Sum of Ultra Maximal Monotone Operators and Operators of Type (D) in Grothendieck Spaces

Suvendu Ranjan Pattanaik
&
Dillip Kumar Pradhan

Department of Mathematics
National Institute of Technology Rourkela, India

ACFPTO-2018
$16^{th} - 18^{th}$ July, 2018
Department of mathematics, Faculty of Science
and
Chiang Mai University Chiang Mai, Thailand

Outline

Sum Problems

In 1970, Rockafellar [4] provided a solution for the maximality of the sum of two maximal monotone operators under the constraint, i.e, one domain must intersect the interior of other one ($domA \cap intdomB \neq \emptyset$) in reflexive spaces.
Crandall and Pazy gave another qualification constraint for maximality of sum of two monotone operators. Interestingly, this qualification constraint is suitable "in certain sense" for handling maximal monotone operator having domain with a empty interior in Hilbert spaces.

Introduction

Brezis, Crandall and Pazy [1, 5]extended it to reflexive spaces.
Let X be a nonzero reflexive Banach space, $A : X \rightrightarrows X^*$ and $B : X \rightrightarrows X^*$ be maximal monotone, and satisfy following constraint qualification conditions (Crandall-Pazy constraint), i.e.,

1. $domA \subset domB$,
2. $|Bx| \leq k(\|x\|)|Ax| + C(\|x\|)$, where $k : [0, \infty) \to [0, 1)$ and $C : [0, \infty) \to [0, \infty)$.

Then $A + B$ is maximal monotone operator [2, Theorem 4.3].
$|C| = \inf_{c \in C} \|c\|$

Problem

The sum of an ultra maximal monotone operator and a (D) type monotone operator with condition (i) and (ii) is maximal in Banach spaces which satisfy Grothendieck property (Grothendieck space) and weakly compactly generated property.

Maximal Monotone Operator

Definition (Monotone Operator)

A set-valued mapping $A : X \rightrightarrows X^*$ is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \mathrm{gra} A.$$

Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$ we say that (x, x^*) is monotonically related to gra$A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \;\; \forall (y, y^*) \in \mathrm{gra} A.$$

And a set valued mapping A is said to maximal monotone if A is monotone and A has no proper monotone extension(in the sense of graph inclusion). In other words A is maximal monotone if for any $(x, x^*) \in X \times X^*$ is monotonically related to gra$A$ then $(x, x^*) \in$ gra.

Ultramaximal Monotone Operator

A monotone operator $A : X \rightrightarrows X^*$ is said to be ultramaximal monotone [3, 6] if A is maximally monotone with respect to $X^{**} \times X^*$.
From the definition of ultramaximal monotone operator it is observed that ultramaximal monotone operators are maximal monotone operators. But the converse is not true for example, Let $A : X \rightrightarrows X^*$ be defined by $graA := X \times \{0\}$. Then $graA = \partial I_X$. But $graA \subsetneq X^{**} \times \{0\}$. This shows that A is not ultramaximal monotone though it is maximal monotone.

Type (D) Monotone Operator

Let $A : X \rightrightarrows X^*$ be maximally monotone. We say A is of dense type or type (D) [4] if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with

$$\inf_{(a,a^*) \in graA} \langle a - x^{**}, a^* - x^* \rangle \geq 0,$$

there exist a bounded net $(a_\alpha, a^*_\alpha)_{\alpha \in \Gamma}$ in $graA$ such that $(a_\alpha, a^*_\alpha)_{\alpha \in \Gamma}$ converges to (x^{**}, x^*) with respect to (weak$^*\times$ strong) norm

Grothendieck space

A real Banach space X is said to be Grothendieck [3] if every weak star convergence sequence is weakly convergent in X^*. Every reflexive Banach space is a Grothendieck space. But the converse is not true for example, the space of bounded nets on some directed set Γ, $l_\infty(\Gamma)$ [3].
A Banach space X is called as weakly compactly generated if there is a weakly compact set K in X such that $X = \overline{\text{span}}(K)$. $L_1(\mu)$ is weakly compactly generated if μ is $\sigma-$finite.

Notation

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

and

$$J_\epsilon(x) = \{x^* \in X^* | \ \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 \leq \langle x, x^* \rangle + \epsilon\}.$$

Gossez's monotone closure of J is defined by

$$\tilde{J} = J_{X^*}^{-1},$$

where J_{X^*} denotes the duality map on X^*.
Let $A : X \rightrightarrows X^*$ be an ultramaximally monotone operator. Then $A + J$ is an ultramaximal monotone operator and $ran(A + J) = X^*$ [6, Corollary 3.6].

Moreau-Yosida regularization

For a maximal monotone operator $B : X \rightrightarrows X^*$ of type (D), the Moreau-Yosida regularization and the resolvent of B, (see [1]) with regularization parameter $\lambda > 0$ are given by $B_\lambda : X \rightrightarrows X^*$ and $R_\lambda : X \rightrightarrows X^{**}$

$$B_\lambda = \big\{(x, x^*) \in X \times X^* : \exists z^{**} \in X^* s.t. \\ 0 \in \lambda x^* + \tilde{J}(z^{**} - x), x^* \in \tilde{B}(z^{**})\big\} \tag{1}$$

$$R_\lambda = \big\{(x, z^{**}) \in X \times X^* : \exists x^* \in X^* s.t. \\ 0 \in \lambda x^* + \tilde{J}(z^{**} - x), x^* \in \tilde{B}(z^{**})\big\}. \tag{2}$$

For a maximal monotone operator $A : X \rightrightarrows X^*$ we will denote $\tilde{A} : X^{**} \rightrightarrows X^*$ as

$\tilde{A} = \{(x^{**}, x^*) \in X^{**} \times X^* : (x^{**}, x^*) \text{ is monotonically related to} graA\}.$

When A is of type (D) $\tilde{A}$ is the unique maximal monotone extension on $X^{**} \times X^*$.

Facts

We recall some properties of B_λ.

Fact

([2, Theorem 3.6] and [5, Theorem 4.4]) Let $B : X \rightrightarrows X^*$ *be maximal monotone of type (D),* $\lambda > 0$. *Then*

1. B_λ *is maximal monotone of type (D).*
2. $dom(B_\lambda) = X$.
3. B_λ *maps bounded sets into bounded sets.*

Conclusion

we have proved that the sum of an ultra maximal monotone operator and an operator of type (D) is maximal monotone operator in Banach spaces which satisfy Grothendieck and weakly compactly generated properties. We extend the results to nonreflexive spaces assuming certain conditions those are automatically satisfied in reflexive spaces.

Bibliography

- Brézis, H., Crandall, M. G., Pazy, A. : Perturbations of Nonlinear Maximal Monotone Sets in Banach Spaces. Communications on pure and applied mathematics. XXIII, 123-144 (1970).
- Crandall, M. G., Pazy, A. : Nonlinear semi-groups of contractions and dissipative sets. J. Functional Anal. 3, 376-418 (1969).
- Bauschke, H.H., Simons, S. : Stronger maximal monotonicity properties of linear operators. Bulletin of the Australian Mathematical Society, 60, 163-174 (1999).
- Gossez, J.-P. : On a convexity property of the range of a maximal monotone operator. Pro- ceedings of the AMS. 55, 359-360 (1976).
- Marques Alves, M., Svaiter, B. F. : On Gossez type (D) maximal monotone operators. J. Convex Anal. 17(3), 1077-1088 (2010).

Bibliography

- Marques Alves, M., Svaiter, B. F. : Moreau-Yosida regularization of maximal monotone operators of type (D). Set-Valued Var. Anal. 19(1), 97-106, 2011.
- Marques Alves, M., Svaiter, B. F. : On the surjectivity properties of perturbations of maximal monotone operators in non-reflexive Banach spaces. J. Convex Anal. 18(1), 209-226 (2011).
- Morrison, T. J. : Functional Analysis An Introduction to Banach Space Theorey. A Wiley Interscience publication, Jhon Wiley & Sons, New York, 2001.
- Rockafellar, R.T. : On the maximality of sums of nonlinear monotone operators. T. Am. Math. Soc. 149, 75-88 (1970).
- Simons, S. : From Hahn-Banach to Monotonicity. Springer-Verlag, 2008.
- Yao, L. : Finer properties of ultramaximally monotone operators on Banach spaces. J. Convex Anal. 23(4), (2016).

Thank You