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On the Seidel matrix of threshold graphs *

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Abstract

Threshold graph has an important role in graph theory and several applied arreas such as computer science, scheduling theory etc. Here threshold graphs with its binary string representation are considered. Let G be a connected threshold graph with adjacency and Seidel matrix A and S respectively. Then $S = J - I - 2A$. We study the spectral properties of S. A recurrence formula for characteristic polynomial of S, multiplicity of the eigenvalues ± 1 of S and eigenvalue bounds are obtained. Characterisation of threshold graphs with at most five distinct Seidel eigenvalue is shown also. We obtain several bounds on Seidel energy of G. It is shown that our bound is better than Haemers' bound in practical. Finally, we prove a very uncommon result for threshold graphs: two threshold graphs may be cospectral on Seidel matrix. Here we define a class of such threshold graphs.

Keywords: threshold graph, Seidel matrix, quotient matrix, Seidel energy.
AMS subject classifications. 05C50

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Seidel matrix of threshold graphs

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Contents

Introduction

- Attention in threshold graphs gained momentum during last 40 years.
- It has many applications in various field.
- Threshold graphs are a special case of cographs and split graphs.
- The motivation for considering threshold graphs comes from the spectral graph theory.
- An important thing is that-a threshold graph with n vertices can always be represented by a finite binary string of length n.
- For $n \geq 2$ there are 2^{n-2} connected threshold graph on n vertices.

Mathematical Setup

Suppose $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_n\}$ is a real sequence and $A_B = (\alpha_{ij})$ is a symmetric matrix whose entries are

$$\alpha_{ij} = \begin{cases} \alpha_i, & \text{for } i > j, \\ \alpha_j, & \text{for } j > i, \\ 0, & \text{otherwise.} \end{cases}$$

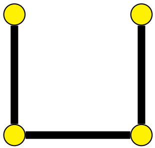
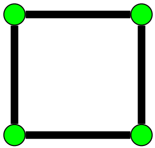
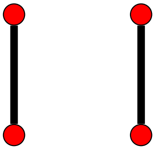
$$A_B = \begin{bmatrix} 0 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_2 & 0 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ \alpha_3 & \alpha_3 & 0 & \alpha_4 & \cdots & \alpha_n \\ \alpha_4 & \alpha_4 & \alpha_4 & 0 & \cdots & \alpha_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_n & \alpha_n & \alpha_n & \alpha_n & \cdots & 0 \end{bmatrix}.$$

The matrix A_B is called the **threshold matrix** associated with the real sequence B.

Definition

A P_4, C_4 and $2K_2$ free graph is called a ***Threshold graph***.

That means a threshold graph does not contain:

Seidel matrix

Let Γ be a threshold graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and binary string $b = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$. Then Clearly A_B is the adjacency matrix of Γ. The **Seidel matrix** (S) is defined by

$$S = J - I - 2A_B.$$

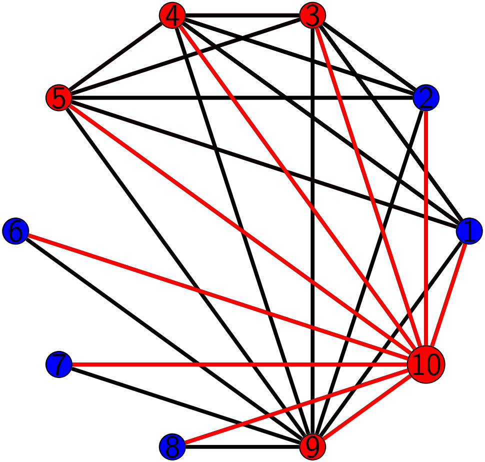
$$\implies S = \begin{vmatrix} 0 & 1-2\alpha_2 & 1-2\alpha_3 & \ldots & 1-2\alpha_n \\ 1-2\alpha_2 & 0 & 1-2\alpha_3 & \ldots & 1-2\alpha_n \\ 1-2\alpha_3 & 1-2\alpha_3 & 0 & \ldots & 1-2\alpha_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1-2\alpha_n & 1-2\alpha_n & 1-2\alpha_n & \ldots & 0 \end{vmatrix},$$

Consider $1 - 2\alpha_i = \beta_i$ for $i = 2, 3, ..., n-1$, then S takes the form

$$S = \begin{vmatrix} 0 & \beta_2 & \beta_3 & \ldots & \beta_n \\ \beta_2 & 0 & \beta_3 & \ldots & \beta_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_n & \beta_n & \beta_n & \ldots & 0 \end{vmatrix}, \qquad (1)$$

Example: Construction of Threshold Graphs

$b = 0011100011 = 0^2 1^3 0^3 1^2$



Adjacency matrix

The adjacency matrix A and the Seidel matrix S of a threshold graph Γ with binary string $b = 0^{s_1}1^{t_1}0^{s_2}1^{t_2}\cdots 0^{s_k}1^{t_k}$ are the square matrices of size n, given by

$$A = \begin{bmatrix} O_{s_1} & J_{s_1\times t_1} & O_{s_1\times s_2} & J_{s_1\times t_2} & \cdots & J_{s_1\times t_k} \\ J_{t_1\times s_1} & (J-I)_{t_1} & O_{t_1\times s_2} & J_{t_1\times t_2} & \cdots & J_{t_1\times t_k} \\ O_{s_2\times s_1} & O_{s_2\times t_2} & O_{s_2} & J_{s_2\times t_2} & \cdots & J_{s_2\times t_k} \\ J_{t_2\times s_1} & J_{t_2\times t_1} & J_{t_2\times s_2} & (J-I)_{t_2} & \cdots & J_{t_2\times t_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ J_{t_k\times s_1} & J_{t_k\times t_1} & J_{t_k\times s_2} & J_{t_k\times t_2} & \cdots & (J-I)_{t_k} \end{bmatrix},$$

Seidel matrix

$$S = \begin{bmatrix} (J-I)_{s_1} & -J_{s_1\times t_1} & J_{s_1\times s_2} & -J_{s_1\times t_2} & \cdots & -J_{s_1\times t_k} \\ -J_{t_1\times s_1} & (I-J)_{t_1} & J_{t_1\times s_2} & -J_{t_1\times t_2} & \cdots & -J_{t_1\times t_k} \\ J_{s_2\times s_1} & J_{s_2\times t_1} & (J-I)_{s_2} & -J_{s_2\times t_2} & \cdots & -J_{s_2\times t_k} \\ -J_{t_2\times s_1} & -J_{t_2\times t_1} & -J_{t_2\times s_2} & (I-J)_{t_2} & \cdots & -J_{t_2\times t_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -J_{t_k\times s_1} & -J_{t_k\times t_1} & -J_{t_k\times s_2} & -J_{t_k\times t_2} & \cdots & (I-J)_{t_k} \end{bmatrix}. \quad (2)$$

where, $O_{m\times n}$, $J_{m\times n}$ are all zero block matrix and all 1 block matrix respectively of size mn.

Quotient matrix

Let $\pi = \{C_1, C_2, \cdots, C_{2k}\}$ be an equitable partition of the vertex set of Γ, such that, C_1 contains first s_1 vertices, C_2 contains next t_1 vertices and so on. Then the adjacency quotient matrix(Q_A) and the Seidel quotient matrix (Q_S) corresponding to the same equitable partition π are the square matrices of size $2k$, given by

$$Q_A = \begin{bmatrix} 0 & t_1 & 0 & t_2 & \dots & t_k \\ s_1 & t_1 - 1 & 0 & t_2 & \dots & t_k \\ 0 & 0 & 0 & t_2 & \dots & t_k \\ \dots & \dots & \dots & & & \\ s_1 & t_1 & s_2 & t_2 & \dots & t_k - 1 \end{bmatrix},$$

Quotient matrix

$$Q_S = \begin{bmatrix} s_1 - 1 & -t_1 & s_2 & -t_2 & s_3 & \dots & -t_k \\ -s_1 & -(t_1 - 1) & s_2 & -t_2 & s_3 & \dots & -t_k \\ s_1 & t_1 & s_2 - 1 & -t_2 & s_3 & \dots & -t_k \\ -s_1 & -t_1 - & -s_2 & -(t_2 - 1) & s_3 & \dots & -t_k \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -s_1 & -t_1 & -s_2 & -t_2 & -s_3 & \dots & -(t_k - 1) \end{bmatrix}.$$

Partition representation

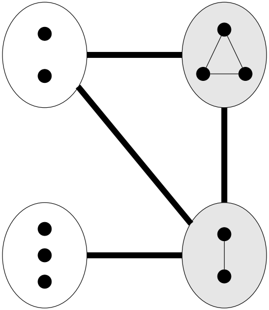


Figure: $b = 0011100011$

Contribution

We prove some important properties of Q_S and derive the formula for the multiplicity of the Seidel eigenvalues ± 1. We characterized all the threshold graphs which have at most five distinct Seidel eigenvalues. Finally, it is shown that the cospectrality may be occured if Seidel matrix is considered instead of adjacency matrix. Few results are given below.

Theorem

Q_S is diagonalizable.

Sketch of proof: Consider $D = diag(s_1, t_1, s_2, t_2, \ldots, s_k, t_k)$. Then Q_S is similar to $D^{\frac{1}{2}} Q_S D^{-\frac{1}{2}}$, a symmetric matrix.

Theorem

Q_A and Q_S are non singular.

Sketch of proof: By *Theorem* 2.3 of Banerjee and Mehatari [9],

Q_A and Q_S are non-singular

we have $det(Q_A) = (-1)^k s_1 t_1 s_2 t_2 \dots s_k t_k$.
Q_S is row equivalent to the following tridiagonal matrix.

$$T = \begin{bmatrix} 1-2s_1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2t_1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & -2s_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2t_2 & -1 & \dots & 0 & 0 \\ \dots & & \dots & & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & -2s_k & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2t_k - 1 \end{bmatrix}$$

By *Algorithm* 2.1 of Mikkawy [14], we see that $det(T) \neq 0$.
Thus, $det(Q_S) \neq 0$.

Characteristic polynomial of S

Theorem

Let $b = \alpha_1\alpha_2\ldots\alpha_n$ be the binary string of a threshold graph and let $b_r = \alpha_1\alpha_2\alpha_3\ldots\alpha_r$. Suppose $\Phi_r(x)$ denote the characteristic polynomial of Seidel matrix of the threshold graph with binary string b_r, then the characteristic polynomial, $\Phi_n(x)$ of the Seidel matrix S given by (1) is obtained by the following recurrence formula

$$\Phi_r(x) = 2(x + \beta_{r-1})\Phi_{r-1}(x) - 2(x + \beta_{r-1})^2\Phi_{r-2}(x),$$

where $\Phi_1(x) = x$ and $\Phi_2(x) = x^2 - 1$.

Sketch of proof: We consider two cases.
Case I. $\beta_r = \beta_{r-1}$.
Case II. $\beta_r = -\beta_{r-1}$.

Spectrum of Q_S

Theorem

$-1 \notin Spec(Q_S)$ if $t_k > 1$.

Sketch of proof: Let us assume that Q_S has the eigenvalue -1. Then there exists a non zero vector $X = (x_1 \ x_2 \ x_3 \ \ldots \ x_{2k})^t$ such that $Q_S X = -X$. Solving the system $Q_S X = -X$, we have $x_i = 0$ for all $i = 1, 2, 3, \ldots, 2k$, if $t_k \neq 1$.

Corollary

If $t_k = 1$, then -1 is a simple eigenvalue of Q_S.

Sketch of proof: Here $X = (0 \ \ldots \ 0 \ 1 \ s_k)$ is an eigenvector corresponding to the eigenvalue -1.

Spectrum of Q_S

Theorem

$1 \notin Spec(Q_S)$ if $s_1 > 1$.

Sketch of proof: Let us assume that Q_S has the eigenvalue 1. Then there exists a non zero vector $X = (x_1\ x_2\ x_3\ \dots\ x_{2k})^t$ such that $Q_S X = X$. Solving the system $Q_S X = X$, we have $x_i = 0$ for all $i = 1, 2, 3, \dots, 2k$, if $s_1 \neq 1$.

Corollary

If $s_1 = 1$, then 1 is a simple eigenvalue of Q_S.

Sketch of proof: Here $X = (t_1\ \ -1\ 0\ \dots\ 0)^t$ is an eigenvector corresponding to the eigenvalue 1.

Q_S has simple eigenvalue

Theorem

All eigenvalues of Q_S are simple.

Sketch of proof: Suppose λ is an eigenvalue of Q_S. Let $X = (x_1\ x_2\ x_3\ \dots\ x_{2k})^t$ be an eigenvector corresponding to λ such that $x_l \neq 0$ and $x_m = 0$ for all $m < l$, where l is minimal. Then $l = 2p, 1 \leq p \leq k$. We already proved that $\lambda = \pm 1$ is a simple eigenvalue. Now we prove the theorem for $\lambda \neq \pm 1$. Then from the relation $Q_S X = \lambda X$, we have $x_{l+i} = c_{l+i} x_l$, where c_{l+i} are the constants depend on λ. Thus we can construct X as $X = x_l (0\ 0\ 0\ \dots\ 0\ 1\ c_{l+1}\ c_{l+2}\ c_{l+3}\ \dots\ c_{2k})^t$. Now if $X^{'} = (x_1^{'}\ x_2^{'}\ x_3^{'}\ \dots\ x_{2k}^{'})$ be the another eigenvector corresponding to λ, then we see that $X^{'}$ is a constant multiple of X. Hence the geometric multiplicity of λ is one. Again Q_S is diagonalizable. Hence algebraic multiplicity of λ is also one.

Properties of Q_S

1. Q_S is diagonalizable.
2. $SP = PQ_S$.
3. Every eigenvalue of Q_S is also an eigenvalue of S.
4. 0 is not an eigenvalue of Q_S.
5. $\pm 1 \notin Spec(Q_S)$ if $s_1 > 1$, and $t_k > 1$.
6. Eigenvalues of Q_S are simple.
7. Eigenvalues of Q_S are real.

Multiplicity of -1

Theorem

Let $n_{-1}(S)$ denotes the multiplicity of the eigenvalue -1 of S. Then

$$n_{-1}(S) = \begin{cases} \sum s_i - k, \ \textit{for } t_k > 1, \\ \sum s_i - k + 1, \ \textit{for } t_k = 1. \end{cases}$$

Sketch of proof: If $t_k > 1$, then $-1 \notin Spec(Q_S)$. Therefore the multiplicity of the eigenvalue -1 in S is exactly $\sum(s_i - 1)$.
If $t_k = 1$, then $-1 \in Spec(Q_S)$ and it is simple.
Thus,

$$n_{-1}(S) = \sum s_i - k + 1.$$

Multiplicity of $+1$

Theorem

Let $n_{+1}(S)$ denotes the multiplicity of the eigenvalue $+1$ of S. Then

$$n_{+1}(S) = \begin{cases} \sum t_i - k, & \text{for } s_1 > 1, \\ \sum t_i - k + 1, & \text{for } s_1 = 1. \end{cases}$$

Sketch of proof: If we take $s_1 > 1$. Then $1 \notin Spec(Q_S)$. Therefore the multiplicity of the eigenvalue 1 in S is exactly $\sum(t_i - 1)$.
If we take $s_1 = 1$. Then $1 \in Spec(Q_S)$ and it is simple.
Thus,

$$n_{+1}(S) = \sum t_i - k + 1.$$

Threshold graphs with at most five distinct Seidel eigenvalue

Corollary

*Let Γ be a threshold graph with binary string **b**. Then Γ has two distinct Seidel eigenvalues if and only if one of the following conditions hold.*
(i) $\boldsymbol{b} = 0^{s_1}1, \; s_1 \geq 1,$
(ii) $\boldsymbol{b} = 01^{t_1}, \; t_1 \geq 1.$

Corollary

*Let Γ be a threshold graph with binary string **b**. Then Γ has four distinct Seidel eigenvalues if and only if one of the following conditions hold.*
(i) $\boldsymbol{b} = 0^{s_1}1^{t_1}, \; s_1 > 1, \; t_1 > 1,$
(ii) $\boldsymbol{b} = 01^{t_1}0^{s_2}1.$

Threshold graphs with at most five distinct Seidel eigenvalue

Corollary

No threshold graph can have three distinct Seidel eigenvalues.

Corollary

Let Γ *be a threshold graph with binary string* $\boldsymbol{b}$ *. Then* Γ *has five distinct Seidel eigenvalues if and only if* $\boldsymbol{b} = 01^{t_1}0^{s_2}1^{t_2}$ *with* $t_2 > 1$ *or* $\boldsymbol{b} = 0^{s_1}1^{t_1}0^{s_2}1$ *with* $s_1 > 1$.

Seidel Energy [E(S)]

Theorem

Let G is a threshold graph with the binary string $b = 0^{s_1}1^{t_1}0^{s_2}\dots0^{s_k}1^{t_k}$, then

$$u \leq E(S) \leq v$$

where
$u = n_{-1}(S) + n_{+1}(S) + \sqrt{(n^2 - 2n + 2k) + 2k(2k-1)|\det Q_S|^{\frac{1}{k}}}$,
$v = n_{-1}(S) + n_{+1}(S) + \sqrt{2k(n^2 - 2n + 2k)}$.

Proof:

$$p_1 \geq p_2^{\frac{1}{2}} \geq p_3^{\frac{1}{3}} \geq \cdots \geq p_n^{\frac{1}{n}},$$

where p_k is the average of products of $k-$element subset of the set $\{a_1, a_2, a_3, \dots, a_n\}$.

Two threshold graphs may be cospectral

Theorem

Let us consider two threshold graphs Γ_1 and Γ_2 on n vertices with the binary string $b_1 = 0^{n-2}1^2$ and $b_2 = 010^{n-3}1$ respectively. Then Γ_1 and Γ_2 are always cospectral on Seidel matrix.

Sketch of proof: The characterastic equation of Q_S for the binary strings b_1 and b_2 are respectively

$$x^2 + (4-n)x + (7-3n) = 0,$$

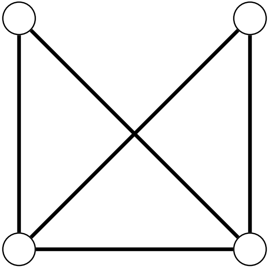
$$(x^2-1)[x^2 + (4-n)x + (7-3n)] = 0.$$

Thus both the strings have same Seidel spectrum $\{-1^{n-3}, 1, \alpha, \beta\}$, where α, β are the roots of the equation $x^2 + (4-n)x + (7-3n) = 0$.

Example of cospectral threshold graphs

Example

If we take $n = 4$ in previous theorem, we get threshold graphs Γ_1 and Γ_2 with binary string $b_1 = 0011$ and $b_2 = 0101$ respectively. In that case, Γ_1 and Γ_2 are not isomorphic (see Figure 2) but they both have eigenvalues $\pm 1, \pm\sqrt{5}$.



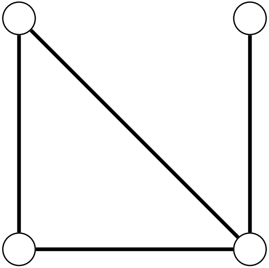


Figure: Non isomorphic cospectral threshold graphs with 4 vertices.

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That's all.