Equidistant Paths in Graphs

Niranjan Balachandran ⋆1 and Sajith P. ⋆⋆2 Tapas Kumar Mishra ⋆⋆⋆3 L. Sunil Chandran †2

[1] Department of Mathematics, Indian Institute of Technology, Bombay 400076, India
[2] Department of Computer Science and Automation, Indian Institute of Science, Bangalore, India
[3] Department of Computer Science and Engineering, National Institute of Technology, Rourkela 769008, India

Abstract. Let n and k be integers, $n \geq k \geq 1$. A graph G is said to admit property $\mathcal{P}_k$ if for any distinct pair $x, y \in V(G)$, there exists k internally vertex disjoint paths between x and y of the same length. Consider the following family of graphs.

$$\mathcal{G}_k^n := \{G_n : G_n \text{ admits property } \mathcal{P}_k\}.$$

There are two interesting directions in the study of $\mathcal{G}_k^n$. Firstly, in the extremal direction, it is interesting to estimate the sparsity of graphs admitting property $\mathcal{P}_k$. That is, estimation of $\nu(n,k) = \min\{|E(G_n)| : G_n \in \mathcal{G}_k^n\}$. The other direction is structural: what properties in the graph ensures admittance of property $\mathcal{P}_k$. In this paper, we tackle the extremal question followed by some structural results on the same.

1 Introduction

Let n and k be integers, $n \geq k \geq 1$. A graph G is said to admit property $\mathcal{P}_k$ if for any distinct pair $x, y \in V(G)$, there exists k internally vertex disjoint paths between x and y of the same length. Consider the following family of graphs.

$$\mathcal{G}_k^n := \{G_n : G_n \text{ admits property } \mathcal{P}_k\}. \tag{1}$$

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2 Some initial observations

It is intimidate that the smallest graph that admits property $\mathcal{P}_2$ is the K_4. In order to extend K_4 to a larger graph admitting $\mathcal{P}_2$, one may try something like a mycielskian of K_4: corresponding to $\{v_1, v_2, v_3, v_4\}$ in the K_4, add $\{u_1, u_2, u_3, u_4\}$ to G such that $V(G) = \{v_1, \dots, v_4\} \cup \{u_1, \dots, u_4\}$ and $E(G) = E(K_4) \cup \{\{u_i, v_j\} : \{v_i, v_j\} \in E(K_4)\}$. It is not hard to verify that G admits property $\mathcal{P}_2$. Repeating the process, one may obtain a graph G_{2^k} on 2^k vertices with $6 \times 3^{k-1}$ edges. This observation yields the bound $\nu(n, 2) \leq 2n^{\log_2 3}$.

One important observation from the above example is that in order to obtain sparser graphs admitting property $\mathcal{P}_2$, we need some structural symmetry in the arrangement of edges. But how much more can we improve $\nu(n, 2)$? As it turns out, the Hamming cube Q_{2^k} also admits $\mathcal{P}_2$. In fact, the Hamming cube Q_{2^k} admits $\mathcal{P}_{\lceil \frac{k}{2} \rceil}$. In order to see this, consider two points $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ at a Hamming distance t and without loss of generality,

⋆ niranj@iitb.ac.in
⋆⋆ sajithiitb@gmail.com
⋆⋆⋆ mishrat@nitrkl.ac.in
† sunil@iisc.ac.in

let the coordinates where the points differ are exactly $1, \dots, t$. We show t equidistant paths of length exactly t in the following manner. Let P_1 denote the path $(x_1, x_2, \dots, x_t, \dots, x_k) \to (y_1, x_2, \dots, x_t, \dots, x_k) \to (y_1, y_2, \dots, x_t, \dots, x_k) \to \dots (y_1, y_2, \dots, y_t, \dots, x_k)$. Let P_2 denote the path $(x_1, x_2, x_3, \dots, x_t, \dots, x_k) \to (x_1, y_2, x_3, \dots, x_t, \dots, x_k) \to (x_1, y_2, y_3 \dots, x_t, \dots, x_k) \to \dots (x_1, y_2, \dots, y_t, \dots, x_k) \to (y_1, y_2, \dots, y_t, \dots, x_k)$. Similarly, let P_i denote the path where we start with $(x_1, x_2, \dots, x_t, \dots, x_k)$, then switch the ith coordinate followed by switching of the successive coordinates cyclically and ending at $(y_1, y_2, \dots, x_{i-1}, y_i, \dots, x_k) \to (y_1, y_2, \dots, y_t, \dots, x_k)$. It is easy to verify that each of the P_i, $1 \leq i \leq t$ are vertex disjoint and of the same length t. Similarly, there are $k - t$ equidistant paths of length exactly $t + 2$, where the first move and the last move is along a coordinate where $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$ have the same value and internal points are all the correction in Hamming weights along that coordinate. This observation yields the bound $\nu(n, \lceil \frac{k}{2} \rceil) \leq n \log n$, where $n = 2^k$.

A lower bound on $\nu(n, k)$ can be obtained by the following simple observation that any graph on n vertices admitting property $\mathcal{P}_k$ must have connectivity at least $k + 1$: otherwise, two adjacent vertices in the graph can never have k equidistant vertex disjoint paths. This gives the following lower bound to $\nu(n, k)$.

$$\nu(n, k) \geq \frac{n(k+1)}{2}. \tag{2}$$

The Hamming cube example gives an upper bound of $2nk$ for $\nu(n, k)$ when $n = 4^k$. We can improve the upper bound using grids in the following way. Consider the graph G_n where the vertices are the points on a $n^{1/k} \times n^{1/k} \times \dots \times n^{1/k}$ grid and a point $(x_1, x_2, \dots, x_k)$ is adjacent to $2k$ points, namely $(x_1 \pm 1, x_2, \dots, x_k)$, $(x_1, x_2 \pm 1, \dots, x_k)$, $\dots$, $(x_1, x_2, \dots, x_k \pm 1)$.