Laplacian State Transfer on Graphs with an Edge Perturbation Between Twin Vertices

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Introduction

- Continuous-time quantum walks play an important role in analysing various quantum transportation phenomena (by Farhi and Gutmann).

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- We discuss two types of quantum state transfer:
  - Perfect state transfer (PST) $\longrightarrow$ Introduced by Bose

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  - Pretty good state transfer (PGST) $\longrightarrow$ Introduced by Chris Godsil

    C. Godsil, *State transfer on graphs*, Discrete Math., **312**(1): 129–147 (2012).

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Note: The adjacency matrix may be considered instead of the Laplacian matrix. However, for regular graphs, both considerations are equivalent.

Examples with adjacency dynamics

The adjacency matrix of path on two vertices

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- Transition matrix is

$$\begin{aligned} U(t) &= \sum_{n \geq 0} \frac{(-it)^n}{n!} A^n \\ &= \cos(t) I - i \sin(t) A = \begin{pmatrix} \cos(t) & -i\sin(t) \\ -i\sin(t) & \cos(t) \end{pmatrix}. \end{aligned}$$

- Hence PST occurs at $\frac{\pi}{2}$.

Laplacian dynamics on complete graphs

- The spectral decomposition of the transition matrix of the complete graph K_n relative to the Laplacian is

$$U_L(t) = \frac{1}{n}J + \exp\left(-int\right)\left(I - \frac{1}{n}J\right), \tag{1}$$

where J is the matrix with all entries 1 and I is the identity matrix.

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- If u_a and u_b are two distinct vertices of K_n then

$$\left|\mathbf{e}_b^T U_L(t)\,\mathbf{e}_a\right| \le \frac{2}{n}.$$

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- In contrast, the main conclusion by Bose et. al. in [1] observes that the complete graph K_{4n} with a missing edge exhibits LPST.

Perturbed Laplacian

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- Consider a perturbation of the Laplacian matrix L with the rank-one matrix $M = (\mathbf{e}_a - \mathbf{e}_b)(\mathbf{e}_a - \mathbf{e}_b)^T$ as

$$L^\alpha = L + \alpha M, \ \alpha \in \mathbb{R},$$

  which is the Laplacian matrix of $G + \alpha\{u_a, u_b\}$ where the edge weight between u_a and u_b is increased by α.

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  which is the Laplacian matrix of $G + \alpha \{u_a, u_b\}$ where the edge weight between u_a and u_b is increased by α.
- Let $N(u)$ denote the set of all neighbours of a vertex u in G. Then

Lemma 1

If $N(u_a) \setminus \{u_b\} = N(u_b) \setminus \{u_a\}$ then the matrices L and M commute.

Proposition (Determining The Transition Matrix):

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- If $U_L(t)$ is the transition matrix of the unperturbed graph, then the transition matrix of the perturbed graph is

$$U_{L^{\alpha}}(t) = U_L(t)\left[I + \frac{1}{2}\left(\exp\left(-2i\alpha t\right) - 1\right)M\right].$$

Theorem (LPST Between Twin Vertices I):

- Suppose the conditions of previous Proposition are satisfied.
- Let the unperturbed graph G exhibit LPST at time τ between the vertices u_p and u_q.
- Then the edge perturbed graph with Laplacian L^{α} exhibits LPST at τ between the vertices u_p and u_q provided one of the following holds:
  1. $p, q \in \{a, b\}$ with $\alpha\tau \in \pi\mathbb{Z}$,
  2. $p, q \notin \{a, b\}$.
- Moreover, if $p \in \{a, b\}$ and $q \notin \{a, b\}$, then there exists no LPST in the perturbed graph between u_p and u_q.

Proof of Previous Theorem:

- Here $\alpha\tau \in \pi\mathbb{Z}$ implies $\exp(-2i\alpha\tau) = 1$, and $U_{L^\alpha}(\tau) = U_L(\tau)$.
- In case $q \notin \{a, b\}$, we have $M\mathbf{e}_q = 0$ and hence

$$U_{L^\alpha}(t)\mathbf{e}_q = U_L(t)\mathbf{e}_q. \tag{2}$$

- Since $N(u_a) \setminus \{u_b\} = N(u_b) \setminus \{u_a\}$, there is an automorphism of G swapping the vertices u_a and u_b, and that fixing all other vertices. Suppose P is the matrix of the automorphism then

$$P\mathbf{e}_a = \mathbf{e}_b \text{ and } P\mathbf{e}_q = \mathbf{e}_q,\ q \notin \{a, b\},$$

and P commutes with L as well as $U_L(t)$. Now using (2)

$$\mathbf{e}_a^T U_{L^\alpha}(t)\mathbf{e}_q = \mathbf{e}_a^T U_L(t)\mathbf{e}_q = \mathbf{e}_a^T U_L(t) P\mathbf{e}_q = \mathbf{e}_a^T P U_L(t)\mathbf{e}_q = \mathbf{e}_b^T U_{L^\alpha}(t)\mathbf{e}_q$$

Since $U_{L^\alpha}(t)$ is unitary and $a \neq b$, there is no LPST between u_p and u_q whenever $p \in \{a, b\}$ and $q \notin \{a, b\}$.

Theorem (LPST Between Twin Vertices II):

- Suppose the conditions of previous Proposition are satisfied.
- Let the unperturbed graph be periodic at u_p at time τ.
- Then the following holds:

  1. If $p \in \{a, b\}$ with $2\alpha\tau \in \pi(2\mathbb{Z} + 1)$, then the edge perturbed graph exhibits LPST between u_a and u_b at τ.
  2. If $p \notin \{a, b\}$, then the edge perturbed graph is also periodic at the vertex u_p at τ.

Corollary (Edge Deleted Complete Graphs):

- The graph K_{4n} on $4n$ vertices with a missing edge exhibits LPST.
- Moreover, removal of any set of pairwise non-adjacent edges from K_{4n} results LPST at $\frac{\pi}{2}$ between the end vertices of every edge removed.

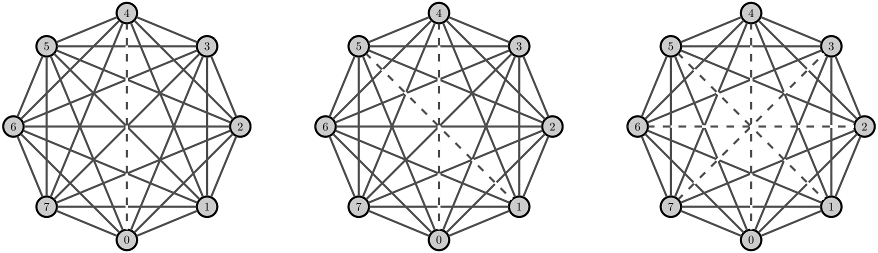

Figure: The complete graph K_8 with disjoint edges removed.

Corollary (Weighted Laplacian Integral Graphs):

- Let G be a Laplacian integral graph having a pair of twins u and v.
- If the edge weight between u and v is set to $\frac{1}{4}$ then the edge perturbed graph exhibits LPST between u and v at time 2π.
- Moreover G is periodic at rest of the vertices at 2π.

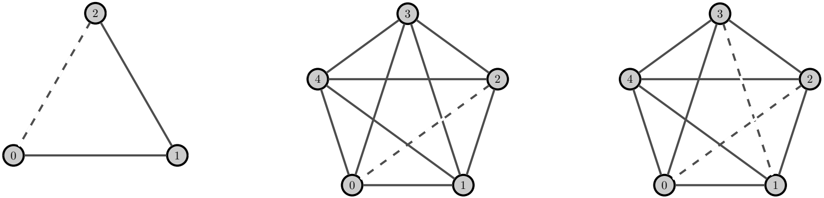

Figure: The complete graphs K_3 and K_5 with disjoint edges perturbed.

Theorem (LPGST on Edge Perturbed Graphs I):

The previous conclusions can further be generalized to have LPGST in certain graphs.

- Suppose the conditions of previous Proposition are satisfied.
- Let the unperturbed graph G exhibit LPGST between the vertices u_p and u_q with respect to a sequence $\tau_k \in \mathbb{R}$.
- Then the edge perturbed graph with Laplacian L^α exhibits LPGST between the vertices u_p and u_q with respect to the sequence $\tau_k \in \mathbb{R}$ provided one of the following holds:
  1. $p, q \in \{a, b\}$ with $\alpha\tau_k \in \pi\mathbb{Z}$,
  2. $p, q \notin \{a, b\}$.
- Moreover, if $p \in \{a, b\}$ and $q \notin \{a, b\}$, then there exists no LPGST in the perturbed graph between u_p and u_q.

Theorem (LPGST on Edge Perturbed Graphs II):

- Suppose the conditions of previous Proposition are satisfied.
- Let the unperturbed graph be almost periodic at u_p with respect to the sequence $\tau_k \in \mathbb{R}$.
- Then the following holds:
  1. If $p \in \{a, b\}$ with $2\alpha\tau_k \in \pi(2\mathbb{Z} + 1)$, then the edge perturbed graph exhibits LPGST between u_a and u_b with respect to τ_k.
  2. If $p \notin \{a, b\}$, then the edge perturbed graph is almost periodic at the vertex u_p with respect to τ_k.

Corollary (Circulant Graphs With Additional Edges):

- Let $k \in \mathbb{N}$, $n = 2^k$ and consider a circulant graph $Cay(\mathbb{Z}_n, S)$. Let $S = \frac{n}{2} - S$ and each divisor d of n satisfies $|S \cap S_n(d)| \equiv 0 \pmod 4$.
- If a new edge is added between a pair of twin vertices in $Cay(\mathbb{Z}_n, S)$ then the resulting graph exhibits LPGST between the end vertices of the newly added edge with respect to a sequence $\tau_k \in \frac{\pi}{2}(4\mathbb{Z} + 1)$.
- Moreover, the perturbed graph is almost periodic at the remaining vertices with respect to τ_k.

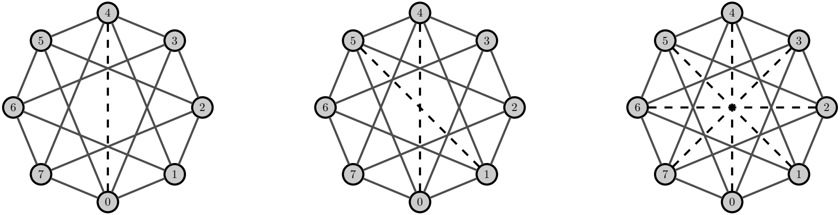

Figure: Edges perturbed circulant graph $Cay(\mathbb{Z}_8, S)$ with $S = \{1, 3, 5, 7\}$.

Conclusions and Futureworks

- The study of state transfer is a rapidly growing area as it contributes to the research in quantum information processing and cryptography.
- Here we have investigated state transfer on edge perturbed graphs, and as a particular case, we have found infinite class of circulant graphs exhibiting LPST and LPGST.
- Cayley graphs appear frequently in communication networks, therefore the study on Cayley graphs in particular have tremendous importance.
- However, investigation into any well known families of graphs shall produce remarkable results.

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Thank You