Empirically, we see that if $\sigma \in$ Aut L, N = order σ then we can divide L into eigen spaces L_j where

$$L_j = \{l \mid \sigma(l) = l e^{2\pi \sqrt{-1}\, j/N}\}. \qquad ...(2.6)$$

The outer automorphisms of super Lie algebra has been studied in detail elsewhere[16]. Here we briefly mention some of the results relevant to our present analysis. We list all the extended Dynkin diagrams (Table III) obtained through outer automorphism. In the Dynkin diagrams, each point can be a white or a grey circle. The different Dynkin diagrams for the same super Lie algebra accounts for the parity of the no. of grey circles, and the different no. of white and grey circles and all possible system of simple roots.

The diagrams which correspond to second order outer automorphism give rise to twisted super Kac-Moody algebras. Here we would like to stress that the extended Dynkin diagrams are more symmetrical than Dynkin diagrams.

3. Chevalley Basis and Super Kac-Moody Algebras

Every super Lie algebra can be written in a Chevalley basis associated to system of simple roots. If $A = a_{ij}$ be a cartan matrix of a super Lie algebra of rank r and L_{-1}, L_0 and L_1 be the vector spaces with bases $\{f_i\}$, $\{h_i\}$, $\{e_i\}$ (i = 1, 2, ..., r) respectively, then we have

$$\begin{aligned} &[e_i, f_i] = \delta_{ij} h_i \\ &[h_i, e_j] = a_{ij} e_j \\ &[h_i, f_i] = - a_{ji} f_j \\ &[h_i, h_j] = 0 \\ &\deg h_i = 0, \deg e_i = \deg f_i = 0 \text{ for even root} \\ &\qquad \deg e_i = \deg f_i = 1 \text{ for odd root.} \end{aligned} \qquad ...(3.1)$$

Conversly, given any 1×1 matrix $A = a_{ij}$ satisfying the above relation (3.1) and having properties of a Cartan matrix for super Lie algebra i. e. $a_{ij} \in \mathbb{Z}$, $a_{ii} = 0$ or 2, $a_{ij} = 0 \Rightarrow a_{ji} = 0$ defines an abstract complex super Lie algebra $G(A)$, which is essentially the super Kac-Moody algebra defined by Cartan matrix A. We can have two conditions.

(i) A is positive definite in the sense that all the principal minors of A are greater than zero. Then $G(A)$ becomes a finite dimensional super Lie algebra.

(ii) We find also that by admitting a single isotropic vector (negative of the highest root vector) in the simple root system, det $A = 0$ and all the proper principal minor of A are greater than zero. That means A is not an invertible matrix and G

(A) becomes infinite dimensional. These are affine super Kac-Moody algebras and they correspond to extended Dynkin diagrams.

4. Canonical Realisation of Super Kac-Moody Algebras

Let G be a super Lie algebra of all $n \times n$ matrices with complex entries on $\mathbb{C}^n$ and $\mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomial. Then a loop algebra $\hat{G}$ is defined as $G(\mathbb{C}[t, t^{-1}])$ i. e. as the complex super Lie algebra of $n \times n$ matrices with Laurant Polynomial as entries. Alternatively, we can say G as the super Lie algebra of maps from unit circle S^1 to super Lie algebra G with finite laurent series and Lie bracket is defined pointwise. The vector space in which G acts is $\mathbb{C}^n$, while the loop algebra $\hat{G}$ acts on $\mathbb{C}([t, t^{-1}])^n$. We know that $\mathbb{C}^n$ has a standard basis $u_1 \ldots u_n$ of $n \times 1$ column vectors in which u_k $(1 \leqq k \leqq n)$ has 1 in the kth row and 0 elsewhere while $\mathbb{C}([t, t^{-1}]^n)$ consists of $n \times 1$ column vectors with Laurent polynomial in t as entries. The vector $v_{nk+j} = t^{-k} u_j$ form a basis of $C([t, t^{-1}])^n$ index by $\mathbb{Z}$. Thus we get a identification of $\mathbb{C}([t, t^{-1}]^n$ with $\mathbb{C}^\infty$. The loop algebra is infinite dimensional. This will become more clear in the following steps.

A Z_2-graded Lie algebra or super Lie algebra involves both commutator and anticommutator. The generators of the super Lie algebra have the following relations.

$$[Q^a\, Q^b] = f_c^{ab}\; Q^c \qquad \ldots(4.1.a)$$

$$[Q^a\; V^\alpha] = F_\beta^{a\alpha}\; V^\beta \qquad \ldots(4.1b)$$

$$\{V^\beta, V^\beta\} = A_a^{\alpha\beta}\, Q^a. \qquad \ldots(4.1.c)$$

$[\,,\,]$, $\{\,,\,\}$ denote commutator and anticommutator respectively. The generators of Z_2 graded Lie algebra also satisfy the following relations.

$$[Q^a, [Q^b, V^\alpha]] + [V^\alpha, [Q^a, Q^b]] + [Q^b, [V^\alpha, Q^a]] = 0 \qquad \ldots(4.2.a)$$

$$[Q^a, \{V, V^\beta\}] + \{[V^\alpha, Q^a], V^\beta\} + \{[V^\beta, Q^a], V^\alpha\} = 0 \qquad \ldots(4.2.b)$$

$$[V^\alpha, \{V^\beta, V^\gamma\}] + [V^\gamma, \{V^\alpha, V^\beta\}] + [V^\beta, \{V^\gamma, V^\alpha\}] = 0 \qquad \ldots(4.2.c)$$

All the above relations can be written in a more compact from i. e., if X^μ denotes all set of generators Q^a's and V^α's, then we set the degree of the generator as $|Q^a| = 0$ and $|V^\alpha| = 1$. Q^a's are called even while V^α's are called odd.
Then defining the graded bracket [,], we have

$$[X^\mu, X^\nu] = X^\mu X^\nu - (-1)^{|X^\mu||X^\nu|} X^\nu X^\mu. \qquad \ldots(4.3)$$

The relations (4.1. a) — (4.1.b) can be written as

$$[X^{\mu}, X^{\nu}] = C^{\mu\nu}_{\bullet} X^{\bullet} \qquad ...(4.4)$$

where the structure constants satisfy

$$C^{\mu\nu}_{\bullet} = (-1)^{|x^{\mu}||x^{\nu}|} C^{\nu\mu}_{\bullet}$$

and the relations (4.2.a ~ 4.2.c) are written as

$$(-1)^{|x^{\mu}||x^{\nu}|} [X^{\mu}, [X^{\nu}, X^{\bullet}]] + (-1)^{|x^{\nu}||x^{\mu}|} [X^{\nu}, [X^{\bullet}, X^{\mu}]] + (-1)^{|x^{\bullet}||x^{\nu}|} [X^{\bullet} [X^{\mu}, X^{\nu}]] = 0. \qquad ...(4.5)$$

Now, if we denote the generator of the loop algebra $\hat{G}$ by X^{μ} it can be given by

$$X^{\mu}_{n} = X^{\mu} \otimes t^{n}\; n \in \mathbb{Z},\; X^{\mu} \in G. \qquad ...(4.6)$$

Hence for loop algebra, the commutation relations can be written as

$$[X^{\mu}_{m}, X^{\nu}_{n}] = [X \otimes t^{m}, X^{\nu} \otimes t^{n}]$$

$$= [X^{\mu}, X^{\nu}] \otimes t^{m+n}$$

$$= C^{\mu\nu}_{\bullet} X^{\bullet} \otimes t^{m+n}$$

$$[X^{\mu}_{m}, X^{\nu}_{n}] = C^{\mu\nu}_{\bullet} X^{\bullet}_{m+n},\; m, n \in \mathbb{Z}. \qquad ...(4.7)$$

This is the loop algebra or untwisted Kac-Moody without central extension.

4b. Central Extension

Central extension of loop algebras constructed from a super Lie algebra plays a vital role in super quantization problem.

Let $\hat{G} = C[t, t^{-1}] \otimes G$ be the loop algebra where G is a super Lie algebra and $C[t, t^{-1}]$ is a ring of Laurent polynomial. Then $\tilde{G} = \hat{G} + ¢k$ is a central extension of G by a 1-dimensional centre $¢k$.

Explicitly,

$$X^{\mu}_{n} = X^{\mu}_{0} \otimes t^{n},\; n \in \mathbb{Z}$$

and

$$[X^{\mu}_{m}, X^{\nu}_{n}] = C^{\mu\nu}_{\alpha} X^{\alpha}_{m+n}.$$

Here we introduce an operator $d_0 = -t\, d/dt$ such that

$$[X^{\mu}_{m}, d_0] = m X^{\mu}_{m}.$$

Now, if $X^{\mu}_{m}, X^{\nu}_{n} \in \hat{G}$, then

$$[X^{\mu}_{m}, X^{\nu}_{n}]_{k.M} = [X^{\mu}, X^{\nu}_{n}] + k\,\Omega(X^{\mu}_{m}, X^{\nu}_{n}) \qquad \ldots(4.8)$$

and

$$[k, X^{\mu}_{m}] = 0 \qquad \ldots(4.9)$$

where Ω is a 2 cocycle and is a bilinear functional satisfying

$$\Omega(X^{\mu}_{m}, X^{\nu}_{n}) = -(-1)^{|X^{\mu}_{m}||X^{\nu}_{n}|}\,\Omega(X^{\mu}_{n}\ X^{\nu}_{m}) \qquad \ldots(4.10)$$

and

$$\Omega(X^{\mu}_{m}, [X^{\nu}_{n}, X^{\alpha}_{1}])$$

$$= \Omega([X^{\mu}_{m}, X^{\alpha}_{1}]\ X^{\alpha}_{1}) + (-1)^{|X^{\mu}_{m}||X^{\nu}_{n}|}\,\Omega(X^{\nu}_{m}\ [X^{\mu}_{n}, X^{\alpha}_{1}]) \qquad \ldots(4.11)$$

The central extension is called trivial if the cocycle Ω is also coboundry, i. e.,

$$\Omega_{c.b}(X^{\mu}_{m}, X^{\nu}_{n}) = F([X^{\mu}_{m}, X^{\nu}_{n}]) \qquad \ldots(4.12)$$

where F is a linear functional on G.

It has been established that there is no non-trivial central extension of a semi-simple Lie algebra. But the same need not be true for Z_2-graded Lie algebra. Only for strongly semi-simple super Lie algebras the central extension is trivial. The computation of 2-cocycles, for super Lie algebras has been discussed by Tripathy and others[11-14]. Here we calculate the 2-cocycles for $\hat{G}$ where G is strongly semi simple.

Using the freedom of adding a coboundry, we can now write

$\Omega(\alpha, d_0) = 0$, where $\alpha \in \hat{G}$. Now with the help of the equation (4.10) and (4.11), we have

$$\Omega([X^{\mu}_{m}, X^{\nu}_{n}], d_0) =$$

$$\Omega(X^{\mu}_{m}, [X^{\nu}_{n} \; d_0]) - (-1)^{|X^{\mu}_{m}||X^{\nu}_{n}|} \Omega(X^{\nu}_{n} \; [X^{\mu}_{m}, d_0])$$

$$= n \; \Omega(X^{\mu}_{m}, X^{\nu}_{n}) - m(-1)^{|X^{\mu}_{m}||X^{\nu}_{n}|} (X^{\nu}_{n}, X^{\mu}_{m})$$

$$= n \, \Omega(X^{\mu}_{m}, X^{\nu}_{n}) + m \, \Omega(X^{\mu}_{m}, X^{\nu}_{n})$$

$$= (m+n) \; \Omega(X^{\mu}_{m}, X^{\nu}_{n}) = 0$$

so that

$$\Omega(X^{\mu}_{m}, X^{\nu}_{n}) \sim \delta_{m,-n}.$$

Actually it can be shown that

$$\Omega(X^{\mu}_{m}, X^{\nu}_{n} = m \, \delta_{m,-n}(X, Y) \qquad \ldots(4.14)$$

where (X, Y) is the invariant normalized product on G.

The untwisted super Kac-Moody algebro is wiitten as

$$[X^{\mu}_{m}, X^{\nu}_{n}] = C^{\mu\nu}_{\omega} \; X^{\omega}_{m+n} + k \, m \, \delta_{m,-n}(X, Y). \qquad \ldots(4.15)$$

If σ is an automornhism of order N of G, we can split G as $\Sigma \, G_k$, $k \in Z_n$ and define the shifted mode $X_{n+k/N}$ and then defining the algebra by the above equation (4.15) we get the twisted super Kac-Moody algebra.

Conclusion

To summarise our results, we have given an exhaustive classification of super Kac-Moody algebras, the construction of loop algebra from a Z_2-graded Lie algebra and its central extension. In a futuie communication, we wish to report the representations of (super) Kac-Moody algebras.

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APPENDIX I

Degeneracy of roots for Uniwisted super Kac-Moody Algebras

The root system of ruper Lie algebra is written in a Cartan-Weyl basis as the following

$$[H^i\ H^i] = 0$$

$$[H^i, E^a] = a^i\ E^a$$

$$[E^a, E^{-a}] = a_i\ H^i$$

$$[E^a, E^b] = \epsilon\ (a, b)\ E^{a+b}$$

and

$$[H^i, V^\alpha] = \alpha^i\ V^\alpha$$

$$[E^a, V^\alpha] = \delta^{a+\alpha}\ M^{\beta,\alpha}\ V^\beta$$

$$\{V^\alpha, V^{-\alpha}\} = f^{\alpha,-\alpha}\ \alpha_i\ H^i$$

$$\{V^\alpha, V^\beta\} = -\ \delta_a^{\alpha+\beta} f^{\alpha,-\alpha}\ M_{-\alpha,\beta}\ E^a \qquad \text{...(A.1)}$$

a's are the even roots; while α_i's are the weights called odd roots. To a given weight, there may correspond several V's. But for the time being we have dropped this degeneracy index.

We can rewrite the root system of the untwisted super Kac-Moody algebra in this basis as

$$[H^i_m\ ,\ H^i_n\ ] = km\,\delta_{m,-n}\,(x.\,y)$$

$$[H^i_m,\ H^i_n\ ] = a^i\,E^a_{m+n}$$

$$[E^a_m\ ,\ E^{-a}_n\ ] = a_i\ H^i_{m+n} + k\ m\ \delta_{m,-n}$$

$$[E^a_m\ ,\ E^b_n\ ] = \epsilon\,(a,\,b)\ E^{a+b}_{m+n}$$

and

$$[H^i_m\quad V^\alpha_n\ ] = \alpha^i\,V^\alpha_{m+n}$$

$$[E^a_m\ ,\ V^\alpha_n\ ] = \delta^{a+\alpha}\ M^{\beta,\alpha}\ V^\beta_{m+n}$$

$$\{V^\alpha_m\ ,\ V^{-\alpha}_n\ \} = f^{\alpha,-\alpha}\ \alpha_i\ H^i + km\ \delta_{m,-n}$$

$$\{V^\alpha_m\ ,\ V^\beta_n\ \} = -\ \ \delta^{\alpha+\beta}_a\ f^{\alpha,-\alpha}\ M_{-\alpha,\beta}\ \ E^a_{m+n}$$

$$[k,\ E^a_n\ ] = \{k,\ V^\alpha_n\ \} = \ [k,\ H^i_n\ \ ] = 0 \tag{A.2}$$

We also here see that do operator distinguishes E^a_n and V^α_n for different n i. e.

$$[d_0,\ E^a_n\ ] = n\ E^a_n\ ,\ [d_0,\ \ V^\alpha_n\ ] = n\ V^\alpha_n\ .$$

We also have

$$[H^i_0\ \ ,\ E^a_n\ ] = \ a^i\ \ E^a_n\ \ ,[H^i_0\ \ ,\ V_n] = \alpha^i\ V^\alpha_u$$

and

$$[H^i_0\ \ ,\ \ H^i_n\ \ ] = 0.$$

Now if we take Cartan subalgebra H^i_0 together with k and d_0 we have step operators

E^a_n corresponding to even root $a \equiv (a,\,0,\,n)$

V_n^α Corresponding to odd root $\alpha = (\alpha, 0, n)$

H_n^i Corresponding to root $n\,\delta = (0, 0, n)$.

We see that each even root and odd root becomes infinitely degenerate. In addition we have infinite number of light like roots $(0, 0, n)$