Proceedings of ICTACEM 2017
International Conference on Theoretical, Applied, Computational and Experimental Mechanics
December 28-30, 2017, IIT Kharagpur, India
ICTACEM-2017/126(paper No.)

Dynamics of axially functionally graded nonuniform beam with geometric nonlinearity

Saurabh Kumar[a], H. Roy[a]* and Anirban Mitra[b]

[a] Department of Mechanical Engineering, National Institute of Technology Rourkela, Odisha, India
[b] Department of Mechanical Engineering, Jadavpur University, Kolkata, West Bengal, India

ABSTRACT

In this article, the nonlinear dynamic behavior of axially functionally graded non-uniform beam is investigated using finite element method. The beam material is graded in the axial direction following the exponential rule and three different taper profiles are selected namely; linear, parabolic and exponential. Euler-Bernoulli beam theory is utilized along with Hamilton's principle to formulate the forced vibration problem. Geometric nonlinearity present in the system is taken care of using von Karman type nonlinear geometrical relations. The beam is considered to be under the action of uniformly distributed external excitation of harmonic nature. The results are validated with those available in the literature and new results are presented showing the effect of material gradient parameter, excitation amplitude, taper profile and taper parameter on the frequency response curves.

Keywords: Functionally graded material, Non-uniform beam, Nonlinear behavior, Frequency-response.

1. INTRODUCTION

Functionally graded materials (FGM) are treated as new generation materials in which the material properties vary continuously along spatial directions and that claim many advantages over traditional materials as well as layered composites. Due to these advantages such materials can be employed in various advanced structures especially in large scale space and aerospace applications. As a result, FGMs came into the attention of many researchers and numerous studies on these materials are carried out. Sankar (2001) provided elasticity solutions for transversely loaded FG beam and also developed a beam theory similar to Euler-Bernoulli beam theory assuming that the plane sections remain plane. A new beam element was developed by Chakraborty et al. (2003) to account the variation of elastic and thermal properties along the thickness direction and studied static, free vibration and wave propagation problems. Agrawal et al (2006) presented a total Lagrangian formulation of a geometrically nonlinear beam element to conduct static and wave propagation analysis on FG beams using first order shear deformation theory.

Aydogdu and Taskin (2007) carried out free vibration analysis on simply-supported FG beams using different higher order shear deformation theories and classical beam theories. Simsek and Kocaturk (2009) investigated the dynamic behaviour of FG beams under action of concentrated moving harmonic load. Simsek (2010) implemented nonlinear dynamic analysis on FG beams subjected to moving harmonic load. Lagrange's equations were used to generate equations of motion of the system and these were solved using Newmark-β method and a direct iteration method. Paul and Das (2015) studied the large amplitude free vibration behaviour of FG beams using variational approach considering Timoshenko beam assumptions and Von Karman strain-displacement relations. Wu et al. (2016) performed similar study on carbon nanotube reinforced FG beams using Ritz method.

A close scrutiny of the literature reveals that the studies are primarily focused on thickness-wise functionally graded materials and studies on axially functionally graded materials are relatively scarce. There are some works on non-uniform beams having axial functional gradation, for example, by Huang et al. (2013), Tang et al. (2014) and Liu et al. (2016), but those too only provide insight into the linear behavior of the structure. Also, it is observed that free vibration studies exist in plenty, whereas studies on forced vibration are comparatively rare. So an effort has been made in the present work to extend the works on beams with in-plane inhomogeneity to the nonlinear domain.

2. MATHEMATICAL FORMULATION

The present nonlinear forced vibration problem is formulated using finite element method (FEM) and the governing differential equations are obtained utilizing Hamilton's principle. The beam is modelled using Euler-Bernoulli theory i.e. the effect of shear deformation and rotary inertia are neglected. Von Karman type nonlinear strain displacement relations are considered to incorporate geometric nonlinearity in the system. The analysis is based on the assumption that the system attains equilibrium at peak amplitude value. This unique assumption helps in reducing the dynamic problem into an equivalent static problem.

The material properties of the beam are considered to be varying exponentially along axial direction using the following relationship,

$$E\left(x\right)=E_0\,\mathrm{e}^{\left\{\kappa_x\left(\frac{x}{L}\right)\right\}},\ \ \rho\left(x\right)=\rho_0\,\mathrm{e}^{\left\{\kappa_x\left(\frac{x}{L}\right)\right\}} \tag{1}$$

Here, κ_x is the gradient parameter along the *x* directions. E_0 and ρ_0 are Young's modulus and material density, respectively, at the left side of the beam. The thickness of the beam is also considered to be varying along the *x*-axis following three different taper profiles namely, linear profile, parabolic profile and exponential profile. The expressions of these taper profiles are as follows,

$$\text{Linear}: h(x) = h_0\left(1+rx\right), \text{ Parabolic}: h(x) = h_0\left(1+rx^{1/2}\right), \text{ Exponential}: h(x) = h_0 \exp\left(rx^2\right) \tag{2}$$

where, h_0 is the root thickness and r is the taper parameter.

Following the assumptions of Euler-Bernoulli beam theory and using Von Karman type geometrical nonlinearity, the strain-displacement relation is expressed as,

$$\bar{\varepsilon}_x = -z\left(w,_{xx}\right) + \left(u,_x\right) + \frac{1}{2}\left(w,_x\right)^2 \tag{3}$$

From above relation, the total strain energy (U), work potential (V) and total kinetic energy of the beam can be expressed as,

$$U = \frac{1}{2} b \int_0^L \left\{\hat{\sigma}\right\}^T \left\{\hat{\varepsilon}\right\} dx \tag{4}$$

$$V = -\int_0^L p(x) w \, dx \tag{5}$$

$$T = \frac{1}{2} b \int_0^L \left(\dot{\bar{u}}^2 + \dot{\bar{w}}^2\right) \rho\left(x\right) dx \tag{6}$$

p(x) in Eq. (6) is the uniformly distributed harmonic excitation which has the expression $\bar{p}(x)e^{i\Omega t}$. $\bar{p}$ is the amplitude of harmonic excitation for uniformly distributed load per unit length, Ω is the frequency of excitation and $i = \sqrt{-1}$.

A two-noded beam element having one node at each end is selected in the analysis. Three degrees of freedom are considered at each node, where u and w are the two displacements along x and z directions respectively and θ is the rotation about y axis.

The nodal displacement vector of the elements can be written as,

$$\left\{q^e\right\} = \left\{u_1 \quad w_1 \quad \theta_1 \quad u_2 \quad w_2 \quad \theta_2\right\}^T \tag{7}$$

The displacement fields at any point in the element are approximated by suitable functions called shape functions and nodal displacements,

$$u = \sum_{i=1}^{2} u_i N_i^u, \; w = \sum_{i=1}^{2} w_i N_i^w + \frac{l_e}{2} \sum_{i=1}^{2} \theta_i N_i^\theta \tag{8}$$

where, l_e is element length and the shape function N_i are given as,

$$\begin{aligned} N_1^u &= \frac{1}{2}\left(1-\xi\right), N_2^u = \frac{1}{2}\left(1+\xi\right), N_1^w = \frac{1}{4}\left(2-3\xi+\xi^3\right), N_2^w = \frac{1}{4}\left(2+3\xi-\xi^3\right) \\ N_1^\theta &= \frac{1}{4}\left(1-\xi-\xi^2+\xi^3\right), N_2^\theta = \frac{1}{4}\left(-1-\xi+\xi^2+\xi^3\right) \end{aligned} \tag{9}$$

An indirect approach is adopted for solving the problem, where the problem is reduced to a static case by assuming that the dynamic system fulfils the force equilibrium conditions at maximum amplitude of excitation. This assumption helps to solve the dynamic problem as an equivalent static problem, where the system response depends on the excitation frequency and amplitude of the harmonic excitation.

The formulation of the large amplitude forced vibration analysis is based on Hamilton's principle which is stated as,

$$\delta\left(\int_{t_1}^{t_2}\left(T-U-V\right)dt\right)=0 \tag{10}$$

The expressions of *T*, *U* and *V*, as in Eq. (4), Eq. (5) and Eq. (6) respectively, are utilized to obtain the governing differential equations for forced vibration analysis of the beam as,

$$\left[M\right]\{\ddot{q}\}+\left[K\right]\{q\}=\{f\} \tag{11}$$

Assuming the beam is subjected to harmonic excitation force ($\{f\}=\left\{\tilde{f}\right\}e^{i\Omega\tau}$), the displacement function is expressed as $\{q\}=\{\tilde{q}\}e^{i\Omega\tau}$. The above equation can now be rewritten as,

$$\left[\left[K\right]-\Omega^2\left[M\right]\right]\{\tilde{q}\}=\left\{\tilde{f}\right\} \tag{8}$$

here, **Ω** is the excitation frequency. The above equation is nonlinear in nature and solved using an iterative scheme.

3. RESULTS AND DISCUSSION

First of all, it is important to establish the validity and correctness of the mathematical formulation and solution methods used in the current work. The results of large amplitude forced vibration analysis are validated with respect to a uniform isotropic beam. The system under consideration is a clamped beam subjected to concentrated harmonic excitation force of amplitudes 0.134 N and 2.0 N at mid-point of the beam. The frequency response curves generated through current method are compared with those published by Ribeiro (2004) and the comparative plot is presented in Figure 1. The abscissa in this figure represents the frequency ratio (Ω/ω_0) where Ω is the frequency of excitation and ω_0 is the first natural frequency and the ordinate represents dimensionless amplitude (w_{max} / h_0). It can be seen that the results show good agreement with published result.

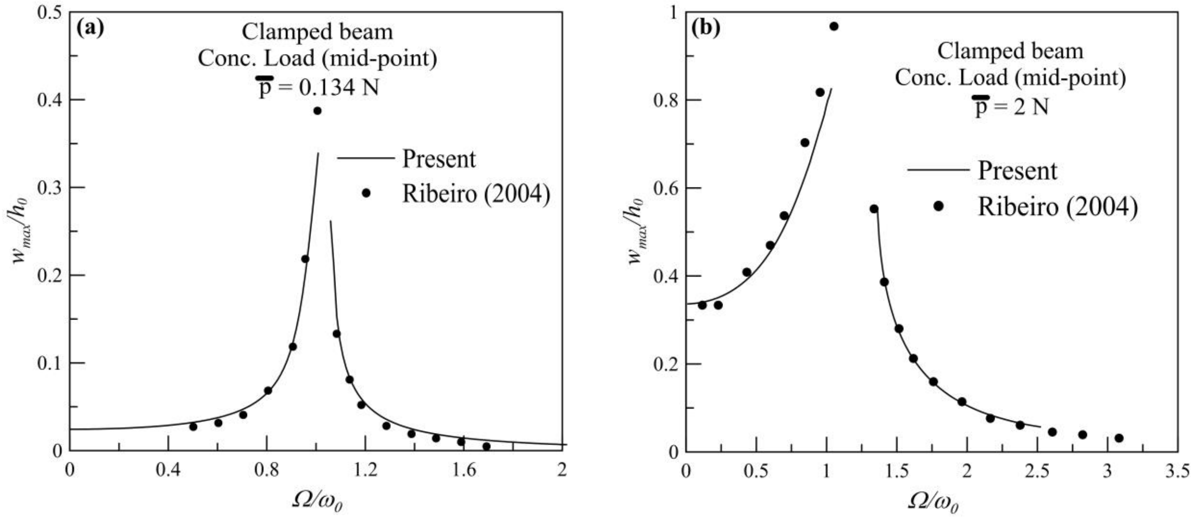

Figure 1. Validation plot for nonlinear forced vibration analysis for isotropic uniform beam subjected to concentrated load at mid-point (a) $\bar{p}$ = 0.134 N (b) $\bar{p}$ = 2 N.

The frequency-response curves obtained from various analyses on beams are presented here. It can be seen in the figures that there are two distinct branches of frequency-response curve which is due to discontinuous behavior of the nonlinear structure. The first branch is known as the increasing branch as, when the excitation frequency is slowly increased from zero, the response follows this curve. On the other hand, when the excitation frequency is slowly decreased from a higher value, the response follows the second branch and hence it is called the decreasing branch.

Frequency response curves for clamped and simply supported AFG beam are drawn in Figure 2, which explain the effect of material gradient on nonlinear forced vibration response. The value of amplitude of external excitation is taken as 100 N/m for clamped beam and 50 N/m for simply supported beam. It is seen that, increasing the material gradient parameter reduces the response amplitude at low amplitude range for both boundary conditions and as the response amplitude increases all the curves try to merge together.

The effect of excitation amplitude is shown in Figure 3 for AFG uniform beams. For clamped boundary condition the values of excitation amplitudes are 10 N/m, 50 N/m and 100 N/m and for simply supported beam the values are 10 N/m, 30 N/m and 50 N/m. The material gradient has been kept constant at 1.0. It is observed that, at low response zone that the response amplitude increases with the excitation amplitude however, at higher response zone, all the response curves try to merge together, irrespective of excitation force. Another observation that can be made by comparing the results of clamped and simply supported beams is that the simply supported beams exhibit higher amplitudes for same frequency ratio even though the excitation amplitude is lower.

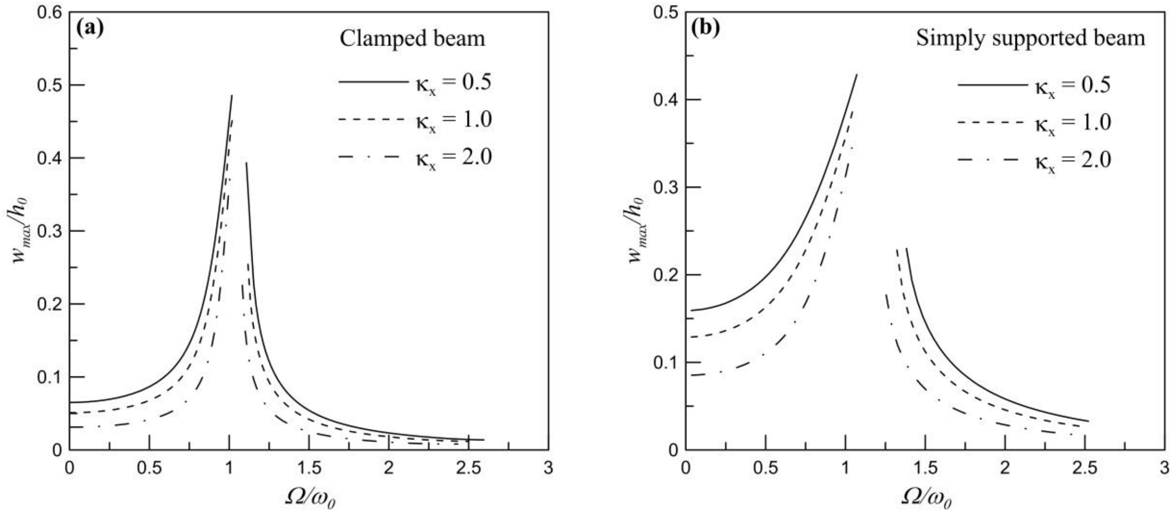

Figure 2. Effect of material gradient on frequency-response of uniform AFG beam (a) clamped (b) simply supported.

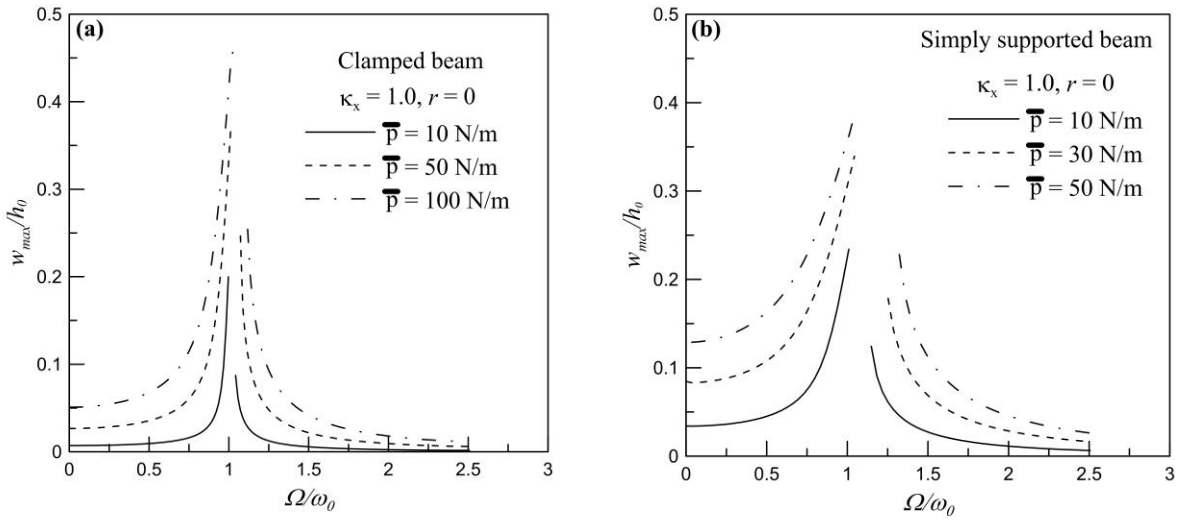

Figure 3. Effect of excitation amplitude on frequency-response of uniform AFG beam (a) clamped (b) simply supported.

The effect of different taper patterns on frequency response curves is also studied using three taper patterns (Linear, parabolic and exponential) and the results are shown in Figure 4. The taper parameter is kept constant at 0.8 for generation of the results. The graphs show that for similar conditions beams with parabolic taper exhibit higher stiffness and beams with exponential taper exhibit lower stiffness indicated by the response amplitudes. Frequency response curves of linearly tapered AFG beam with different taper parameters are shown in Figure 5 for clamped and simply supported boundary conditions. Three taper parameters are selected (0.2, 0.5 and 0.8) for generating the results. It is seen that as the taper parameter is increased, the response amplitude decreases in low amplitude region but all the curves try to merge together as the response amplitude increases.

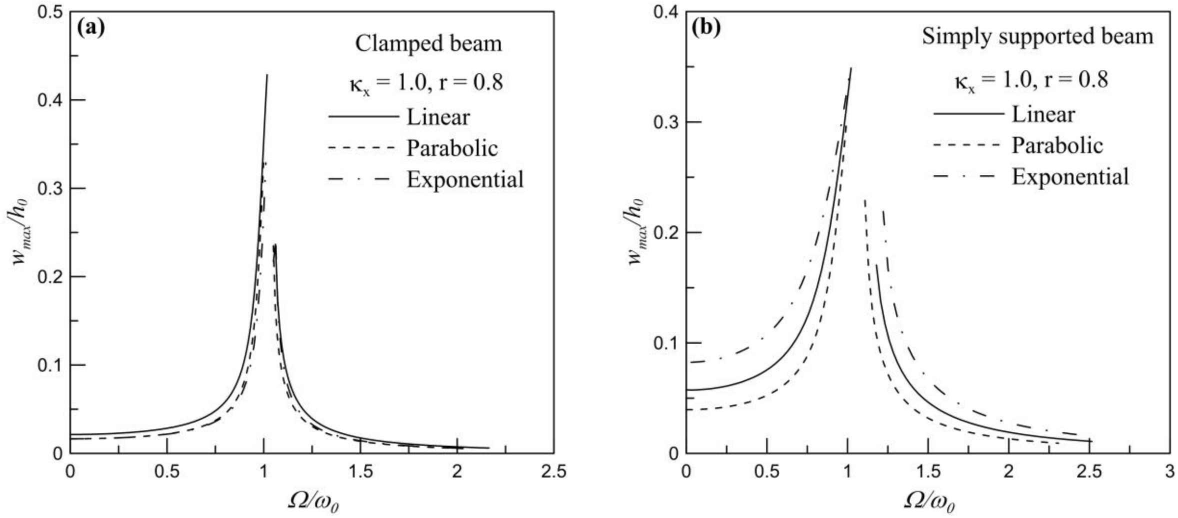

Figure 4. Effect of taper profile on frequency-response of AFG beam (a) clamped (b) simply supported.

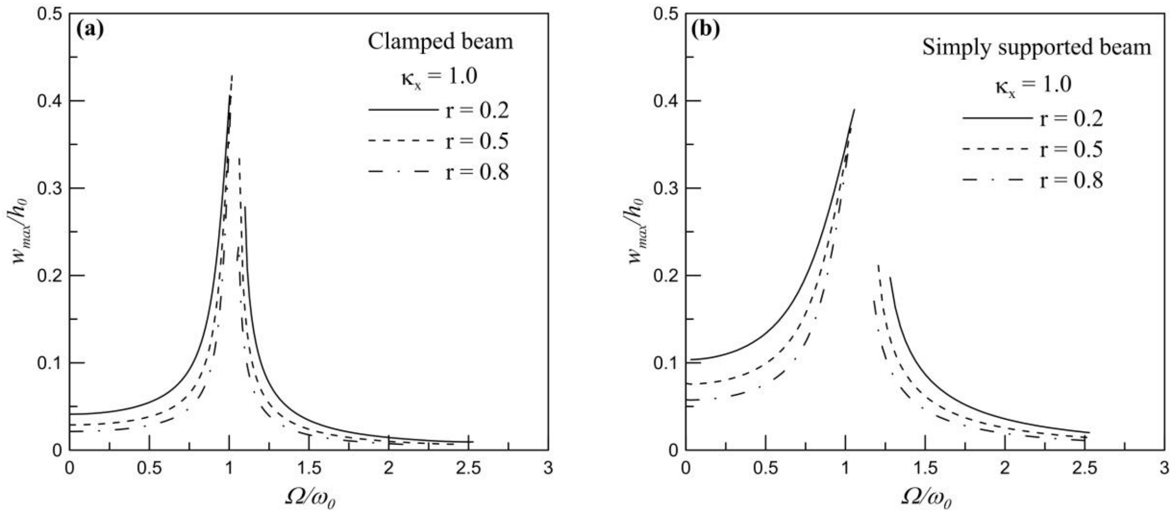

Figure 5. Effect of taper parameter on frequency-response of AFG beam (a) clamped (b) simply supported.

4. CONCLUSION

In this paper, geometrically nonlinear forced vibration analysis on AFG non-uniform beam is carried out using finite element method. The formulation is based on Euler-Bernoulli beam theory and the governing equations are generated using Hamilton's principle. von Karman type nonlinear strain displacement relations are used to account for the nonlinearity present in the system. The results are first validated with those already available in the literature. The effect of different parameters like material gradient, excitation amplitude, taper profile and taper parameter is studied in detail. It is seen that the response amplitude decreases with increase in material gradient and the reason behind it can be attributed to the enhancement of overall system stiffness. The general trend is that, at low response zone the response amplitude increases with the excitation force however, as the amplitude increases, the increasing branches of all the response curves merge together, irrespective of excitation force. For similar conditions, beams with parabolic taper exhibits higher stiffness compared to exponential taper, as indicated by the response curves. It is also seen that as the taper parameter is increased, the response amplitude decreases in low amplitude region but the curves try to merge together as the response amplitude increases.

REFERENCES

1. A. Chakraborty, S. Gopalakrishnan, and J. N. Reddy, "A new beam finite element for the analysis of functionally graded materials." *Int. J. Mech. Sci.* ***45*****(3)**, pp. 519-539, 2003.
2. A. Paul, and D. Das, "Free vibration analysis of pre-stressed FGM Timoshenko beams under large transverse deflection by a variational method." *Eng. Sci. Tech. Int. J.* ***19*****(2)**, pp. 1003-1017, 2016.
3. A. Y. Tang, J. X. Wu, X. F. Li, and K. Y. Lee, "Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams." *Int. J. Mech. Sci.* ***89***, pp. 1-11, 2014.
4. B. V. Sankar, "An elasticity solution for functionally graded beams." *Compos. Sci. Tech.* ***61*****(5)**, pp. 689-696, 2001.
5. H. L. Wu, J. Yang, and S. Kitipornchai, "Nonlinear vibration of functionally graded carbon nanotube-reinforced composite beams with geometric imperfections." *Compos. Part B: Eng*, ***90***, pp. 86-96, 2016.
6. M. Aydogdu, and V. Taskin, "Free vibration analysis of functionally graded beams with simply supported edges." *Mater. Des.* ***28*****(5)**, pp. 1651-1656, 2007.
7. M. Şimşek, "Non-linear vibration analysis of a functionally graded Timoshenko beam under action of a moving harmonic load." *Compos. Struct.* ***92*****(10)**, pp. 2532-2546, 2010.
8. M. Şimşek, and T. Kocatürk, "Free and forced vibration of a functionally graded beam subjected to a concentrated moving harmonic load." *Compos. Struct.* ***90*****(4)**, pp. 465-473, 2009.
9. P. Liu, K. Lin, H. Liu, and R. Qin, "Free transverse vibration analysis of axially functionally graded tapered Euler-Bernoulli beams through spline finite point method." *Shock Vib.* **doi:10.1155/2016/5891030**, *2016.*
10. P. Ribeiro, "Non-linear forced vibrations of thin/thick beams and plates by the finite element and shooting methods," *Compos. Struct.* **82(17)**, pp. 1413-1423, 2004.
11. S. Agarwal, A. Chakraborty and S. Gopalakrishnan, "Large deformation analysis for anisotropic and inhomogeneous beams using exact linear static solutions." *Compos. Struct.* ***72*****(1)**, pp. 91-104, 2006.
12. Y. Huang, L. E. Yang, and Q. Z. Luo, "Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section." *Compos. Part. B Eng.* ***45*****(1)**, pp. 1493-1498, 2013.