*Published in International Journal of Mathematics and Mathematical Sciences,*
*Vol. 2005, No. 8 Year 2005 Page 1189 – 1200*
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Author's postprint version

# CO-BALANCING NUMBERS AND CO-BALANCERS

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**Key words**: Triangular numbers, Balancing numbers and balancers, Pronic numbers, Recurrence relations

**Abstract**: Co-balancing numbers and co-balancers are defined and introduced. Many properties of co-balancing numbers are explored. A link between the Pythagorean triplets and the co-balancing numbers is also established.

## 1. INTRODUCTION

Recently, Behera and Panda [1] introduced *balancing numbers* $n \in \mathbb{Z}^+$ as solutions of the equation

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r), \qquad (1)$$

calling $r \in \mathbb{Z}^+$, the *balancer* corresponding to the balancing number *n*. 6, 35 and 204 are examples of balancing numbers with balancers 2, 14 and 84 respectively. Behera and Panda [1] also proved that a positive integer *n* is a balancing number if and only if $n^2$ is a triangular number, that is, $8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2, in [1], 1 is accepted as a balancing number being the positive square root of the square triangular number 1.

In [3] and [4] Subramaniam has explored many interesting properties of square triangular numbers without linking to balancing numbers because of their unavailability in literature till that time. In a latter paper [5], he introduced the concept of almost square triangular numbers (triangular numbers that differ from a square by unity) and linked them with the square triangular numbers. In this paper we introduce co-balancing numbers that are linked to a third category of triangular numbers that can be expressed as product of two consecutive natural numbers (approximately as the arithmetic mean of squares of two consecutive natural numbers i.e. $[n^2+(n+1)^2]/2 \approx n(n+1)$). Indeed, in what follows, we introduce the co-balancing numbers in a more natural way like the balancing numbers.

By slightly modifying equation (1), we call $n \in \mathbb{Z}^+$ a *co-balancing number* if

$$1+2+\cdots+n = (n+1)+(n+2)+\cdots+(n+r) \qquad (2)$$

for some $r \in \mathbb{Z}^+$. Here, we call *r* the *co-balancer* corresponding to the co-balancing number *n*.

The first three co-balancing numbers are 2, 14 and 84 with co-balancers 1, 6 and 35 respectively.

It is clear from (2) that, $n$ is a co-balancing number with co-balancer $r$ if and only if

$$n(n+1) = \frac{(n+r)(n+r+1)}{2},$$

which when solved for $r$ gives

$$r = \frac{-(2n+1)+\sqrt{8n^2+8n+1}}{2}. \tag{3}$$

It follows from (3) that $n$ is a co-balancing number if and only if $8n^2+8n+1$ is a perfect square that is, $n(n+1)$ is a triangular number. Since $8\times0^2+8\times0+1=1$ is a perfect square, we accept 0 as a co-balancing number, just like Behera and Panda [1] accepted 1 as a balancing number, though, by definition, a co-balancing number should be grater than 1.

From the above discussion it is clear that if $n$ is a co-balancing number, then both $n(n+1)$ and $n(n+1)/2$ are triangular numbers. Thus our search for co-balancing number is confined to the pronic triangular numbers, i.e., triangular numbers that are also pronic numbers. It is worth mentioning here that a positive integer is called a *pronic number* if it is expressible in the form $n(n+1)$ for some positive integer $n$. Since $n < \sqrt{n(n+1)} < n+1$, it follows that if $T$ is a pronic triangular number then $[\sqrt{T}]$ must be a co-balancing number, where $[\cdot]$ denote the greatest integer function. For example, $T = 6$ is a pronic triangular number and therefore $[\sqrt{6}] = 2$ is a co-balancing number.

## 2. SOME FUNCTIONS OF CO-BALANCING NUMBERS

In this section we introduce some functions of co-balancing numbers that also generate co-balancing numbers. For any two co-balancing numbers $x$ and $y$, we consider the following functions:

$$f(x) = 3x + \sqrt{8x^2+8x+1} + 1,$$

$$g(x) = 17x + 6\sqrt{8x^2+8x+1} + 8,$$

$$h(x) = 8x^2 + 8x + 1 + (2x+1)\sqrt{8x^2+8x+1} + 1,$$

$$t(x,y) = \tfrac{1}{2}\Big[2(2x+1)(2y+1) + (2x+1)\sqrt{8y^2+8y+1} + (2y+1)\sqrt{8x^2+8x+1} + \sqrt{8x^2+8x+1}\sqrt{8y^2+8y+1} - 1\Big].$$

We first prove that the above functions always generate co-balancing numbers.

***Theorem* 2.1:** *For any two co-balancing numbers $x$ and $y$, $f(x)$, $g(x)$, $h(x)$ and $t(x, y)$ are all co-balancing numbers.*

***Proof***: Suppose that $u = f(x)$. Then $x < u$ and

$$x = 3u - \sqrt{8u^2 + 8u + 1} + 1.$$

Since $x$ and $u$ are non-negative integers, $8u^2 + 8u + 1$ must be a perfect square, and hence $u$ is a co-balancing number.

Since $f(f(x)) = g(x)$, it follows that $g(x)$ is also a co-balancing number.

We can also directly verify that $8h^2(x) + 8h(x) + 1$ and $8t^2(x, y) + 8t(x, y) + 1$ are perfect squares so that $h(x)$ and $t(x, y)$ are co-balancing numbers. But these verifications would involve lengthy algebra. To avoid algebraic complications, we provide relatively easy proofs of these results in Section 6 using Theorem 6.1.

Next we show that for any co-balancing number $x$, $f(x)$ is not merely a co-balancing number, but it is the co-balancing number next to $x$.

***Theorem* 2.2** *If $x$ is any co-balancing number, then the co-balancing number next to $x$ is $f(x) = 3x + \sqrt{8x^2 + 8x + 1} + 1$ and consequently the previous one is $\tilde{f}(x) = 3x - \sqrt{8x^2 + 8x + 1} + 1$.*

*Proof:* The proof of the fact that $f(x) = 3x + \sqrt{8x^2 + 8x + 1} + 1$ is the co-balancing number next to $x$ is exactly same as the proof of Theorem 3.1 of [1], and hence it is omitted. Since $f(\tilde{f}(x)) = x$, it follows that $\tilde{f}(x)$ is the largest co-balancing number less than $x$.

## 3. RECURRANCE RELATIONS FOR CO-BALANCING NUMBERS

For $n = 1, 2, \cdots$, let $b_n$ be the $n^{\text{th}}$ co-balancing number. We set $b_1 = 0$. The next two co-balancing numbers are $b_2 = 2$ and $b_3 = 14$.

Behera and Panda [1], while accepting 1 as a balancing number, have set $B_0 = 1$, $B_1 = 6$, and so on, using the symbol $B_n$ for the $n^{\text{th}}$ balancing number. To standardize the notation at par with Fibonacci numbers, we relabel the balancing numbers by setting $B_1 = 1$, $B_2 = 6$ and so on.

Theorem 2.2 suggests

$$b_{n+1} = 3b_n + \sqrt{8b_n^2 + 8b_n + 1} + 1$$

and

$$b_{n-1} = 3b_n - \sqrt{8b_n^2 + 8b_n + 1} + 1.$$

Adding the last two equations we arrive at the conclusion that the co-balancing numbers obey the second-order linear recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2. \qquad (4)$$

An immediate consequence of (4) is the following:

***Theorem 3.1*** *Every co-balancing number is even.*

***Proof****:* The proof is based on mathematical induction. The first two co-balancing numbers $b_1 = 0$ and $b_2 = 2$ are even. Assume that $b_n$ is even for $n \le k$. Using (4) one can easily see that $b_{k+1}$ is also even.

Using the recurrence relation (4) we can derive some other interesting relations among the co-balancing numbers.

***Theorem 3.2***

(a) $(b_n - 1)^2 = 1 + b_{n-1}b_{n+1}$,

(b) *for* $n > k \geq 2$,

$$b_n = b_k + B_k b_{n-k+1} - B_{k-1} b_{n-k},$$

(c) $b_{2n} = B_n b_{n+1} - b_n (B_{n-1} - 1)$,

(d) $b_{2n+1} = (B_{n+1} + 1) b_{n+1} - B_n b_n$.

***Proof:*** From (4) we have

$$\frac{b_{n+1} + b_{n-1} - 2}{b_n} = 6.$$

Replacing $n$ by $n-1$ we obtain

$$\frac{b_n + b_{n-2} - 2}{b_{n-1}} = 6,$$

which implies

$$\frac{b_{n+1} + b_{n-1} - 2}{b_n} = \frac{b_n + b_{n-2} - 2}{b_{n-1}},$$

which when rearranged gives

$$(b_n - 1)^2 - b_{n-1}b_{n+1} = (b_{n-1} - 1)^2 - b_{n-2}b_n.$$

Now iterating recursively we obtain

$$(b_n - 1)^2 - b_{n-1}b_{n+1} = (b_2 - 1)^2 - b_1 b_3 = (2-1)^2 - 0 \times 14 = 1,$$

from which (a) follows.

The proof of (b) needs an important link between balancing numbers and co-balancing numbers, which is to be established in the next section after Theorem 4.1. Till then, we postpone the proof of (b).

The proof of (c) follows from (b) by replacing $n$ by $2n$ and $k$ by $n$. Similarly the proof of (d) follows from (b) by replacing $n$ by $2n$+1 and $k$ by $n$+1.

## 4. GENERATING FUNCTION FOR CO-BALANCING NUMBERS

In Section 3 we developed the recurrence relation $b_{n+1} = 6b_n - b_{n-1} + 2$ for co-balancing numbers. Using this recurrence relation we first obtain the generating function for co-balancing numbers and then establish a very interesting link between balancing numbers and co-balancing numbers.

Recall that the ordinary generating function ([3], p.29) for a sequence $\{x_n\}_{n=0}^{\infty}$ of real numbers is defined as

$$g(s) = \sum_{n=0}^{\infty} x_n s^n .$$

From [1] we know that the generating function for the sequence of balancing numbers $\{B_n\}_{n=0}^{\infty}$ with the definition $B_0 = 1$, $B_1 = 6, \cdots$ is

$$g(s) = \frac{1}{1-6s+s^2} .$$

But in accordance with the new convention $B_1 = 1$, $B_2 = 6, \cdots$ one can easily see that the generating function for the sequence of balancing numbers $\{B_n\}_{n=1}^{\infty}$ takes the form

$$g(s) = \frac{s}{1-6s+s^2} . \tag{5}$$

***Theorem* 4.1** *The generating function for the sequence of co-balancing numbers* $\{b_n\}_{n=1}^{\infty}$ is

$$f(s) = \frac{2s^2}{(1-s)(1-6s+s^2)} \tag{6}$$

*and consequently for* $n \geq 2$

$$b_n = 2(B_1 + B_2 + ... + B_{n-1}) .$$

***Proof***: From (4) for $n = 1, 2, \cdots$ we have $b_{n+2} - 6b_{n+1} + b_n = 2$. Multiplying both sides by $s^{n+2}$ and summing over $n = 1$ to $n = \infty$, we obtain

$$\sum_{n=1}^{\infty} b_{n+2} s^{n+2} - 6s \sum_{n=1}^{\infty} b_{n+1} s^{n+1} + s^2 \sum_{n=1}^{\infty} b_n s^n = 2s^2 \sum_{n=1}^{\infty} s^n ,$$

which in terms of $f(s)$ can be expressed as

$$\left(f(s) - 2s^2\right) - 6sf(s) + s^2 f(s) = 2s^3 \big/ (1-s) .$$

Thus

$$f(s) = \frac{2s^2}{(1-s)(1-6s+s^2)} = \frac{2s}{1-s} \cdot \frac{s}{1-6s+s^2} = \frac{2s}{1-s} \cdot g(s)$$

$$= 2(s + s^2 + \cdots) g(s).$$

Now for $n \geq 2$, the coefficient of $s^n$ in $f(s)$ can be obtained by collecting the coefficient of $s^r$ from $g(s)$ and the coefficient of $s^{n-r}$ from $2(s + s^2 + \cdots)$ for $r = 1, 2, \ldots, n-1$. While the coefficient of $s^r$ in $g(s)$ is $B_r$, the coefficient of $s^{n-r}$ in $2(s + s^2 + \cdots)$ is 2. Hence

$$b_n = 2(B_1 + B_2 + ... + B_{n-1}) .$$

This completes the proof.

The following corollary and Theorem 3.1 are direct consequences of Theorem 4.1.

***Corollary* 4.2**: $B_n = \dfrac{b_{n+1} - b_n}{2}$.

We are now in a position to prove Theorem 3.2(b).

***Proof of Theorem 3.2(b)*:** The proof is based on induction on *k*. It is easy to see that the assertion is true for $n > k = 2$. Assume that the assertion is true for $n > r \geq k \geq 2$, that is

$$b_n = b_r + B_r b_{n-r+1} - B_{r-1} b_{n-r}\,. \tag{7}$$

From [1] we know that the balancing numbers obey the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1}\,.$$

Applying this relation, (4), (7) and Corollary 4.2 to (6) we obtain

$$\begin{aligned} &b_{r+1} + B_{r+1} b_{n-r} - B_r b_{n-r-1} \\ &\quad = b_{r+1} + (6B_r - B_{r-1}) b_{n-r} - B_r (6b_{n-r} - b_{n-r+1} + 2) \\ &\quad = b_{r+1} - 2B_r + B_r b_{n-r+1} - B_{r-1} b_{n-r} \\ &\quad = b_r + B_r b_{n-r+1} - B_{r-1} b_{n-r} = b_n. \end{aligned}$$

Thus the assertion is also true for $k = r+1$. This completes the proof of Theorem 3.2(b).

## 5. BINET FORM FOR CO-BALANCING NUMBERS

From Section 4 we know that the co-balancing numbers satisfy the recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2$$

which is a second order linear non-homogeneous difference equation with constant coefficients. Substituting $c_n = b_n + 1/2$ we see that $c_n$ obey the recurrence relation

$$c_{n+1} = 6c_n - c_{n-1}$$

which is homogeneous. The general solution of this equation is

$$c_n = A\lambda_1^n + B\lambda_2^n \tag{8}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$ are the two roots of the auxiliary equation

$$\lambda^2 - 6\lambda + 1 = 0\,.$$

Substituting $c_1 = 1/2$ and $c_2 = 5/2$ into (8) we obtain

$$A = \frac{1}{\sqrt{\lambda_1}(\lambda_1 - \lambda_2)} \text{ and } B = \frac{1}{\sqrt{\lambda_2}(\lambda_1 - \lambda_2)}$$

where $\sqrt{\lambda_1} = 1 + \sqrt{2}$ and $\sqrt{\lambda_2} = 1 - \sqrt{2}$. Thus

$$c_n = A\lambda_1^n + B\lambda_2^n = \frac{\lambda_1^{n-1/2} - \lambda_2^{n-1/2}}{\lambda_1 - \lambda_2};\ n = 1, 2, \cdots$$

which implies

$$b_n = \frac{\lambda_1^{n-1/2} - \lambda_2^{n-1/2}}{\lambda_1 - \lambda_2} - \frac{1}{2};\ n = 1, 2, \cdots .$$

The above discussion proves the following theorem:

***Theorem*** **5.1** *If $b_n$ is the $n^{th}$ co-balancing number then its Binet form is*

$$b_n = \frac{\lambda_1^{n-1/2} - \lambda_2^{n-1/2}}{\lambda_1 - \lambda_2} - \frac{1}{2};\ n = 1, 2, \cdots$$

*where* $\lambda_1 = 3 + \sqrt{8},\ \lambda_2 = 3 - \sqrt{8},\ \lambda_1^{1/2} = 1 + \sqrt{2}$ and $\lambda_2^{1/2} = 1 - \sqrt{2}$ .

## 6.RELATIONS AMONG BALANCING NUMBERS, CO-BALANCING NUMBERS, BALANCERS AND CO-BALANCERS

Let $B$ be any balancing number with balancer $R$ and $b$ any co-balancing number with co-balancer $r$. Then by definition, the pairs $(B, R)$ and $(b, r)$ satisfy respectively

$$1+2+ \cdots + (B-1) = (B+1) + (B+2) + \cdots + (B+R) \qquad (9)$$

and

$$1+2+ \cdots + b = (b+1) + (b+2) + \cdots + (b+r). \qquad (10)$$

Solving (9) for $B$ and (10) for $b$ we find

$$B = \frac{(2R+1) + \sqrt{8R^2 + 8R + 1}}{2} \qquad (11)$$

and

$$b = \frac{(2r-1) + \sqrt{8r^2 + 1}}{2} . \qquad (12)$$

We infer from (11) that if $R$ is a balancer then, $8R^2 + 8R + 1$ is a perfect square and from (12) we conclude that if $r$ is a co-balancer then, $8r^2 + 1$ is a perfect square.

The above discussion proves the following theorem:

***Theorem*** **6.1** *Every balancer is a co-balancing number and every co-balancer is a balancing number.*

For $n = 1, 2, \cdots$ as usual let $B_n$ be the $n^{\text{th}}$ balancing number and $b_n$, the $n^{\text{th}}$ co-balancing number. We also denote by $R_n$, the balancer corresponding to $B_n$ and $r_n$, the co-balancer corresponding to $b_n$. What we are going to prove now is much stronger than Theorem 6.1.

***Theorem*** **6.2** *For* $n = 1, 2, \cdots$, $R_n = b_n$ and $r_{n+1} = B_n$ .

***Proof***: We know that if $B$ is a balancing number with balancer $R$ then

$$R = \frac{-(2B+1)+\sqrt{8B^2+1}}{2} \text{ (see[1], p.98).}$$

Thus

$$R_{n+1} = \frac{-(2B_{n+1}+1)+\sqrt{8B_{n+1}^2+1}}{2}, \tag{13}$$

$$R_{n-1} = \frac{-(2B_{n-1}+1)+\sqrt{8B_{n-1}^2+1}}{2}, \tag{14}$$

Also, from Theorem 3.1 and Corollary 3.2 of [1] we have

$$B_{n+1} = 3B_n + \sqrt{8B_n^2+1}, \tag{15}$$

and

$$B_{n-1} = 3B_n - \sqrt{8B_n^2+1}. \tag{16}$$

Substituting (15) and (16) into (13) and (14) respectively we obtain

$$R_{n+1} = \frac{2B_n + \sqrt{8B_n^2+1} - 1}{2},$$

$$R_{n-1} = \frac{-14B_n + 5\sqrt{8B_n^2+1} - 1}{2}.$$

Adding the last two equations we get

$$\begin{aligned} R_{n+1} + R_{n-1} &= \frac{-12B_n + 6\sqrt{8B_n^2+1} - 2}{2} \\ &= 6 \cdot \frac{-(2B_n+1)+\sqrt{8B_n^2+1}}{2} + 2 \\ &= 6R_n + 2. \end{aligned}$$

This gives

$$R_{n+1} = 6R_n - R_{n-1} + 2.$$

Thus $R_n$ satisfies the same recurrence relation as that of $b_n$. Further, since $R_1 = b_1 = 0$ and $R_2 = b_2 = 2$, it follows that $R_n = b_n$ for $n = 1, 2, \cdots$. This proves the first part of the theorem.

We prove the second part of the theorem in a similar way. Using (3) we obtain

$$r_{n+1} = \frac{-(2b_{n+1}+1)+\sqrt{8b_{n+1}^2+8b_{n+1}+1}}{2} \tag{17}$$

and

$$r_{n-1} = \frac{-(2b_{n-1}+1)+\sqrt{8b_{n-1}^2+8b_{n-1}+1}}{2}. \tag{18}$$

Substituting

$$b_{n+1} = 3b_n + \sqrt{8b_{n+1}^2+8b_{n+1}+1} + 1$$

into (17) and

$$b_{n-1} = 3b_n + \sqrt{8b_{n-1}^2 + 8b_{n-1} + 1} + 1$$

into (18) we obtain

$$r_{n+1} = \frac{2b_n + \sqrt{8b_{n+1}^2 + 8b_{n-1} + 1} + 1}{2}$$

and

$$r_{n-1} = \frac{-14b_n + 5\sqrt{8b_{n-1}^2 + 8b_{n-1} + 1} - 7}{2}.$$

Adding the last two equations we get

$$r_{n+1} + r_{n-1} = \frac{-12b_n + 6\sqrt{8b_n^2 + 8b_n + 1} - 6}{2}$$

$$= 6 \cdot \frac{-(2b_n + 1) + \sqrt{8b_n^2 + 8b_n + 1}}{2} = 6r_n.$$

Thus $r_n$ satisfies the same recurrence relation as that of $B_n$. Further, since $B_1 = r_2 = 1$ and $B_2 = r_3 = 6$ it follows that $B_n = r_{n+1}$ for $n$ = 1, 2,⋯ . This completes the proof of the theorem.

***Corollary* 6.3** *Every balancer is even.*

***Proof:*** Directly follows from Theorem 3.1 and Theorem 6.2.

***Corollary* 6.4** $R_{n+1} = R_n + 2B_n$.

***Proof:*** Directly follows from Corollary 4.2 and Theorem 6.2.

We are now in a position to prove that $h(x)$ and $t(x, y)$ are co-balancing numbers as stated in Theorem 2.1.

We first show that if $x$ is a co-balancing number then

$$h(x) = 8x^2 + 8x + 1 + (2x + 1)\sqrt{8x^2 + 8x + 1} + 1$$

is also a co-balancing number.

From Theorem 3.1 of [1] we know that if $y$ is a balancing number then $u = 2y\sqrt{8y^2 + 1}$ is also a balancing number and the balancer corresponding to $u$ is

$$R = \frac{-(2u + 1) + \sqrt{8u^2 + 1}}{2} = 8y^2 - 2y\sqrt{8y^2 + 1}. \tag{19}$$

If $x$ is the balancer corresponding to the balancing number $y$ then from (11) we find

$$y = \frac{(2x + 1) + \sqrt{8x^2 + 8x + 1}}{2},$$

so that

$$8y^2+1 = 24x^2+24x+4(2x+1)\sqrt{8x^2+8x+1}+5 \\ = \left(2(2x+1)+\sqrt{8x^2+8x+1}\right)^2. \tag{20}$$

Substitution of (20) into (19) gives

$$R = 24x^2+24x+4(2x+1)\sqrt{8x^2+8x+1}+4 \\ -2\left[\frac{2(2x+1)+\sqrt{8x^2+8x+1}}{2}\right]\cdot\left[2(2x+1)+\sqrt{8x^2+8x+1}\right] \\ = 8x^2+8x+1+(2x+1)\sqrt{8x^2+8x+1} = h(x).$$

Thus for any balancer $x$, $h(x)$ is always a balancer. Since by Theorem 6.1 every balancer is a co-balancing number the result follows.

We next prove that if $x$ and $y$ are co-balancing numbers then

$$t(x,y) = \tfrac{1}{2}\Big[2(2x+1)(2y+1)+(2x+1)\sqrt{8y^2+8y+1} \\ +(2y+1)\sqrt{8x^2+8x+1}+\sqrt{8x^2+8x+1}\sqrt{8y^2+8y+1}-1\Big]$$

is also a co-balancing number. From Theorem 4.1 of [1] we know that if $u$ and $v$ are balancing numbers then

$$w = u\sqrt{8v^2+1}+v\sqrt{8u^2+1}$$

is also a balancing number. Let $s$, $x$ and $y$ be the balancers corresponding to the balancing numbers $w$, $u$ and $v$ respectively. Then

$$s = \frac{-(2w+1)+\sqrt{8w^2+1}}{2} \\ = \frac{1}{2}\left[8uv+\sqrt{(8u^2+1)(8v^2+1)}-2u\sqrt{8v^2+1}-2v\sqrt{8u^2+1}-1\right]. \tag{21}$$

Now substituting

$$u = \frac{(2x+1)+\sqrt{8x^2+8x+1}}{2}$$

and

$$v = \frac{(2y+1)+\sqrt{8y^2+8y+1}}{2}$$

into (21) we find that

$$s = \tfrac{1}{2}\Big[2(2x+1)(2y+1)+(2x+1)\sqrt{8y^2+8y+1}$$
$$+(2y+1)\sqrt{8x^2+8x+1}+\sqrt{8x^2+8x+1}\sqrt{8y^2+8y+1}-1\Big]$$
$$= t(x,y).$$

Again since every balancer is a co-balancing number by Theorem 6.1, the result follows.

***Remark:*** $t(x, x) = h(x)$.

## 7. AN APPLICATION OF CO-BALANCING NUMBERS TO THE DIOPHANTINE EQUATION $x^2+(x+1)^2 = y^2$

As we know, the Diophantine equation $x^2+(x+1)^2 = y^2$, $x, y \in \mathbb{Z}^+$, is a particular case of the equation $x^2 + y^2 = z^2$, $x, y, z \in \mathbb{Z}^+$. Any solution $(x, y, z)$ of the later equation is called a Pythagorean triplet. Behera and Panda [1] established a link between the solutions of the equation $x^2+(x+1)^2 = y^2$ and balancing numbers. Here we are going to obtain an easy relation between the solutions of this equation with co-balancing numbers.

Let $b$ be any co-balancing number and $r$ its co-balancer and $c = b + r$. Then equation (2) can be re-written as

$$1+2+\cdots+b = (b+1)+(b+2)+\cdots+c,$$

from which we find $b$ in terms of $c$ as

$$b = -1+\sqrt{2c^2+2c+1}.$$

Thus $2c^2+2c+1$ is a perfect square and also

$$2c^2+2c+1 = c^2+(c+1)^2.$$

This suggests that the Diophantine equation $x^2+(x+1)^2 = y^2$ has the solution

$$x = b+r,\ \ y = \sqrt{2c^2+2c+1}.$$

Take for example $b = 14$, so that $r = 6$ and $c = b + r = 20$. Further $2c^2+2c+1 = 841 = 29^2$ and we have

$$20^2+21^2 = 29^2.$$

Similarly for $b = 84$, we have $119^2 + 120^2 = 169^2$.

**Acknowledgement**: It is a pleasure to thank the anonymous referees for their valuable comments and suggestions, which resulted in an improved presentation of the paper.

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