Dimensional Analysis of Non-stationary Fractal Functions on the Sierpinski Gasket

Sangita Jha

Fractals and Related Fields IV

Department of Mathematics
NIT Rourkela

September 6, 2022

Fractal Interpolation Functions (FIFs)

Iterated Function System (IFS): $\{X; w_i, i = 1, 2, \ldots, N-1\}$, w_i are continuous maps on X.

Attractor: Hutchinson map on $H(X)$ is defined as $W(A) = \cup_{i=1}^{N-1} w_i(A)$. The unique fixed point is known as the**Attractor** of the IFS.

- Interpolation data: $\{(x_i, y_i), i = 1, 2, \ldots, N\}$, with increasing abscissae, and $L_i : I = [a, b] \mapsto I_i = [x_i, x_{i+1}], i \in \{1, 2, \ldots, N-1\}$ be contractive homeomorphisms such that $L_i(x_1) = x_i,\ L_i(x_N) = x_{i+1}$.
- Let $K = I \times \mathbb{R}$ and $w_i(x, y) = (L_i(x), F_(x, y))$, where $F_i : K \mapsto \mathbb{R}$ be such that $F_i(x_1, y_1) = y_i, F_i(x_N, y_N) = y_{i+1}$ and

$$|F_i(x, y) - F_i(x, y')| \leq \alpha_i |y - y'|, \quad (x, y), (x, y') \in K, \quad 0 \leq \alpha_i < 1.$$

Theorem 1 (Barnsley).

The IFS $\mathcal{I} = \{K; w_i : i = 1, 2, \ldots, N\}$ admits a unique attractor G. Further, G is the graph of a continuous function $f : I \mapsto \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i = 1, 2, \ldots, N$.

Sequence of Transformations and Trajectories

Consider a sequence of transformations $\{T_i\}_{i\in\mathbb{N}},\ T_i : X \to X$.

Invariant Set: A subset $\mathcal{P}$ of X is called an invariant set of the sequence $\{T_i\}_{i\in\mathbb{N}}$ if for all $i \in \mathbb{N}$ and $\forall\ x \in \mathcal{P}$, $T_i(x) \in \mathcal{P}$.

Lemma:(Levin, Dyn, Viswanathan) Let $T_i : X \to X$. Suppose there exists a $y \in X$ such that

$$d(T_i(x), y) \leq cd(x, y) + M,$$

for all $x \in X,\ c \in [0, 1)$ and $M > 0$. Then the ball $B_r(y)$ of radius $r = \frac{M}{1-c}$ centered at y is an invariant set for $\{T_i\}_{i\in\mathbb{N}}$.

For $W_k = \{w_{1,k}, w_{2,k}, \ldots, w_{n_k,k}\}$, consider the sequence of set valued maps

$$W_k(A) = \bigcup_{i=1}^{n_k} w_{i,k}(A), A \in H(X). \quad (1)$$

Forward and Backward Trajectories: Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on X. We define forward and backward procedures

$$\Phi_k := T_k \circ T_{k-1} \circ \ldots T_1 \text{ and } \Psi_k := T_1 \circ T_2 \circ \ldots T_k.$$

Convergence of Trajectories

Theorem 2 (Levin, Dyn, Viswanathan).

Let $\{W_k\}_{k\in\mathbb{N}}$ *be a family of set-valued maps as described in* (1)*, where the elements are collections* $W_k = \{w_{i,k} : i \in \mathbb{N}_{n_k}\}$ *of contractions on a complete metric space* (X, d)*. Assume that*

(i) *there exists a nonempty closed invariant set* $\mathcal{P} \subset X$ *for* $w_{i,k}, i \in \mathbb{N}_{n_k}, k \in \mathbb{N}$ *and*

(ii) $\sum_{k=1}^{\infty} \prod_{j=1}^{k} \mathrm{Lip}(W_j) < \infty.$

Then the backward trajectories $\{\Psi_k(A)\}$ *converges for any initial* $A \subseteq \mathcal{P}$ *to a unique attractor* $G \subseteq \mathcal{P}$*.*

Results

Result1:(Levin, Dyn, Viswanathan) Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on a complete metric space X such that T_k has Lipschitz constant c_k. If $\lim_{k\to\infty}\prod_{i=1}^k c_i = 0$, then $\{\Phi_k(x)\}$, $\{\Phi_k(y)\}$ are asymptotically similar for all $x, y \in X$, and so are $\{\Psi_k(x)\}$, $\{\Psi_k(y)\}$ for all $x, y \in X$.

Result 2:(Navascues, Verma) Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on a complete metric space X. If there exists $x_* \in X$ such that the sequence $\{d(x_*, T_k(x_*))\}$ is bounded, and $\sum_{k=1}^{\infty}\prod_{i=1}^k c_i < \infty$ then the sequence $\{\Psi_k(x)\}$ converges for all $x \in X$ to a unique limit $\overline{x}$.

Non-stationary fractal functions on SG

- Let $V_0 = \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle on $\mathbb{R}^2$ and $u_i(x) = \frac{1}{2}(x + p_i)$, where $i = 1, 2, 3$, three contractions of the plane which constitutes an IFS.
- The Sierpiński gasket (abbreviated as SG) is the attractor of this IFS:

$$SG = u_1(SG) \cup u_2(SG) \cup u_3(SG).$$

- For fix $n \in \mathbb{N}$, consider the iterations $u_i = u_{i_1} u_{i_2} \dots u_{i_n}$ for any sequence $i = (i_1, i_2, \dots, i_n) \in I^n := \{1, 2, 3\}^n$. The union of images of V_0 under these iterations constitutes the set of n-th stage vertex V_n of SG.
- Let $B : V_n \to \mathbb{R}$ be a given function. We find an IFS whose attractor is the graph of a continuous function on SG such that $f|_{V_n} = B$. For $k \in \mathbb{N}$, define maps $W_{w,k} : SG \times \mathbb{R} \to SG \times \mathbb{R}$ by

$$W_{w,k}(x, z) = \Big(u_w(x), F_{w,k}(x, z)\Big), \quad w \in I^n$$

- $F_{w,k}(x,z) : SG \times \mathbb{R} \to \mathbb{R}$ need to satisfy the following conditions:

$$\|F_{w,k}(.,z_1) - F_{w,k}(.,z_2)\| \leq c_{w,k}\ |z_1 - z_2|$$

  and $F_{w,k}(p_j, B(p_j)) = B(u_i(p_j))$ for every $w \in I^n, j \in I$, where $c := \sup_{k\in\mathbb{N}} \max_{w\in I^n} c_{w,k} < 1$.
- Consider $F_{w,k}(x,z) = \alpha_{w,k}(x)z + q_{w,k}(x)$, where $\alpha_{w,k} : SG \to \mathbb{R}$ and $q_{w,k} : SG \to \mathbb{R}$ are continuous functions with $\|\alpha\|_\infty := \sup_{k\in\mathbb{N}} \max\{\|\alpha_{w,k}\|_\infty : w \in I^n\} < 1$ and $\|q\|_\infty := \sup_{k\in\mathbb{N}} \max\{\|q_{w,k}\|_\infty : w \in I^n\} < \infty$.
- Let $K = SG \times \mathbb{R}$. We get a sequence of IFSs $\mathcal{I}_k := \{K; W_{w,k} : w \in I^n\}$.

Theorem 3.

Let $n \in \mathbb{N}$ and $B : V_n \to \mathbb{R}$ be given. The sequence of IFSs $\{K; W_{w,k} : w \in I^n\}$ defined above produces a continuous function $g_* : SG \to \mathbb{R}$ which satisfies $g_*|_{V_n} = B$.

Idea of the proof

- Let $\mathcal{C}^*(SG,\mathbb{R}) = \left\{g \in \mathcal{C}(SG,\mathbb{R}) : g|_{V_0} = B|_{V_0}\right\}$.
- For $k \in \mathbb{N}$, we define a mapping $T_k : \mathcal{C}^*(SG,\mathbb{R}) \to \mathcal{C}^*(SG,\mathbb{R})$ by

$$(T_k g)(x) = F_{w,k}(u_w^{-1}(x), g(u_w^{-1}(x)))\ \forall\, x \in u_w(SG), w \in I^n.$$

- One can check that T_k is a contraction map and the sequence $\{\|T_k g - g\|_\infty\}$ is bounded.
- Using Result 2, the backward trajectories $\Phi_k := T_1 \circ T_2 \circ \cdots \circ T_k$ of $\{T_k\}$ converge for every $g \in \mathcal{C}^*(SG)$ to a unique attractor $g_* \in \mathcal{C}^*(SG)$.

Oscillation Spaces

- For $g : SG \to \mathbb{R}$, we define total oscillation of order n by

$$R(n,g) = \sum_{w \in \{1,2,3\}^n} R_g[u_w(SG)],$$

where $R_g[u_w(SG)] = \sup\{|g(x_1) - g(x_2)| : x_1, x_2 \in u_w(SG)\}$.

- Let

$$\mathcal{C}^\beta(SG) := \left\{ f : SG \to \mathbb{R} : \text{ f is continuous and } \sup_{n \in \mathbb{N}} \frac{R(n,g)}{2^{n\left(\frac{\log 3}{\log 2} - \beta\right)}} < \infty \right\},$$

where $0 \le \beta \le 1$.

Theorem 4 (Deliu,Jawerth).

Let $f : I \to \mathbb{R}$ be a continuous function and let $0 < \gamma < 1$. Then we have

$$\overline{\dim}_B(Graph(f)) = 2 - \gamma \iff f \in \cap_{\theta < \gamma} \mathcal{V}^\theta(I) \setminus \cup_{\beta > \gamma} \mathcal{V}^\beta(I).$$

Theorem 5.

Let $g : SG \to \mathbb{R}$ *be a continuous function and let* $0 < \gamma < 1$. *Then* $\overline{\dim}_B(\mathrm{Gr}(g)) = 1 - \gamma + \frac{\log 3}{\log 2}$ *if and only if* $g \in \Big(\cap_{\alpha<\gamma} \mathcal{C}^{\alpha}(SG)\Big)\setminus\Big(\cup_{\beta>\gamma} \mathcal{C}^{\beta}(SG)\Big)$.

Idea of the proof:

- Let $\overline{\dim}_B(\mathrm{Gr}(f)) = 1 - \gamma + \frac{\log 3}{\log 2}$. Let $\epsilon > 0$ be given. As $\overline{\dim}_B(\mathrm{Gr}(g)) = 1 + \overline{\lim}_{n\to\infty} \frac{\log R(n,f)}{n \log 2}$, we obtain

(1) $\exists\ n^* \in \mathbb{N}$ such that $R(n,g) \leq 2^{n(\frac{\log 3}{\log 2} - \gamma + \epsilon)}$ for every $n > n^*$,

(2) a sequence (n_k) with $n_k \to \infty$ and $R(n,g) \geq 2^{n_k(\frac{\log 3}{\log 2} - \gamma - \epsilon)}$.

- Using item (1) and the boundedness of g, we obtain $g \in \cap_{\alpha<\gamma}\mathcal{C}^{\alpha}(SG)$.
- The only if part follows from the oscillation $R(n,g)$ and the definition of box-dimension.

Dimension Result

Theorem 6.

Let $f, b_k, \alpha_{w,k}$ $(w \in I^n,\ k \in \mathbb{N}) \in \mathcal{C}^\beta(SG)$ *be such that* $b_k|_{V_0} = f|_{V_0}$. *Assume that* $\|b\|_\infty = \sup_{k\in\mathbb{N}} \|b_k\|_\infty < \infty$. *Then, for*

$$\max\left\{ \|\alpha\|_\infty + \frac{3^n}{2^{n\left(\frac{\log 3}{\log 2}-\beta\right)}} \sup_{w\in I^n, k\in\mathbb{N}} \sup_{m\in\mathbb{N}} \frac{R(m,\alpha_{w,k})}{2^{m\left(\frac{\log 3}{\log 2}-\beta\right)}}, \frac{3^n\|\alpha\|_\infty}{2^{n\left(\frac{\log 3}{\log 2}-\beta\right)}} \right\} < 1,$$

there exists a non-stationary fractal function $f_*^\alpha \in \mathcal{C}^\beta(SG)$. *Furthermore,* $\overline{\dim}_B(Gr(f_*^\alpha)) \le 1 - \beta + \frac{\log 3}{\log 2}$.

Idea of the proof

- Consider the space $\mathcal{C}_f^{\beta}(SG) = \{g \in \mathcal{C}^{\beta}(SG) : g|_{V_0} = f|_{V_0}\}$.
- Define a sequence of mappings $T_k : \mathcal{C}_f^{\beta}(SG) \to \mathcal{C}_f^{\beta}(SG)$ by

$$(T_k g)(x) = f(x) + \alpha_{w,k}(u_w^{-1}(x))\ (g - b_k)(u_w^{-1}(x))$$

  for all $x \in u_w(SG),\ w \in I^n$.
- For $g, h \in \mathcal{C}_f^{\beta}(SG)$, one can obtain that T_k is a contraction map on $\mathcal{C}_f^{\beta}(SG)$ and $\{\|T_k g - g\|_{\mathcal{C}^{\beta}}\}$ is bounded.
- The backward trajectories $\Phi_k(g) := T_1 \circ T_2 \circ \cdots \circ T_k(g)$ of $\{T_k\}$ converges for every $g \in \mathcal{C}_f^{\beta}(SG)$ to a unique attractor $f_*^{\alpha} \in \mathcal{C}_f^{\beta}(SG)$.
- Since $f_*^{\alpha} \in \mathcal{C}^{\beta}(SG)$, Theorem 5 yields that $\overline{\dim}_B(\text{Gr}(f_*^{\alpha}) \leq 1 - \beta + \frac{\log 3}{\log 2}$.

Following the work of Falconer(2011), we define

$$\mathcal{X}_\beta(SG) := \{f \in \mathcal{C}(SG) : \overline{\dim}_B(\mathsf{Gr}(f) \leq \beta\}.$$

Again, following the result of (Falconer, 2011), we have

Theorem 7.

Let $\beta \in \left[\frac{\log 3}{\log 2}, 1 + \frac{\log 3}{\log 2}\right)$*. Then*

$$X_\beta(SG) = \cap_{k\in\mathbb{N}} \mathcal{C}^{\beta+\frac{1}{k}}(SG).$$

Moreover, $(X_\beta(SG), d)$ *is a Banach space, where*

$$d(f,g) = \sum_{k\in\mathbb{N}} \min\left\{2^{-k}, \|f-g\|_{\mathcal{C}^{\beta+\frac{1}{k}}}\right\}.$$

Theorem 8.

Let $f, b_k, \alpha_{w,k}$ $(w \in I^n,\ k \in \mathbb{N}) \in X_\beta(SG)$ *be such that* $b_k|_{V_0} = f|_{V_0}$. *Assume that* $\|b\|_\infty = \sup_{k\in\mathbb{N}} \|b_k\|_\infty < \infty$. *Then, for*

$$\sup_{l\in\mathbb{N}} \left\{ \|\alpha\|_\infty + \frac{3^n}{2^{n\left(\frac{\log 3}{\log 2} - \beta - \frac{1}{l}\right)}} \sup_{w\in I^n, k\in\mathbb{N}} \sup_{m\in\mathbb{N}} \frac{R(m, \alpha_{w,k})}{2^{m\left(\frac{\log 3}{\log 2} - \beta - \frac{1}{l}\right)}}, \frac{3^n \|\alpha\|_\infty}{2^{n\left(\frac{\log 3}{\log 2} - \beta - \frac{1}{l}\right)}} \right\} < 1,$$

there exists a non-stationary fractal function $f_*^\alpha \in X_\beta(SG)$. *Furthermore,* $\overline{\dim}_B(\mathit{Gr}(f_*^\alpha)) \leq 1 - \beta + \frac{\log 3}{\log 2}$.

Energy

- Consider the vertex set V_0 and define $\mathcal{G}_0$ as the complete graph on it.
- After constructing graph $\mathcal{G}_{n-1}$ with vertex set V_{n-1} for some $n \geq 1$, we define the graph $\mathcal{G}_n$ on V_n as follows: for any $x, y \in V_n$, $x \sim_n y$ holds if and only if $x = u_i(x_1), y = u_i(y_1)$ with $x_1 \sim_{n-1} y_1$ and $i \in I$.
- Equivalently, $x \sim_n y$ if and only if there exists $i \in I^m$ such that $x, y \in u_i(V_0)$.
- For $n = 0, 1, 2, \ldots,$ we define graph energies $\mathcal{E}_n$ on $\mathcal{G}_n$ by

$$\mathcal{E}_n(f) := \left(\frac{5}{3}\right)^n \sum_{x \sim_m y} (f(x) - f(y))^2.$$

- Note that the graph energy sequence $\{\mathcal{E}_n\}$ satisfies $\mathcal{E}_{n-1}(f) = \min \mathcal{E}_n(\tilde{f})$, where the minimum is taken over all $\tilde{f}$ satisfying $\tilde{f}|_{V_{n-1}} = f$ for any $f : V_*(:= \cup_{m=0}^{\infty} V_m) \to \mathbb{R}$ and for any $n \geq 1$. Then for each function f on V_*, we observe that $\{\mathcal{E}_n(f)\}_{n=0}^{\infty}$ is an increasing sequence. The energy of f on V_* is defined as

$$\mathcal{E}(f) := \lim_{n \to \infty} \mathcal{E}_n(f).$$

Harmonic Function

If f is continuous and satisfies $\mathcal{E}_{n-1}(f) = \mathcal{E}_n(f)$ for all $n \geq 1$, then we call f is a harmonic function on SG. Next we consider the well-known "$\frac{1}{5} - \frac{2}{5}$" rule of harmonic functions.

Lemma 9.

Consider a harmonic function h on SG, and (i, j, k) as a permutation of $(1, 2, 3)$. Then

$$h(p_{ij}) = \frac{2}{5}h(p_i) + \frac{2}{5}h(p_j) + \frac{1}{5}h(p_k).$$

In general, for any $w = w_1 \dots w_n \in \{1, 2, 3\}^n$, we have

$$h(p_{wij}) = \frac{2}{5}h(p_{wi}) + \frac{2}{5}h(p_{wj}) + \frac{1}{5}h(p_{wk}),$$

where wi, wj, wk and wij to be the word $w_1 \dots w_n i,\ w_1 \dots w_n j,\ w_1 \dots w_n k$ and $w_1 \dots w_n ij$, respectively.

Note that h is constant on SG if it is constant on V_0. Let us define dom$(\mathcal{E}) = \{g \in \mathcal{C}(SG) : \mathcal{E}(g) < \infty\}$. The space $(\text{dom}(\mathcal{E}), \|.\|_{\mathcal{E}})$ is complete with $\|g\|_{\mathcal{E}} := \|g\|_\infty + \sqrt{\mathcal{E}(g)}$.

Theorem 10.

Let $n \in \mathbb{N}$. Let germ function $f \in$ dom$(\mathcal{E})$ and $b_k \in$ dom$(\mathcal{E})$ with $b_k|_{V_0} = f|_{V_0}$. Assume that $\mathcal{E}(b) := \sup_{k\in\mathbb{N}} \mathcal{E}(b_k) < \infty$ and $\mathcal{E}(\alpha) := \sup\limits_{w \in I^n, k \in \mathbb{N}} \mathcal{E}(\alpha_{w,k}) < \infty$. If $\|\alpha\|_{\mathcal{E}} < \frac{1}{2\sqrt{5^n}}$ then $f_*^\alpha \in$ dom$(\mathcal{E})$.

Thank you