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# SEQUENCE BALANCING AND COBALANCING NUMBERS

G. K. Panda
Department of Mathematics
National Institute of Technology
Rourkela – 769 008
Orissa, India
Email–gkpanda_nit@rediffmail.com

**Abstract:** The concept of balancing and cobalancing numbers is generalized to an arbitrary sequence; thereby sequence balancing numbers and sequence cobalancing numbers are introduced and defined. It is proved that there does not exist any sequence balancing number in the Fibonacci sequence and the only sequence cobalancing number in the Fibonacci sequence is $F_2 = 1$. Higher order balancing and cobalancing numbers are also introduced. A result on nonexistence of third order balancing and cobalancing numbers is also proved. A conjecture on the nonexistence of solutions of higher order balancing and cobalancing numbers is also stated at the end.

Key words: Triangular number, Pronic number, Balancing number, Cobalancing number

## 1. INTRODUCTION

Behera and Panda [1] defined balancing numbers $n$ as solutions of the Diophantine equation $1+2+ \dots +(n-1) = (n+1) + (n+2) +\dots+ (n+r)$, calling $r$ the *balancer* corresponding to $n$. They also established many important results on balancing numbers. Later on, Panda [12] identified many beautiful properties of balancing numbers, some of which are equivalent to the corresponding results on Fibonacci numbers, and some others are more interesting than the corresponding results on Fibonacci numbers. Subsequently, Liptai [7] added another interesting result to the theory of balancing numbers by proving that the only balancing number in the Fibonacci sequence is 1.

Behera and Panda [1] proved that the square of any balancing number is a triangular number. It is also true that if $r$ is a balancer, then $r^2 + r$ is a triangular number. Subramaniam [14, 15] explored many interesting properties of square triangular numbers without linking them to balancing numbers because of their unavailability in the literature at that time. In [16] he introduced the concept of almost square triangular numbers (triangular numbers that differ from a square by unity) and linked them with the square triangular numbers. Panda and Ray [11] introduced cobalancing numbers as solutions of the Diophantine equation $1+2+\ldots+n=(n+1)+(n+2)+\ldots+(n+r)$ calling $r\in\mathbb{Z}^+$ the cobalancer corresponding to $n$. The cobalancing numbers are linked to a third category of triangular numbers that are expressible as the product of two consecutive natural numbers (approximately as the arithmetic mean of squares of two consecutive natural numbers i.e. $[n^2+(n+1)^2]/2\approx n(n+1)$ ).

The definitions of balancing and cobalancing numbers are closely related to the sequence of natural numbers. In what follows, we define sequence balancing and cobalancing numbers, in which the sequence of natural numbers is replaced by an arbitrary sequence of real numbers.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We call a number $a_m$ of this sequence a *sequence balancing number* if

$$a_1+a_2+\ldots+a_{m-1}=a_{m+1}+a_{m+2}+\ldots+a_{m+r}$$

for some natural number $r$. Similarly, we call $a_m$ a *sequence cobalancing number* if

$$a_1+a_2+\ldots+a_m=a_{m+1}+a_{m+2}+\ldots+a_{m+r}$$

for some natural number $r$. For example, if we take $a_n=2n$ then the sequence balancing numbers of this sequence are 12, 70, 408, ... which are twice the sequence of balancing numbers, and the sequence cobalancing numbers are 4, 28, 168, ... which are twice the sequence of cobalancing numbers. Similarly, if we take $a_n=n/2$ then the sequence balancing numbers of this sequence are 3, 17.5, 102, ... which are half the sequence of balancing numbers, and the sequence cobalancing numbers are 1, 7, 42, ... which are half the sequence of cobalancing numbers.

## 2. SEQUENCE BALANCING AND COBALANCING NUMBERS IN CERTAIN SEQUENCES

In this section we investigate sequence balancing and cobalancing numbers in some number sequences.

Throughout this section $B_n$ is the $n^{\text{th}}$ balancing number, $R_n$ is the $n^{\text{th}}$ balancer, $b_n$ is the $n^{\text{th}}$ cobalancing number and $r_n$ is the $n^{\text{th}}$ cobalancer, where $n\in\mathbb{Z}^+$.

**2.1 Sequence balancing and cobalancing numbers in the sequence of odd natural numbers.** Let $a_n=2n-1$. Then any sequence balancing number $2m-1$ of this sequence satisfies

$$1+3+\ldots+(2m-3)=(2m+1)+(2m+3)+\ldots+(2(m+s)-1)$$

for some natural number $s$. This is equivalent to

$$(m-1)^2 + m^2 = (m+s)^2,$$

which is a particular case of the Pythagorean equation. Putting $y = m+s$ we see that the above equation reduces to

$$\frac{m(m-1)}{2} = \frac{y-1}{2} \cdot \frac{y+1}{2}.$$

Since $\frac{y-1}{2}$ and $\frac{y+1}{2}$ must be consecutive integers it follows that $\frac{m(m-1)}{2}$ is a pronic triangular number. Hence, $\frac{y-1}{2}$ must be a cobalancing number ([11], 1189). Putting $b = \frac{y-1}{2}$ we see that $y = 2b+1$ and consequently

$$m = \frac{1+\sqrt{8b^2+8b+1}}{2}.$$

Since the cobalancing numbers $b$ and cobalancers $r$ are related by

$$r = \frac{-(2b+1)+\sqrt{8b^2+8b+1}}{2},$$

it follows that $2m-1 = 2r+2b+1$ is the required sequence balancing number. For example for $b=2$, $r=1$, $2m-1 = 2r+2b+1 = 7$ and we have $1+3+5=9$; similarly for $b=14$, $r=6$, $2m-1 = 2r+2b+1 = 41$ and we have $1+3+\ldots+39 = 43+45+\ldots+57$.

Thus the sequence of sequence balancing numbers in the sequence of odd natural numbers is given by $\{2b_{n+1}+2r_{n+1}+1\}_{n=1}^{\infty}$. Indeed we can express these sequence balancing numbers in terms of the balancing numbers. For this, we need the following results.

**THEOREM 2.1.1** ( [1], p.98). *For* $n$ = 1, 2, ..., $R_n = \frac{-(2B_n+1)+\sqrt{8B_n^{\,2}+1}}{2}$.

**THEOREM 2.1.2** ( [1], p.101). For $n$ = 2, 3,..., $B_{n-1} = 3B_n - \sqrt{8B_n^2+1}$.

**THEOREM 2.1.3** ([11], p.1196). *For* $n$ = 1, 2,... , $R_n = b_n$ and $r_{n+1} = B_n$.

We are now in a position to prove that the sequence of sequence balancing numbers in the sequence of odd natural numbers is also given by the more convenient form $\{B_{n+1}+B_n\}_{n=1}^{\infty}$.

**THEOREM 2.1.4.** *The sequence of sequence balancing numbers in the sequence of odd natural numbers is given by* $\{B_{n+1}+B_n\}_{n=1}^{\infty}$, i.e., $2b_{n+1}+2r_{n+1}+1 = B_{n+1}+B_n$ *for* $n$ = 1, 2,... .

**PROOF**. For $n$ = 2, 3, ... we have

$$B_n - B_{n-1} = B_n - \left[3B_n - \sqrt{8B_n^2+1}\right]$$
$$= 2\left[\frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2}\right]+1$$
$$= 2R_n + 1,$$

and thus,

$$B_n + B_{n-1} = 2B_{n-1} + 2R_n + 1$$
$$= 2r_n + 2b_n + 1.$$

This completes the proof.

Let the $n^{\text{th}}$ sequence balancing number in the sequence of odd natural numbers be denoted by $x_n$. Then $x_n$ can be more conveniently calculated by a recurrence relation.

**THEOREM 2.1.5.** *The sequence* $\{x_n\}_{n=1}^{\infty}$ *satisfies the recurrence relation* $x_{n+1} = 6x_n - x_{n-1}$ *for* $n \geq 2$.

**PROOF**. Since the sequence of balancing numbers satisfies the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$ ( [1], p.101), and $x_n = B_{n+1} + B_n$, it follows that $\{x_n\}_{n=1}^{\infty}$ satisfies the same recurrence relation as that of $\{B_n\}_{n=1}^{\infty}$.

We next investigate the existence of sequence cobalancing numbers in this sequence. Any sequence cobalancing number $2m-1$ of this sequence satisfies

$$1+3+\ldots+(2m-1) = (2m+1)+(2m+3)+\ldots+(2(m+s)-1)$$

for some natural number $s$. This is equivalent to $2m^2 = (m+s)^2$, which is impossible since 2 is not a square. Hence, we have the following important result.

**THEOREM 2.1.6.** *There does not exist any sequence cobalancing numbers in the sequence of odd natural numbers.*

**2.2 Sequence balancing and cobalancing numbers in the sequence $a_n = n+1$.** Any sequence balancing number $m+1$ of this sequence satisfies

$$2+3+\ldots+m = (m+2)+(m+3)+\ldots+(m+s+1)$$

for some natural number $s$. Putting $y = m+s+1$ we see that the above equation is equivalent to

$$\frac{y(y+1)}{2} = (m+1)^2 - 1,$$

showing that $\frac{y(y+1)}{2}$ is a triangular number differing from a square by 1. Such a triangular number is called an almost square triangular number

(ASTN) [16]. Indeed, a triangular number $T$ for which $T+1$ is a perfect square is called a $\beta$-ASTN. Thus, $\frac{y(y+1)}{2}$ is a $\beta$-ASTN. Let $\{\beta_n\}_{n=1}^{\infty}$ be the sequence of $\beta$-ASTN's. Then the following theorem gives the totality of $\beta$-ASTN's.

**THEOREM 2.2.1** ([16], p.196, [1])**.** $\beta_{2n-1} = (B_{n+1} - 4B_n)^2 - 1$ and $\beta_{2n} = (B_{n+1} - 2B_n)^2 - 1$.

Now putting

$$\beta = \frac{y(y+1)}{2} = (m+1)^2 - 1$$

we see that $m+1 = \sqrt{\beta+1}$. Thus, a simple use of Theorem 2.2.1 yields the following result.

**THEOREM 2.2.2.** *If $z_n$ denotes the $n^{th}$ sequence balancing number of the sequence $a_n = n+1$, then $z_{2n-1} = B_{n+1} - 4B_n$ and $z_{2n} = B_{n+1} - 2B_n$ for $n$ = 1, 2,… .*

Thus $B_3 - 4B_2 = 35 - 4\times 6 = 11$ satisfies $2+3+\ldots+10 = 12+13+14+15$, and $B_3 - 2B_2 = 35 - 2\times 6 = 23$ satisfies $2+3+\ldots+22 = 24+25+\ldots+32$ and so on.

We next investigate the existence of sequence cobalancing numbers in this sequence. Any sequence cobalancing number $m + 1$ of this sequence satisfies

$$1+3+\ldots+(m+1) = (m+2)+(m+3)+\ldots+(m+s+1)$$

for some natural number $s$, and once again putting $y = m+s+1$ we see that the above equation is equivalent to

$$(m+1)(m+2) = \frac{y(y+1)}{2} + 1.$$

Thus, we must search for those pronic numbers that are 1 more than triangular numbers. One such number is 56 since

$$56 = 7\times 8 = \frac{10\times 11}{2} + 1.$$

Hence, $m + 1 = 7$ is a sequence cobalancing number in this sequence and we have $2+3+\ldots+7 = 8+9+10$. Again since

$$1892 = 43\times 44 = \frac{61\times 62}{2} + 1,$$

it follows that $m + 1 = 43$ is also a sequence cobalancing number in this sequence and we have $2+3+\ldots+43 = 44+45+\ldots+61$.

**2.3 Sequence balancing and cobalancing numbers in the Fibonacci sequence.** A sequence balancing number $F_m$ in the Fibonacci sequence would satisfy

$$F_1 + F_2 + \ldots + F_{m-1} = F_{m+1} + F_{m+2} + \ldots + F_{m+s}$$

for some $s$. But it is well known that

$$F_1 + F_2 + \ldots + F_{m-1} = F_{m+1} - 1$$

so that

$$F_1 + F_2 + \ldots + F_{m-1} < F_{m+1}$$

for each natural number $m$. Hence, there does not exist any sequence balancing number in the Fibonacci sequence. Similarly, a sequence cobalancing number $F_m$ in the Fibonacci sequence would satisfy

$$F_1 + F_2 + \ldots + F_m = F_{m+1} + F_{m+2} + \ldots + F_{m+s}$$

for some $s$. In view of

$$F_{m+1} < F_1 + F_2 + \ldots + F_m < F_{m+1} + F_{m+2}$$

for $m > 2$, it follows that no Fibonacci number $F_n$ for $n > 2$ can be a sequence balancing number. For $n \le 2$, we have

$$F_1 + F_2 = 1 + 1 = 2 = F_3.$$

Hence, the only sequence balancing number in the Fibonacci sequence is $F_2 = 1$.

The above discussion proves the following theorems.

**THEOREM 2.3.1.** *There does not exist any sequence balancing number in the Fibonacci sequence.*

**THEOREM 2.3.2.** *The only sequence cobalancing number in the Fibonacci sequence is* $F_2 = 1$.

## 3. HIGHER ORDER BALANCING AND COBALANCING NUMBERS

Let $k$ be any natural number. We call the sequence balancing numbers of the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n^k$, the balancing numbers of order $k$. Similarly, we call the sequence cobalancing numbers of this sequence, the cobalancing numbers of order $k$. Thus, balancing and cobalancing numbers of order one are the usual balancing and cobalancing numbers, respectively. We also call a balancing number of order two a balancing square and a balancing number of order three a balancing cube. Similarly, we also call a cobalancing number of order two a cobalancing square and a cobalancing number of order three a cobalancing cube.

We first prove the following result on balancing cubes and cobalancing cubes.

**THEOREM 3.1.** *There does not exist any balancing cube or cobalancing cube.*

We need the following theorem to prove Theorem 3.1.

**THEOREM 3.2** ([8], p. 277). The only solutions of the Diophantine equation $\left[\frac{x(x-1)}{2}\right]^2 = \frac{y(y-1)}{2}$ in positive integers are $(x, y) = (1, 1),\ (2, 2),\ (4, 9)$.

**PROOF OF THEOREM 3.1.** Any balancing cube $n^3$ must satisfy

$$1^3 + 2^3 + \ldots + (n-1)^3 = (n+1)^3 + (n+2)^3 + \ldots + (n+r)^3$$

for some natural number $r$, which is equivalent to

$$\left[\frac{m(m+1)}{2}\right]^2 = \frac{n^2(n^2+1)}{2} \quad \text{(where } m = n+r).$$

Now by Theorem 3.2, the only possible solutions of this equation are $(m+1, n^2+1) = (1, 1), (2, 2)$ and $(4, 9)$. $m+1 = 1$ and $n^2+1 = 1$ implies $m = n = 0$ which is not possible since $m > n > 0$. Again $m+1 = 2$ and $n^2+1 = 2$ implies $m = n = 1$ which is not possible since $m > n$. Lastly, $m+1 = 4$ and $n^2+1 = 9$ implies $m = 3$ and $n^2 = 8$, which is again impossible. Hence, no balancing cube exists.

If $n^3$ is a cobalancing cube, then it satisfies

$$1^3 + 2^3 + \ldots + n^3 = (n+1)^3 + (n+2)^3 + \ldots + (n+r)^3$$

for some natural number $r$, which is equivalent to

$$\left[\frac{(n+r)(n+r+1)}{2}\right]^2 = 2\left[\frac{n(n+1)}{2}\right]^2$$

which has no solution in positive integers since 2 is not a square. Hence, no cobalancing cube exists.

In connection with the higher order balancing and cobalancing numbers, the author, after exhaustive verification of special cases feels that the following is true.

**CONJECTURE 3.3.** *There exists no balancing number or cobalancing number of order k for $k \geq 2$. More precisely, the Diophantine equations*

$$1^k + 2^k + \ldots + (n-1)^k = (n+1)^k + (n+2)^k + \ldots + (n+r)^k$$

*and*

$$1^k + 2^k + \ldots + n^k = (n+1)^k + (n+2)^k + \ldots + (n+r)^k$$

*have no solutions in $(n, r)$ in positive integers if $k \geq 2$.*

## 4. CONCLUSION

The work on higher order cobalancing numbers is related to some classical unsolved problem in Diophantine equations. In this context we recall the works of Bernstein (see [2], [3] and [4]) on pyramidal Diophantine equations. These works, in turn, are particular cases of a problem due to Ërdös [6], namely whether the Diophantine equation

$$m(m+1)(m+2)\dots(m+k-1)=2n(n+1)(n+2)\dots(n+k-1)$$

has any solution for $k>2$ and $m+k+1<n$. Makowski [9] answered Ërdös' question in the negative for a particular case with the use of results of Segal [13]. The existence of cobalancing squares is equivalent to the existence of solution to the Diophantine equation

$$m(m+1)(m+2)=2n(n+1)(n+2),$$

which is a particular case of the previous Diophantine equation. Mordell [10] looked at particular cases of nearly pyramidal numbers (i.e. any number differing from a pyramidal number by 1) as did Boyd and Kisilevsky [5], but the scope of generalization is wide open.

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## REFERENCES

[1] A. Behera and G. K. Panda, *On the square roots of triangular numbers,* The Fibonacci Quarterly, 37(2)(1999), 98 – 105.

[2] L. Bernstein, *The generalized Pellian equation*, Transactions of the American Mathematical Society, 127(1968), 76 – 89.

[3] L. Bernstein, *The linear Diophantine equation in n variables and its application*, The Fibonacci Quarterly, 6(1968), 1 – 62.

[4] L. Bernstein, *Explicit solutions of pyramidal Diophantine equations*, Canadian Mathematical Bulletin, 15 (1972), 177 – 184.

[5] D.W. Boyd, and H.H. Kiisilevsky, *The Diophantine equation* $u(u+1)(u+2)(u+3)=v(v+1)(v+2)$, Pacific Journal of Mathematics, 40 (1972), 23 – 32.

[6] P. Ërdös, *Problem* 4638, The American Mathematical Monthly, 62 (1955), 370 – 371.

[7] K. Liptai, *Fibonacci balancing numbers*, The Fibonacci Quarterly, 42(4) (2004), 330 – 340

[8] M. Luo, *On the Diophantine equation* (*x*(*x* 1)/2)^2= (*y*(*y*–1)/2), The Fibonacci Quarterly, 34(3) (1996), 277 – 279.

[9] A. Makowski, *Problem* 4638, The American Mathematical Monthly, 70 (1963), 1015.

[10] L. J. Mordell, *Integer solutions of the equation* $ey^2 = ax^3 + bx^2 + cx + d$, Proceedings of the London Mathematical Society, 21 (1923), 415 – 419.

[11] G. K. Panda and P. K. Ray, *Cobalancing numbers and cobalancers*, International Journal of Mathematics and Mathematical Sciences, 2005(8), (2005), 1189 – 1200.

[12] G. K. Panda, *Some fascinating properties of balancing numbers*, to appear in "Applications of Fibonacci Numbers" Vol. 10, Kluwer Academic Pub., 2006.

[13] S.L. Segal, *A note on pyramidal numbers*, The American Mathematical Monthly, 69 (1962), 637 – 638.

[14] K. B. Subramaniam, *A simple computation of square triangular numbers*, International Journal of Mathematics Education in Science and Technology, 23(5) (1992), 790 – 793.

[15] K. B. Subramaniam, *A divisibility property of square triangular numbers*, International Journal of Mathematics Education in Science and Technology, 26(2) (1995), 284 – 286.

[16] K. B. Subramaniam, *Almost square triangular numbers*, The Fibonacci Quarterly, 37(3) (1999), 194 – 197.

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