STOCHASTIC COMPARISONS OF THE SERIES AND PARALLEL SYSTEMS WITH HETEROGENEOUS POWER LOMAX DISTRIBUTED COMPONENTS

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ABSTRACT

In this paper, the authors investigated stochastic comparisons of the series and parallel systems under the assumption that the component lifetimes have independent heterogeneous power-Lomax distributions. The comparisons are established based on the hazard rate ordering, reversed hazard rate ordering and likelihood ratio ordering. Several examples are presented to illustrate the established results.

Keywords**:** Series and parallel systems, Hazard rate ordering, Reversed hazard rate ordering, Likelihood ratio ordering.

Stochastic Comparison of the Lifetimes of Series and Parallel Systems with Heterogeneous Power Lomax Distributed Components

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Outline of the presentation

Introduction (Stochastic comparison)

- In the last few decades, stochastic comparisons and the inequalities associated to it have expanded its spectrum into various fields of mathematics and statistics.
- The simplest and common method which appears in mind naturally to compare two random variables is by comparing them with respect to their corresponding means.
- However, this type of comparison is mostly based only on two fixed numbers say the means, and hence this comparisons are often not very useful and informative. Further, it is noteworthy to mention that the means may not exist always.
- But there are many instances when the means of two random variables are equal then to compare these two random variables one may approach by using their dispersion.
- Similar to the comparison of means, this comparison also depends only on two single values e.g., standard deviations. Again this comparison fails when the standard deviation of the random variables does not exist.
- Moreover, there are situations when the median of a random variable X is smaller than that of Y and the mean of X is larger than mean of Y.

- However, the mentioned scenarios can be handled if the random variables are stochastically ordered.
- In the recent past, several researchers have developed different notions of stochastic orders between two random variables.
- Some of the well-known partial orderings amongst the random variables are based on their survival functions, hazard rate functions, reversed hazard rate functions and other suitable characteristics of probability distributions.
- The comparison of random variables with respect to different stochastic orders are known as stochastic comparison.
- The advantage of using these methods is that provide more information than those depending only on some fixed numerical entities.

Notion of order statistics

- The order statistics play an important role in reliability theory, since there is a connection between the order statistic and the lifetime of a k-out-of-n system.
- Let $X_1, X_2, \ldots, X_n$ be n independent identically distributed random observations are taken from a population having the cumulative distribution function (CDF) F and probability density function (PDF) f.
- The ordered sample values $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ are called their order statistic, where $X_{k:n}$ is the k^{th} order statistic of random variables $X_1, X_2, \ldots, X_n$ for $k \in \mathcal{I}_n = \{1, 2, \ldots, n\}$.
- In general, the k^{th} order statistic $X_{k:n}$ represents the lifetime of $(n-k+1)$-out-of-n system.
- The lifetime of a k-out-of-n system is represented by the $(n-k+1)^{th}$ order statistic $X_{n-k+1:n}$.

- It can be recalled that, a k-out-of-n system works if and only if at least k out of n components work.
- When $k = 1$ and $k = n$, the k-out-of-n system reduces to the parallel (1-out-of-n) and series (n-out-of-n) systems, respectively.
- Particularly, the lifetime of a parallel system is represented by the largest or maximum order statistic $X_{n:n} = \max\{X_1, X_2, \ldots, X_n\}$, whereas the lifetime of a series system is represented by the smallest or minimum order statistic $X_{1:n} = \min\{X_1, X_2, \ldots, X_n\}$.
- The largest or maximum order statistic is often applied in stochastic modeling of floods and of other meteorological phenomena, whereas the smallest or minimum order statistic can be helpful in reliability and survival analysis etc.

Some definitions

- The distribution function of $X_{n:n}$ is given by

$$F_{X_{n:n}}(x) = P(\max(X_1, X_2, \ldots, X_n) \le x) = F^n(x). \tag{1}$$

- The distribution function of $X_{1:n}$ is given by

$$F_{X_{1:n}}(x) = P(\min(X_1, X_2, \ldots, X_n) \le x) = 1 - [1 - F(x)]^n. \tag{2}$$

- The distribution function of $X_{k:n}$ is given by

$$F_{X_{k:n}}(x) = \sum_{j=k}^{n} \binom{n}{j} [F(x)]^j [1 - F(x)]^{n-j}. \tag{3}$$

It can be noticed that there are several developments in ordering results between two series and parallel systems. A brief literature review is presented below.

- Khaledi and Kochar (2006) considered the problem of stochastic comparison between two series and parallel systems, when the lifetime of the components are assumed to follow independent Weibull random observations. The authors have obtained various stochastic ordering results between the lifetimes of the smallest and largest order statistics.
- Khaledi *et al.* (2011) studied conditions under which the series and parallel systems consisting of components with lifetimes from scale family of distributions are ordered in terms of the hazard rate and the reversed hazard rate orderings, respectively.
- Hazra *et al.* (2017) proposed sufficient conditions for which several stochastic ordering results hold between the maximum order statistics arising from a location-scale family of distributions.

- Zhang *et al.* (2019) considered the problem of stochastic comparison between two series and parallel systems, where the component lifetimes follow resilience-scale model. The authors established hazard rate and reversed hazard rate orderings between the lifetimes of the series and parallel systems.
- Balakrishnan *et al.* (2020) dealt with the parallel systems having exponentiated models as the component lifetime distributions. Sufficient conditions have been established by the authors, under which various stochastic orders such that the likelihood ratio, dispersive and the star orders are preserved.
- Chowdhury *et al.* (2021) considered the problem of stochastic comparison between two parallel systems, when the lifetimes of the components follow log-Lindley distribution. The authors have obtained the usual stochastic ordering between the lifetimes of the parallel systems.

Motivation

- The hazard rate (or the instantaneous failure rate) function of the power-Lomax distribution is decreasing and upside down bathtub. The decreasing hazard rate is more likely to occur in the early age of a product. Further, there are some situations, where a component constructed by metal gets harder as time passes and the corresponding hazard rate function decreases. The upside down bathtub shape hazard rate function is often seen when studying the life cycle of an industrial product, or the entire life span of a biological entity. Generally, there is a high failure rate in infancy which decreases to a certain level, where it remains essentially constant for some time, and then increases from a point onwards due to aging.
- From the literature review presented above, it is noticed that nobody has considered the power-Lomax distribution as the component lifetimes in the study of stochastic comparison of two series and parallel systems. Due to the shapes of the hazard rate function of the power-Lomax distribution, this distribution can be used as an alternative distribution to various other distributions studied in the literature so far.

Literature gap and proposed problem

- There is an extensive literature on the problems of stochastic comparison of series and parallel systems for independent heterogeneous distributed components such as exponential, Weibull, Pareto, gamma, generalized exponential etc. However, nobody has considered the power-Lomax distribution and studied stochastic comparison results in the sense of various stochastic orders such as the hazard rate, reversed hazard rate and the likelihood ratio orderings.
- A random variable X is said to have power-Lomax distribution if its cumulative distribution function (CDF) is given by

$$F_X(x) = 1 - \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha} \; ; \; x > 0 \; ; \; \alpha, \; \beta, \; \lambda > 0, \tag{4}$$

where α and β are two shape parameters and λ is a scale parameter. The corresponding probability density function (PDF) of the power-Lomax distribution is given by

$$f_X(x) = \frac{\alpha\beta}{\lambda} x^{\beta-1} \left(1 + \frac{x^\beta}{\lambda}\right)^{-\alpha-1} \; ; \; x > 0 \; ; \; \alpha, \; \beta, \; \lambda > 0. \tag{5}$$

Basic definitions and lemmas

Let X and Y be two nonnegative random variables with the PDFs $f_X(.)$ and $g_Y(.)$, the CDFs $F_X(.)$ and $G_Y(.)$, the survivals functions $\bar{F}_X(.) \equiv 1 - F_X(.)$ and $\bar{G}_Y(.) \equiv 1 - G_Y(.)$, the hazard rate functions $r_X(.) \equiv f_X(.)/\bar{F}_X(.)$ and $r_Y(.) \equiv g_Y(.)/\bar{G}_Y(.)$, the reversed hazard rate functions $\tilde{r}_X(.) \equiv f_X(.)/F_X(.)$ and $\tilde{r}_Y(.) \equiv g_Y(.)/G_Y(.)$, respectively. The following standard widely recognized definitions might be acquired in Shaked and Shanthikumar (2007). A random variable X is stated to be smaller than Y in the sense of the

- **hazard rate** ordering (denoted as $X \leq_{hr} Y$), if $\bar{G}_Y(x)/\bar{F}_X(x)$ is increasing, for all $x \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of positive real numbers;
  or, equivalently, if $r_X(x) \geq r_Y(x)$, for all $x \in \mathbb{R}_+$.
- **reversed hazard rate** ordering (denoted as $X \leq_{rhr} Y$), if $G_Y(x)/F_X(x)$ is increasing, for all $x \in \mathbb{R}_+$;
  or, equivalently, if $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$, for all $x \in \mathbb{R}_+$.
- **likelihood ratio** ordering (denoted as $X \leq_{lr} Y$), if $g_Y(x)/f_X(x)$ is increasing, for all $x \in \mathbb{R}_+$. It is worthwhile to mention here that

$$X \leq_{hr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{rhr} Y.$$

Let $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y}_n = (y_1, \dots, y_n) \in \mathbb{R}^n$ be two vectors, with $x_{1:n} \leq \dots \leq x_{n:n}$ representing the increasing arrangements in terms of order statistics of the vector $\mathbf{x}_n$'s components. Below, present some definitions which can be seen in Marshall et al. (1979). A vector $\mathbf{x}_n$ is supposed to be

- **majorized** via the vector $\mathbf{y}_n$ (denoted as $\mathbf{x}_n \overset{m}{\preccurlyeq} \mathbf{y}_n$) if $\sum_{i=1}^{j} x_{i:n} \geq \sum_{i=1}^{j} y_{i:n}$, for all $j = 1, \dots, n-1$ and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$.
- **weakly supermajorized** via the vector $\mathbf{y}_n$ (denoted as $\mathbf{x}_n \overset{w}{\preccurlyeq} \mathbf{y}_n$) if $\sum_{i=1}^{j} x_{i:n} \geq \sum_{i=1}^{j} y_{i:n}$, for all $j = 1, \dots, n$.

- **weakly submajorized** via the vector $\mathbf{y}_n$ (denoted as $\mathbf{x}_n \preccurlyeq_w \mathbf{y}_n$) if $\sum_{i=j}^{n} x_{i:n} \leq \sum_{i=j}^{n} y_{i:n}$, for all $j = 1, \ldots, n$.
  It is well known that

$$\mathbf{x}_n \preccurlyeq_w \mathbf{y}_n \Leftarrow \mathbf{x}_n \overset{m}{\preccurlyeq} \mathbf{y}_n \Rightarrow \mathbf{x}_n \overset{w}{\preccurlyeq} \mathbf{y}_n.$$

- Let $\mathcal{A} \subseteq \mathbb{R}$. A function $\phi : \mathcal{A}^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) on $\mathcal{A}^n$ if

$$\mathbf{x}_n \overset{m}{\preccurlyeq} \mathbf{y}_n \text{ implies } \phi(\mathbf{x}_n) \leq (\geq)\ \phi(\mathbf{y}_n),\ \textit{for all } \mathbf{x}_n, \mathbf{y}_n \in \mathcal{A}^n.$$

Now, consider the following lemmas, which will be utilized to prove the upcoming results in this presentation.

- **Lemma** 1. (Marshall et al. (1979), Theorem 3.A.8). A real valued function ϕ on $\mathbb{R}^n$, satisfies

$$\mathbf{x}_n \overset{w}{\preccurlyeq} \mathbf{y}_n \Rightarrow \phi(\mathbf{x}_n) \leq (\geq)\ \phi(\mathbf{y}_n),$$

if and only if ϕ is decreasing and Schur-convex (Schur-concave) on $\mathbb{R}^n$. Similarly, ϕ satisfies

$$\mathbf{x}_n \preccurlyeq_w \mathbf{y}_n \Rightarrow \phi(\mathbf{x}_n) \leq (\geq)\ \phi(\mathbf{y}_n),$$

if and only if ϕ is increasing and Schur-convex (Schur-concave) on $\mathbb{R}^n$.

- **Lemma** 2. (Marshall et al. (1979), Proposition 3.C.1). If $\mathcal{I} \subseteq \mathbb{R}$ is an interval and $u : \mathcal{I} \to \mathbb{R}$ is convex, then

$$l(\mathbf{x}_n) = \sum_{i=1}^{n} u(x_i),$$

is Schur-convex on $\mathcal{I}^n$.

Results obtained

Denote $\boldsymbol{\alpha}_n = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta}_n = (\beta_1, \ldots, \beta_n)$, $\boldsymbol{\alpha}_n^* = (\alpha_1^*, \ldots, \alpha_n^*)$, $\boldsymbol{\beta}_n^* = (\beta_1^*, \ldots, \beta_n^*)$, $\boldsymbol{\lambda}_n = (\lambda_1, \ldots, \lambda_n)$, and $\boldsymbol{\lambda}_n^* = (\lambda_1^*, \ldots, \lambda_n^*)$, where the suffix n indicates the dimension of the vector. For the above proposed lemmas, here proved the following theorems

Theorem 1

Let $X_1, \ldots, X_n$ $(X_1^*, \ldots, X_n^*)$ be the independent random variables with $X_i \sim \mathcal{PL}(\alpha_i, \beta, \lambda)$ $(X_i^* \sim \mathcal{PL}(\alpha_i^*, \beta, \lambda))$, $i = 1, \ldots, n$. Then, for any β, $\lambda > 0$, we have

$$\boldsymbol{\alpha}_n \preccurlyeq_w \boldsymbol{\alpha}_n^* \Rightarrow X_{1:n}^* \leq_{hr} X_{1:n}.$$

Proof.

- Under the set-up, the hazard rate function of $X_{1:n}$ is given by

$$r_{X_{1:n}}(x) = \frac{\beta x^{\beta-1}}{(\lambda + x^{\beta})} \sum_{i=1}^{n} f(\alpha_i), \tag{6}$$

where $f(\alpha_i) = \alpha_i$.

□

- In order to apply Lemma 1, we have to show that the function $r_{X_{1:n}}(x)$ given by (6) is Schur-convex and increasing in α_i's, where $i = 1, \ldots, n$.
- The partial derivative of $r_{X_{1:n}}(x)$ with respect to α_i is

$$\frac{\partial r_{X_{1:n}}(x)}{\partial \alpha_i} = \frac{\beta x^{\beta-1}}{(\lambda + x^{\beta})} \geq 0. \tag{7}$$

So, $r_{X_{1:n}}(x)$ is increasing in each α_i. Further, to prove the Schur-convexity, it follows from Lemma 2 that it is required to establish $\frac{\partial^2 f(\alpha_i)}{\partial^2 \alpha_i} \geq 0$, which is obvious. So, Lemma 1 implies the required result.

An illustration of Theorem 1 is provided below.

Example 1

Let $\{X_1, X_2, X_3\}$ and $\{X_1^*, X_2^*, X_3^*\}$ be two sets of independent random observations such that $X_i \sim \mathcal{PL}(\alpha_i, \beta, \lambda)$ and $X_i^* \sim \mathcal{PL}(\alpha_i^*, \beta, \lambda)$, $i = 1,\ 2,\ 3$. Take $\beta = 2$ and $\lambda = 3$. Assume $\boldsymbol{\alpha}_3 = (6, 2, 4)$ and $\boldsymbol{\alpha}_3^* = (10, 12, 8)$. Clearly, $\boldsymbol{\alpha}_3 \preccurlyeq_w \boldsymbol{\alpha}_3^*$. Now, plot the difference $r_{X_{1:3}}(x) - r_{X_{1:3}^*}(x)$ in Figure 1, which shows that $X_{1:3}^* \leq_{hr} X_{1:3}$.

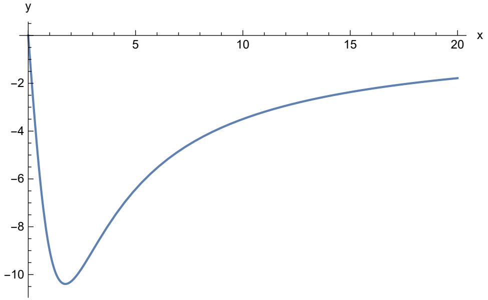

Figure 1: The graph presents the plot of $r_{X_{1:3}}(x) - r_{X_{1:3}^*}(x)$ as in Example 1.

Theorem 2

Let $X_1, \ldots, X_n$ $(X_1^*, \ldots, X_n^*)$ *be independent random variables with* $X_i \sim \mathcal{PL}(\alpha, \beta, \lambda_i)$ $(X_i^* \sim \mathcal{PL}(\alpha, \beta, \lambda_i^*))$, $i = 1, \ldots, n$. *Then, for any* α, $\beta > 0$, *we have*

$$\boldsymbol{\lambda}_n \overset{w}{\preccurlyeq} \boldsymbol{\lambda^*}_n \Rightarrow X_{1:n}^* \leq_{hr} X_{1:n}.$$

Proof.

- The hazard rate function of $X_{1:n}$ is given by

$$r_{X_{1:n}}(x) = \alpha\beta x^{\beta-1} \sum_{i=1}^{n} f(x; \lambda_i), \tag{8}$$

  where $f(x; \lambda_i) = \frac{1}{(\lambda_i + x^\beta)}$.

- To prove the stated result, it is sufficient to show that $r_{X_{1:n}}(x)$ in (5) is Schur-convex and decreasing in λ_i's.

□

- The partial derivative of $r_{X_{1:n}}(x)$ with respect to λ_i is

$$\frac{\partial r_{X_{1:n}}(x)}{\partial \lambda_i} = -\frac{\alpha\beta x^{\beta-1}}{(\lambda_i + x^{\beta})^2} \leq 0. \tag{9}$$

So, $r_{X_{1:n}}(x)$ is decreasing in each λ_i. Now, to show the Schur-convexity of $r_{X_{1:n}}(x)$, it is required to establish that $\frac{\partial^2 f(x;\, \lambda_i)}{\partial^2 \lambda_i} \geq 0$.

- In this sequel, obtain

$$\frac{\partial^2 f(x; \lambda_i)}{\partial^2 \lambda_i} = \frac{2}{(\lambda_i + x^{\beta})^3} \geq 0. \tag{10}$$

Hence, the required result follows from Lemma 1 and Lemma 2. This completes the proof of the theorem.

The following counterexample demonstrates that the described hazard rate ordering may not hold if the conditions in Theorem 2 are not hold.

Counterexample 1

Consider three independent random variables X_1, X_2, X_3*, such that* $X_i \sim \mathcal{PL}(\alpha, \beta, \lambda_i)$*,* $i = 1,\ 2,\ 3$*. Assume* $\lambda_1 = 3.5$*,* $\lambda_2 = 9.3$*, and* $\lambda_3 = 20.5$*. Take another set of three independent random variables* X_1^*, X_2^*, X_3^**, such that* $X_i^* \sim \mathcal{PL}(\alpha, \beta, \lambda_i^*)$*,* $i = 1,\ 2,\ 3$*. Here, let* $\lambda_1^* = 7.2$*,* $\lambda_2^* = 10.8$*, and* $\lambda_3^* = 12.4$*. Also, take* $\alpha = 3$ *and* $\beta = 15$*. Clearly,* $\boldsymbol{\lambda}_3 \overset{w}{\not\preccurlyeq} \boldsymbol{\lambda}_3^*$*. Now, plot the difference* $r_{X_{1:3}}(x) - r_{X_{1:3}^*}(x)$ *in Figure 2, which shows that* $X_{1:3}^* \not\leq_{hr} X_{1:3}$*.*

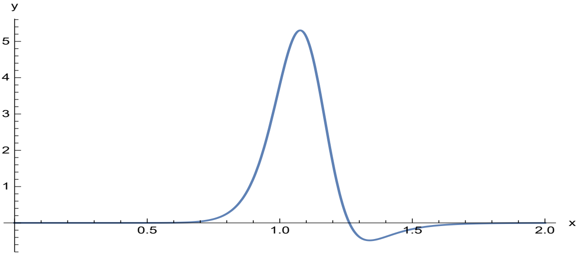

Figure 2: The graph presents the plot of $r_{X_{1:3}}(x) - r_{X_{1:3}^*}(x)$.

Theorem 3

Let $\{X_1, \ldots, X_n\}$ $(\{X_1^*, \ldots, X_n^*\})$ *be two sets of independent heterogeneous random variables such that* $X_i \sim \mathcal{PL}(\alpha_i, \beta, \lambda)$ $(X_i^* \sim \mathcal{PL}(\alpha_i^*, \beta, \lambda))$*, where* $i = 1, \ldots, n$*. Then, for any* β*,* $\lambda > 0$*, we have*

$$\boldsymbol{\alpha}_n \overset{w}{\preccurlyeq} \boldsymbol{\alpha^*}_n \Rightarrow X_{n:n} \leq_{rhr} X_{n:n}^*.$$

Proof.

The proof is omitted here for the sake of brevity. □

Theorem 4

Consider $X_1, \ldots, X_n$ $(X_1^*, \ldots, X_n^*)$ *be independent random variables such that* $X_i \sim \mathcal{PL}(\alpha, \beta, \lambda_i)$ $(X_i^* \sim \mathcal{PL}(\alpha, \beta, \lambda_i^*))$*,* $i = 1, \ldots, n$*. Then, for any* α*,* $\beta > 0$*, we have*

$$\max \boldsymbol{\lambda^*}_n \leq \min \boldsymbol{\lambda}_n \Rightarrow X_{1:n}^* \leq_{lr} X_{1:n}.$$

Proof.

The proof is omitted here for the sake of brevity. □

Conclusion and future work

In this presentation, the stochastic comparison results between the lifetimes of the series and parallel systems in the sense of the various stochastic orderings, such as hazard rate, reversed hazard rate and likelihood ratio orderings have been derived. It has been assumed that the component lifetimes of the systems follow power-Lomax distributions.
Next, two problems which can be studied in future are discussed.

- Here, the stochastic comparison results have been studied for the independent component lifetimes. The same problem can be studied for the dependent heterogeneous components with the help of Archimedean copula.
- Further, one may consider the mixture models of heterogeneous power-Lomax distributions and study some stochastic comparison results in the sense of various stochastic orderings.

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Thank you for your attention!