Published in Fibonacci Numbers and Their Applications
Vol. 10 Year 2006 (To appear)
Author's Postprint
Archived in http://dspace.nitrkl.ac.in/dspace

SOME FASCINATING PROPERTIES OF BALANCING NUMBERS

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Key words: Triangular numbers, Balancing numbers, Fibonacci numbers, Recurrence relation
1991 Mathematics Subject Classification (A. M. S.): 11 A 05, 11 B 39

Abstract: The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the fascinating triangular numbers. In a recent study Behera and Panda tried to find the solutions of the Diophantine equation $1+2+\cdots+(n-1) = (n+1) + (n+2) + \cdots + (n+r)$ and found that the square of any $n \in \mathbb{Z}^+$ satisfying this equation is a triangular number. It can be also shown that if $r \in \mathbb{Z}^+$ satisfies the above equation then $r^2 + r$ is also a triangular number. If a pair (n, r) constitutes a solution of the above equation then n is called a *balancing number* and r is called the *balancer* corresponding to n. In the joint paper "*On the square roots of triangular numbers*" published in "*The Fibonacci Quarterly*" in 1999, Behera and Panda introduced balancing numbers and studied many important properties of these numbers. In this paper we establish some other interesting arithmetic-type, de-Moivre's-type and trigonometric-type properties of balancing numbers. We also establish a most important property concerning the greatest common divisor of two balancing numbers.

1. INTRODUCTION

Recently, Behera and Panda [3] introduced *balancing numbers* $n \in \mathbb{Z}^+$ as solutions of the equation

$$1+2+\cdots+(n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

calling $r \in \mathbb{Z}^+$, the *balancer* corresponding to the balancing number n. For example 6, 35 and 204 are balancing numbers with balancers 2, 14 and 84 respectively. It is also proved in [3] that a positive integer n is a balancing number if and only if n^2 is a triangular number, that is $8n^2 + 1$ is a perfect square. Though the definition of balancing number suggests that it must be grater than 2, Behera and Panda [3] accepted 1 as a balancing number being the positive square root of the square triangular number 1.

Behera and Panda [3], while accepting 1 as a balancing number, have set $B_0 = 1$, $B_1 = 6$, and so on, using the symbol B_n for the n^{th} balancing number. To standardize the notation at par with Fibonacci numbers, we relabel the balancing numbers by setting $B_1 = 1$, $B_2 = 6$ and so on.

Some results established by Behera and Panda [3] can be stated with this new convention as follows:

The second order linear recurrence:

$$B_{n+1} = 6B_n - B_{n-1};\ n = 2, 3, \cdots \qquad \ldots(1)$$

The non-linear first order recurrence:

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1};\ n = 1, 2, \cdots \qquad \ldots(2)$$

The relation:

$$B_n = B_{r+1} \cdot B_{n-r} - B_r B_{n-r-1};\ r = 1, 2, \cdots, n-2. \qquad \ldots(3)$$

The Binet form:

$$B_n = \frac{\lambda_1{}^n - \lambda_2{}^n}{\lambda_1 - \lambda_2},\ n = 1, 2, \cdots \qquad \ldots(4)$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

The interesting relation:

$$B_{n+1}.B_{n-1} = (B_n + 1)(B_n - 1). \qquad \ldots(5)$$

Now using (1) we can set $B_0 = B_2 - 6B_1 = 6 - 6 \times 1 = 0$.

In the next section we establish some arithmetic-type properties and other interesting properties of balancing numbers.

2. SOME INTERESTING RESULTS ON BALANCING NUMBERS

Throughout this section F_n is the n^{th} Fibonacci number, L_n is the n^{th} Lucas number, B_n is the n^{th} Balancing number and $C_n = \sqrt{8B_n^2 + 1}$ where $n \in \mathbb{Z}^+$. Some of the following results suggest that C_n is associated with B_n in the way L_n is associated with F_n.

We know that if x and y are real or complex numbers, then $(x+y)(x-y) = x^2 - y^2$. In the following theorem we prove an analogous property of balancing numbers. This theorem also generalizes equation (5).

Theorem 2.1: *If m and n are natural numbers and $m > n$, then $(B_m + B_n)(B_m - B_n) = B_{m+n} \cdot B_{m-n}$.*

Proof: Using the Binet form (4) and keeping in mind that $\lambda_1 \lambda_2 = 1$, we have

$$B_{m+n} \cdot B_{m-n} = \frac{(\lambda_1^{m+n} - \lambda_2^{m+n})(\lambda_1^{m-n} - \lambda_2^{m-n})}{(\lambda_1 - \lambda_2)^2}$$

$$= \frac{(\lambda_1^{2m} + \lambda_2^{2m}) - (\lambda_1^{m+n}\lambda_2^{m-n} + \lambda_1^{m-n}\lambda_2^{m+n})}{(\lambda_1 - \lambda_2)^2}$$

$$= \frac{(\lambda_1^{2m} + \lambda_2^{2m}) - (\lambda_1^{2n} + \lambda_2^{2n})}{(\lambda_1 - \lambda_2)^2}$$

$$= \frac{(\lambda_1^{2m} + \lambda_2^{2m} - 2\lambda_1^{m}\lambda_2^{m}) - (\lambda_1^{2n} + \lambda_2^{2n} - 2\lambda_1^{n}\lambda_2^{n})}{(\lambda_1 - \lambda_2)^2}$$

$$= \left[\frac{\lambda_1^{m} - \lambda_2^{m}}{\lambda_1 - \lambda_2}\right]^2 - \left[\frac{\lambda_1^{n} - \lambda_2^{n}}{\lambda_1 - \lambda_2}\right]^2$$

$$= B_m^{\ 2} - B_n^{\ 2} = (B_m + B_n)(B_m - B_n).$$

Remark: The Fibonacci numbers satisfy a similar property (see [4], p.59)

$$F_{m+n} \cdot F_{m-n} = F_m^{\ 2} - (-1)^{m+n} F_n^{\ 2}.$$

The identity of Theorem 1 looks more symmetric than this result.

We know that if n is a natural number, then $1 + 3 + \cdots + (2n-1) = n^2$, $2 + 4 + \cdots + 2n = n(n+1)$ and $1 + 2 + \cdots + 2n = n(2n+1)$. In the following theorem we prove three properties of balancing numbers similar to the above three identities.

Theorem 2.2**:** *If n is a natural number then*

(a) $B_1 + B_3 + \cdots + B_{2n-1} = B_n^{\ 2}$,

(b) $B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}$,

(c) $B_1 + B_2 + \cdots + B_{2n} = B_n (B_n + B_{n+1})$.

Proof: From Theorem 2.1 we have

$$B_{m+n} \cdot B_{m-n} = B_m^{\ 2} - B_n^{\ 2}$$

where $m > n$. Replacing m by $n+1$ in the above identity and keeping in mind that $B_1 = 1$ we obtain

$$B_{2n+1} = B_{n+1}^{\ 2} - B_n^{\ 2},$$

from which (a) follows.

Replacing n by $2n$ and r by n in equation (3) we find

$$B_{2n} = B_{n+1} \cdot B_n - B_n B_{n-1},$$

from which (b) follows.

The identity (c) directly follows from (a) and (b).

The complex identity $(\cos x + i\sin x)^n = \cos nx + i\sin nx$ is known as the de-Moivre's formula (see [1]). The following theorem looks like de-Moivre's formula.

Theorem 2.3: *If n and r are natural numbers, then* $(C_n + \sqrt{8}B_n)^r = C_{nr} + \sqrt{8}B_{nr}$.

Proof: Using the Binet form (4) we obtain

$$C_n^{\ 2} = 8B_n^{\ 2} + 1 = 8\left[\frac{\lambda_1^{\ n} - \lambda_2^{\ n}}{\lambda_1 - \lambda_2}\right]^2 + 1$$

$$= 8\left[\frac{\lambda_1^{\ 2n} + \lambda_2^{\ 2n} - 2}{(2\sqrt{8})^2}\right] + 1$$

$$= \frac{\lambda_1^{\ 2n} + \lambda_2^{\ 2n} + 2}{4} = \left[\frac{\lambda_1^{\ n} + \lambda_2^{\ n}}{2}\right]^2.$$

Hence

$$C_n = \frac{\lambda_1^{\ n} + \lambda_2^{\ n}}{2}.$$

Now

$$C_n + \sqrt{8}B_n = \frac{\lambda_1^{\ n} + \lambda_2^{\ n}}{2} + \sqrt{8}\,\frac{\lambda_1^{\ n} - \lambda_2^{\ n}}{2\sqrt{8}} = \lambda_1^{\ n}. \qquad \ldots(6)$$

Thus

$$(C_n + \sqrt{8}B_n)^r = (\lambda_1^{\ n})^r = \lambda_1^{\ nr} = C_{nr} + \sqrt{8}B_{nr}.$$

Remark: The Fibonacci numbers satisfy a similar property

$$\left[\frac{L_n + \sqrt{5}F_n}{2}\right]^r = \frac{L_{rn} + \sqrt{5}F_{rn}}{2}.$$

Corollary 2.4: *If n and r are natural numbers, then* $(C_n - \sqrt{8}B_n)^r = C_{nr} - \sqrt{8}B_{nr}$.

Proof: Since

$$C_n - \sqrt{8}B_n = \frac{\lambda_1^{\ n} + \lambda_2^{\ n}}{2} - \sqrt{8}\,\frac{\lambda_1^{\ n} - \lambda_2^{\ n}}{2\sqrt{8}} = \lambda_2^{\ n},$$

the result follows.

The following theorem looks like the trigonometric identity $\sin(x+y) = \sin x\cos y + \cos x\sin y$.

Theorem 2.5: *If m and n are natural numbers, then* $B_{m+n} = B_m C_n + C_m B_n$.

Proof: If *m* and *n* are natural numbers, then using equation (6) we obtain

$$\begin{aligned}(C_m+\sqrt{8}B_m)(C_n+\sqrt{8}B_n) &= \lambda_1^{\,m}\lambda_1^{\,n} = \lambda_1^{\,m+n} \\ &= (C_{m+n}+\sqrt{8}B_{m+n}).\end{aligned} \qquad \ldots(7)$$

On the other hand,

$$\begin{aligned}&(C_m+\sqrt{8}B_m)(C_n+\sqrt{8}B_n) \\ &\quad = (C_mC_n+8B_mB_n)+\sqrt{8}(B_mC_n+8C_mB_n).\end{aligned} \qquad \ldots(8)$$

Comparing equations (7) and (8) we get

$$C_{m+n}+\sqrt{8}B_{m+n} = (C_mC_n+8B_mB_n)+\sqrt{8}(B_mC_n+C_mB_n). \qquad \ldots(9)$$

Equating the rational and irrational parts from both sides of equation (9) we obtain

$$C_{m+n} = C_mC_n+8B_mB_n$$

and

$$B_{m+n} = B_mC_n+C_mB_n.$$

Remark: The corresponding property for Fibonacci numbers

$$F_{m+n} = \tfrac{1}{2}\left[F_mL_n+L_mF_n\right],$$

does not look like the trigonometric identity $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

The following corollary looks like the trigonometric identity $\sin(x-y)=\sin x\cos y-\cos x\sin y$.

Corollary 2.6**:** *If m and n are natural numbers and m > n, then* $B_{m-n}=B_mC_n-C_mB_n$.

Proof: Same as Theorem 2.5.

The following corollary resembles the trigonometric identity $\sin 2x = 2\sin x\cos x$.

Corollary 2.7**:** *If n is a natural number, then* $B_{2n}=2B_nC_n$.

Proof: Directly follows from Theorem 2.5 with $m=n$.

Remark: The corresponding property for Fibonacci numbers $F_{2n}=F_nL_n$ (see [4]) does not look like $\sin 2x = 2\sin x\cos x$.

For any two integers *m* and *n*, let us denote the greatest common divisor of *m* and *n* by (*m*, *n*). We know that F_m divides F_n if and only if *m* divides *n* and $(F_m,\ F_n)=F_{(m,\ n)}$. The following results show that the balancing numbers also enjoy these beautiful properties.

Theorem 2.8: *If m and n are natural numbers, then* B_m *divides* B_n *if and only* if *m* divides *n*.

To prove Theorem 2.8 we need the following lemmas.

Lemma 2.9: *If m and n are natural numbers, then* $(B_n, C_n) = 1$.

Proof: Since $C_n^{\ 2} = 8B_n^{\ 2} + 1$, it follows that $(B_n^{\ 2}, C_n^{\ 2}) = 1$ and thus $(B_n, C_n) = 1$.

Lemma 2.10: *If n and k are natural numbers, then* B_k *divides* B_{nk}.

Proof: The proof is based on induction. The hypothesis is trivial for $n = 1$. Assume that it is true for $n = r$. We need only to show that it is also true for $n = r + 1$, that is, B_k divides $B_{(r+1)k}$. Since $B_{(r+1)k} = B_{rk+k} = B_{rk}C_k + C_{rk}B_k$ by Theorem 2.5, $(B_k, C_k) = 1$ by Lemma 2.9 and B_k divides B_{rk} by assumption, it follows that B_k divides $B_{(r+1)k}$.

Lemma 2.11: *If n and k are natural numbers, then* $(B_k, C_{nk}) = 1$.

Proof: By Lemma 2.9, $(B_{nk}, C_{nk}) = 1$. Since B_k divides B_{nk} by Lemma 2.10, it follows that $(B_k, C_{nk}) = 1$.

Lemma 2.12: *If n and k are natural numbers and* B_k *divides* B_n, *then k divides n.*

Proof: Certainly $n \geq k$. If $n = k$ then the proof is trivial. Assume that $n > k$. Then by Euclid's division lemma ([2], Theorem 2.1), there exists integers q and r such that $q \geq 1, 0 \leq r < k$ and $n = qk + r$. By Theorem 2.5, $B_n = B_{qk+r} = B_{qk}C_r + C_{qk}B_r$. Since B_k divides B_{qk} by Lemma 2.10, and $(B_k, C_{qk}) = 1$ by Lemma 2.11, it follows that B_k divides B_r. Since $r < k$, it follows that $B_r = 0$ and hence $r = 0$. Thus $n = qk$ and therefore *k* divides *n*.

It can now be readily seen that Theorem 2.8 directly follows from Lemmas 2.10 and 2.12.

The following theorem tells something more than Theorem 2.8.

Theorem 2.13: *If m and n are natural numbers, then* $(B_m, B_n) = B_{(m, n)}$.

Proof: If $m = n$, the proof is trivial; else let us assume without loss of generality that $m < n$. By Euclid's division lemma, there exists integers q_1 and r_1 such that $q_1 \geq 1, 0 \leq r_1 < m$ and $n = qm + r_1$. Now by Theorem 2.5

$$\begin{aligned}(B_m, B_n) &= (B_m, B_{q_1m+r_1}) \\ &= (B_m, B_{q_1m}C_{r_1} + C_{q_1m}B_{r_1}).\end{aligned}$$

Since B_m divides $B_{q_1 m}$ by Lemma 2.10 and $(B_m, C_{q_1 m}) = 1$ by Lemma 2.11, it follows that $(B_m, B_n) = (B_m, B_{r_1})$ and $(m, n) = (m, q_1 m + r_1) = (m, r_1)$. If $r_1 > 0$, then there exists integers q_2 and r_2 such that $q_2 \geq 1, 0 \leq r_2 < r_1$ and $m = q_2 r_1 + r_2$. Now again by Theorem 2.5,

$$\begin{aligned}(B_m, B_n) &= (B_m, B_{r_1}) \\ &= (B_{q_2 r_1 + r_2}, B_{r_1}) \\ &= (B_{q_2 r_1} C_{r_2} + C_{q_2 r_1} B_{r_2}, B_{r_1}) \\ &= (B_{r_2}, B_{r_1}),\end{aligned}$$

and $(m, r_1) = (q_2 r_1 + r_2, r_1) = (r_2, r_1)$. This process may be continued till a newly arising r_i does not equal to zero. Since $r_1 > r_2 > \cdots$, it follows that $r_i \leq m - i$, so that after at most m steps some r_i will be equal to zero. If $r_{k-1} > 0$, and $r_k = 0$, then we have

$$(B_m, B_n) = (B_{r_{k-2}}, B_{r_{k-1}}) = (B_{q_k r_{k-1}}, B_{r_{k-1}}) = B_{r_{k-1}}$$

and $(m, n) = (r_{k-2}, r_{k-1}) = (q_k r_{k-1}, r_{k-1}) = r_{k-1}$. Thus $(B_m, B_n) = B_{r_{k-1}} = B_{(m,n)}$ and the proof is complete.

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