# Existence and Concentration of Solutions for a class of elliptic PDEs involving $p$-biharmonic Operator

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**Abstract**

The perturbed Schrodinger equation with $p$-biharmonic operator and real valued parameter has been considered. We use variational technique to guarantee the existence of solution.

**Keywords:** $p$-Laplacian; $p$-biharmonic; elliptic PDE; Sobolev space.

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## 1. Introduction

The problem of perturbed Schrodinger equation has been studied. The potential function $V(x)$ is a real valued continuous function on $\mathbb{R}^N$ satisfying some conditions.Further, to study the existence of nontrivial solution and concentration of solutions (as $\lambda \to \infty$), we make a few assumptions on the nonlinear function $f$. We further have to deal with lack of compact embedding since the domain considered is $\mathbb{R}^N$. We will prove the following results.

**Theorem 1.1.** *Assume the conditions (V1)-(V3), (F1), (F2) to hold. Then there exists $\Lambda_0 > 0$ such that for each $\lambda > \Lambda_0$, problem has at least one non trivial solution $u_\lambda$.*

**Theorem 1.2.** *Let $u_n = u_{\lambda_n}$ be a solution of the problem corresponding to $\lambda = \lambda_n$. If $\lambda_n \to \infty$, then*

$$||u_n||_{\lambda_n} \leq c,$$

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*for some $c > 0$*

$$u_n \to \tilde{u} \text{ in } L^q(\mathbb{R}^N),$$

*and up to a subsequence for $p \leq q < p_*$. This $\tilde{u}$ is a solution of the problem*

$$\begin{aligned} \Delta_p^2 u - \Delta_p u = f(x,u), \text{ in } \Omega \\ u = 0, \text{ on } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{1.1}$$

*and $u_n \to \tilde{u}$ in $W^{2,p}(\mathbb{R}^N)$.*

The paper has been organized as follows. In section 2, we discuss the notations which will be used in the theorems. In section 3, we give the proof of Theorem 1.1 and in section 4, we prove the Theorem 1.1.

## 2. Preliminaries and Notations

We will denote a Sobolev space of order 2 as $W^{2,p}(\mathbb{R}^N)$, which is given by

$$W^{2,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N : |\nabla u|, \Delta u \in L^p(\mathbb{R}^N)\}$$

endowed with the norm

$$||u||^p_{W^{2,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + |u|^p)dx.$$

Let

$$X = \left\{ u \in W^{2,p}(\mathbb{R}^N) : \int_{\mathbb{R}^n} (|\Delta u|^p + |\nabla u|^p + V(x)|u|^p)dx < \infty \right\}$$

be endowed with the norm

$$||u||^p = \int_{\mathbb{R}^n} (|\Delta u|^p + |\nabla u|^p + V(x)|u|^p)dx.$$

For $\lambda > 0$, we set

$$E_\lambda = \{u \in W^{2,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + \lambda V(x)|u|^p)dx < \infty\}$$

with

$$||u||^p_\lambda = \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + \lambda V(x)|u|^p)dx.$$

It is easy to verify that for $\lambda \geq 1$ $(E_\lambda, ||\cdot||_\lambda)$ is a closed in $X$ and

$$||u|| \leq ||u||_\lambda$$

. We will denote $\mu$ to be the Lebesgue measure on $\mathbb{R}^N$.

**Lemma 2.1.** *If (V1)-(V2) hold, then there exists positive constants $\lambda_0, c_0$ such that*

$$||u||_{W^{2,p}(\mathbb{R}^N)} \leq c_0||u||_\lambda; \ \textit{for all } u \in E_\lambda, \lambda \geq \lambda_0.$$

The lemma shows that $E_\lambda \hookrightarrow W^{2,p}(\mathbb{R}^N)$. By the Sobolev embedding results for $p < N$ we have $\hookrightarrow L^q(\mathbb{R}^{\mathbb{N}})$, for $q \in [p, p_*]$.

$$||u||_q \leq c_q||u||_{W^{2,p}(\mathbb{R}^N)} \leq c_0 c_q ||u||_\lambda,$$

for all $\lambda \geq \lambda_0$, $q \in [p, p_*]$.

## 3. Existence of non trivial solutions

We propose the following lemma.

**Lemma 3.1.** *Suppose that (V1)-(V3), (F1),(F2) are satisfied. Then there exists $\Lambda_0 > 0$ such that for every $\lambda \geq \Lambda_0$, $J_\lambda$ is bounded below in $E_\lambda$.*

*Proof.* Since $\xi_i(x) \in L^{\frac{p}{p-\gamma_i}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose $R_\epsilon > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |\xi_i(x)|^{\frac{p}{p-\gamma_i}} dx\right)^{\frac{p-\gamma_i}{p}} < \epsilon, \ \ 1 \leq i \leq m. \tag{3.1}$$

Since $u_n \to u_0$ in $L^p_{loc}(\mathbb{R}^N)$, there exists $N_0 \in \mathbb{N}$ such that

$$\left(\int_{B_{R_\epsilon}} |u_n - u_0|^p dx\right)^{\frac{\gamma_i}{p}} < \epsilon \tag{3.2}$$

for $n \geq N_0$ and for all $1 \leq i \leq m$. Therefore,

$$\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |f(x, u_n - u_0)||u_n - u_0| dx \to 0, \text{ as } n \to \infty. \tag{3.3}$$

From (3.3), we have

$$\int_{\mathbb{R}^N} |f(x, u_n - u_0)||u_n - u_0| dx \to 0, \text{ as } n \to \infty. \tag{3.4}$$

This shows that $u_n \to u_0$ in $E_\lambda$. □

***Proof of the Theorem*** 1.1. By lemmas above, it follows that $c_\lambda = \inf_{E_\lambda} J_\lambda(u)$ is a critical value of $J_\lambda$, that is there exists a critical point $u_\lambda \in E_\lambda$ such that $J_\lambda(u_\lambda) = c_\lambda$. Therefore, $u_\lambda$ is a solution for the problem for $\lambda > \Lambda_0$. Now we will show that $u_\lambda \neq 0$. □

## 4. Limiting case $\lambda \to \infty$

We consider the concentration of solutions for the problem as $\lambda \to \infty$. Since $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset E_\lambda$ for all $\lambda > 0$, restrict $J_\lambda$ on $W^{2,p}(\Omega) \bigcap W_0^{1,2}(\Omega)$.

$$\tilde{c} = \inf_{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)} J_\lambda|_{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)},$$

where $\Omega$ is given in the condition (V3) and $J_\lambda|_{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)}$ is a restriction of $J_\lambda$ on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, that is

$$J_\lambda|_{W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)} = \frac{1}{p} \int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx - \int_\Omega F(x,u) dx,$$

for $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Similar to the proof of the theorem 1.1, it can be seen that $\tilde{c} < 0$ is achieved and

$$c_\lambda \leq \tilde{c} < 0, \text{ for all } \lambda > \Lambda_0.$$

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