Some Fascinating Sum Formula Involving Balancing Number

Gopal Krishna Panda

Abstract: The problem of finding sums of specific sequences of natural numbers had been a fascination to mathematicians. In this connection, there is an interesting story about the famous German mathematician Carl Friedrich Gauss when he was just eight years old and was in primary school. One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. There are binary recurrence sequences having sum formulas resembling that for natural numbers, e.g. the sum of first n odd positive integers is equal to n^2, while the sum of first n odd balancing numbers is equal to the square of the n^{th} balancing numbers. The talk focuses on certain sum formulas involving balancing numbers. In each formula, if the balancing numbers is replaced by its index, it reduces to a known formula for natural numbers.

CERTAIN CURIOUS SUMS INVOLVING BALANCING NUMBERS

Gopal Krishna Panda
Professor, Department of Mathematics
and Dean(SR)
National Institute of Technology
Rourkela-769008
Odisha, INDIA

DART-2016, WUHAN UNIVERSITY, PRC

The speaker is thankful to
the organizers of DART-2016
for their invitation to the
conference and local hospitality.

The Diophantine equation

$$\boldsymbol{y^2 - dx^2 = 1}$$

is known as Pell's equation and the particular case

$$\boldsymbol{y^2 - 8x^2 = 1}$$

has the solutions

$$x = B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}, y = C_n = \frac{\alpha^n + \beta^n}{2}, n = 1,2,\cdots$$

B_n's are known as balancing numbers while C_n $'s$ are called Lucas-balancing numbers. The Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers.

The problem of finding sums of specific sequences of natural numbers had been a fascination to mathematicians. In this connection, there is a interesting story about the famous German mathematician Carl Friedrich Gauss when he was just eight years old and was in primary school.

One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple. This led to the discovery of the famous formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

The sequence of natural numbers is enriched with the following sum formulas.

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$1 + 3 + \cdots + (2n - 1) = n^2$$

$$2 + 4 + \cdots + 2n = n(n+1)$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

A natural question is : *"Is it possible to find such sum formulas for specific integer sequences?"* Panda [8] answered this question in affirmative, at least for the sequence of balancing numbers. He showed that

$$B_1 + B_3 + \cdots + B_{2n-1} = {B_n}^2$$

and

$$B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}$$

where B_n is the n^{th} balancing numbers. These formulas exactly look like corresponding sum formulas for natural numbers.

What is a balancing number?

Every balancing number n is associated with a balancer r and such pairs (n, r) are solutions of the Diophantine equation

$$\mathbf{1 + 2 + \cdots + (n - 1) = (n + 1) + \cdots + (n + r)}.$$

The balancing numbers can also be obtained from the binary recurrence

$$B_{n+1} = 6B_n - B_{n-1}$$

with initial values $B_0 = 0, B_1 = 1$.

The Binet form for balancing numbers is

$$B_n = \frac{\alpha^n - \beta^n}{4\sqrt{2}}$$

where $\alpha = 3 + 2\sqrt{2}, \beta = 3 - 2\sqrt{2}$. The balancing numbers satisfy the identities

$$B_{n+r} \cdot B_{n-r} = B_n^2 - B_r^2$$

for $n \geq r$. In particular, for $n \geq 1$, one has

$$B_{n+1} \cdot B_{n-1} = B_n^2 - 1.$$

$$B_{2n-1} = B_n^2 - B_{n-1}^2$$

Panda and Rout [7] proved that the class of sequences obtained from the binary recurrences

$$x_{n+1} = Ax_n - x_{n-1}$$

with initial values $x_0 = 0, x_1 = 1,$ where $A > 2$ is any arbitrary integer, also satisfies

$$x_1 + x_3 + \cdots + x_{2n-1} = {x_n}^2$$

and

$$x_2 + x_4 + \cdots + x_{2n} = x_n x_{n+1}$$

It is important to note that the natural numbers are solutions of the above binary recurrence corresponding to $A = 2$. Hence, the sequences arising out of the above binary recurrences are called natural sequences.

Panda and Ray [6] obtained the following interesting sum formula for balancing numbers.

$$B_1 + B_2 + \cdots + B_n = \frac{R_{n+1}}{2}$$

where R_n denotes balancer corresponding to the the n^{th} balancing number.

The balancers' sequence R_n coincides with the sequence of cobalancing numbers. Every cobalancing number n is associated with a cobalancer r and by definition such pairs (n, r) are solutions of the Diophantine equation

$$\mathbf{1} + \mathbf{2} + \cdots \boldsymbol{n} = (\boldsymbol{n} + \mathbf{1}) + \cdots + (\boldsymbol{n} + \boldsymbol{r}).$$

The cobalancing numbers can also be calculated from the binary recurrence

$$b_{n+1} = 6b_n - b_{n-1} + 2$$

with initial values $b_0 = b_1 = 0$.

An important observation about the sum formulas for balancing numbers is that

$$n|1+2+\cdots+2n-1 \text{ and } n|1+2+\cdots+2n.$$

A similar result also holds for the sum of balancing numbers, see [8].

$$B_1+B_2+\cdots+B_{2n-1}=B_n(B_{n-1}+B_n)$$

and

$$B_1+B_2+\cdots+B_{2n}=B_n(B_{n+1}+B_n).$$

These formulas resembles sum formulas for natural numbers in the sense that they continue to hold if B_n is replaced by n.

For the integer sequences obtained from the class of binary recurrences

$$\boldsymbol{x_{n+1} = Ax_n - x_{n-1}}$$

with initial values $\boldsymbol{x_0 = 0}$, $\boldsymbol{x_1 = 1}$, where $\boldsymbol{A > 2}$, we also have

$$x_1 + x_2 + \cdots + x_{2n-1} = x_n(x_{n-1} + x_n)$$

and

$$x_1 + x_2 + \cdots + x_{2n} = x_n(x_{n+1} + x_n).$$

So far as the sum of squares of balancing numbers is concerned, we do not have a formula resembling that for natural numbers. The sum formula

$$B_1{}^2 + B_2{}^2 + \cdots + B_n{}^2 = \frac{B_{2n+1} - 2n - 1}{32}$$

can be derived from the Binet form for balancing numbers. For example, when $n = 3$

$$B_1{}^2 + B_2{}^2 + B_3{}^2 = 1^2 + 6^2 + 35^2 = \mathbf{1262}$$

while

$$\frac{B_7 - 6 - 1}{32} = \frac{40391 - 6 - 1}{32} = \mathbf{1262}.$$

The following are some interesting sum formulas with alternating signs:

$$\sum_{k=1}^{2n-1} (-1)^{k-1} {B_k}^2 = \frac{B_{2n-1}B_{2n}}{6}$$

$$\sum_{k=0}^{2n-1} (-1)^{k-1} {C_k}^2 = 8 \cdot \frac{B_{2n-1}B_{2n}}{6}$$

$\boldsymbol{C_n = \sqrt{8{B_n}^2 + 1}}$ is the n^{th} Lucas-balancing number, can be calculated recurrently as $\boldsymbol{C_{n+1} = 6C_n - C_{n-1}}$ with initial values $\boldsymbol{C_0 = 1, C_1 = 3.}$ The Binet form is $C_n = \frac{\alpha^n + \beta^n}{2}$.

Verification of $\sum_{k=1}^{2n-1}(-1)^{k-1}\, B_k{}^2 = \frac{B_{2n-1}B_{2n}}{6}$

$B_1{}^2 - B_2{}^2 + B_3{}^2 = 1^2 - 6^2 + 35^2 = \mathbf{1190},$

$\frac{B_3 B_4}{B_2} = \frac{35 \cdot 204}{6} = \mathbf{1190}.$

Verification of $\sum_{k=0}^{2n-1}(-1)^{k-1}\, C_k{}^2 = 8 \cdot \frac{B_{2n-1}B_{2n}}{6}$

$-C_0{}^2 + C_1{}^2 - C_2{}^2 + C_3{}^2 = 1^2 - 3^2 + 19^2 - 99^2 = \mathbf{9520},$

$8 \cdot \frac{B_3 B_4}{B_2} = 8 \cdot \frac{35 \cdot 204}{6} = \mathbf{9520}$

$$\sum_{k=1}^{2n-1} (-1)^{k-1} B_{2k} = \frac{B_{2n}C_{2n-1}}{3}$$

Proof: Using the Binet form for balancing numbers

$$\sum_{k=0}^{2n-1}(-1)^k B_{2k} = \frac{1}{4\sqrt{2}}\sum_{k=0}^{2n-1}(-1)^k(\alpha^{2k} - \beta^{2k})$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{-\alpha^{4n}+1}{\alpha^2+1} - \frac{-\beta^{4n}+1}{\beta^2+1}\right]$$

$$= \frac{1}{24\sqrt{2}}[-\alpha^{4n-1} + \beta + \alpha^{4n-1} - \alpha]$$

$$= -\frac{1}{6}[1 + B_{4n-1}]$$

$$= -\frac{1}{3}[B_{2n}C_{2n-1}]$$

In a similar manner, the following identities can be verified.

$$\sum_{k=0}^{2n-1} (-1)^{k-1} C_{2k} = \frac{8B_{2n}B_{2n-1}}{3}$$

$$\sum_{k=1}^{2n-1} (-1)^{k-1} {B_{2k}}^2 = \frac{B_{4n-2}B_{4n}}{34}$$

$$\sum_{k=0}^{2n-1} (-1)^{k-1} {C_{2k}}^2 = 8 \cdot \frac{B_{4n-2}B_{4n}}{34}$$

It is easy to verify the following sum formula for natural numbers

$$\sum_{k=0}^{2n-1} (-1)^{k-1}\, 2k(k+1) = (2n)^2$$

The following formula (which can be easily proved by induction) in terms balancing numbers is similar to the above formula

$$\sum_{k=1}^{2n-1} (-1)^{k-1}\, B_2 B_k B_{k+1} = {B_{2n}}^2$$

Verification:

$$B_2(B_1B_2 - B_2B_3 + B_3B_4) = 6(1 \cdot 6 - 6 \cdot 35 + 35.204) = \mathbf{41616}$$

$${B_4}^2 = 204^2 = \mathbf{41616}$$

Well-known sum formula for natural numbers:

$$\sum_{k=1}^{n} k^2 \cdot 2k = \frac{n^2(n+1)^2}{2}.$$

Sum formula for balancing numbers resembling the above formula

$$\sum_{k=1}^{n} {B_k}^2 \cdot B_{2k} = \frac{{B_n}^2 {B_{n+1}}^2}{B_2}$$

Proof (by induction): The statement is true for $n = 1$. For $n = m + 1$,

$$\sum_{k=1}^{m+1} B_k{}^2 B_{2k} = \frac{B_m{}^2 B_{m+1}{}^2}{B_2} + B_{m+1}{}^2 B_{2m+2}$$

$$= \frac{B_{m+1}{}^2}{B_2}\left(B_m{}^2 + B_{2m+2}B_2\right)$$

$$= \frac{B_{m+1}{}^2 B_{m+2}{}^2}{B_2}$$

by induction statement is true for all n.

Well-known sum formula for natural numbers:

$$\sum_{k=1}^{n} (-1)^{n-k} \cdot k^2 = \frac{n(n+1)}{2}.$$

Sum formula for balancing numbers resembling the above formula is

$$\sum_{k=1}^{n} (-1)^{n-k} {B_k}^2 = \frac{B_n B_{n+1}}{B_2}$$

Well-known sum formula for natural numbers:

$$\sum_{k=1}^{n} k(k+2)(2k+2) = \frac{n(n+1)(n+2)(n+3)}{2}.$$

Sum formula for balancing numbers resembling the above formula

$$\sum_{k=1}^{n} B_k B_{k+2} B_{2k+2} = \frac{B_n B_{n+1} B_{n+2} B_{n+3}}{B_2}$$

Proof: Statement is true for $n = 1$. Assuming the statement true for $n = m$ and using the index reduction formula

$$B_a B_b - B_c B_d = B_{a+r} B_{b+r} - B_{c+r} B_{d+r}$$

which holds when $a + b = c + d$, we have

$$\sum_{k=1}^{m+1} B_k B_{k+2} B_{2k+2} = \frac{B_m B_{m+1} B_{m+2} B_{m+3}}{B_2} + B_{m+1} B_{m+3} B_{2m+4}$$

$$= \frac{B_{m+1} B_{m+3}}{B_2} [B_m B_{m+2} + B_2 B_{2m+4}]$$

$$= \frac{B_{m+1} B_{m+3}}{B_2} [B_m B_{m+2} - B_{-2} B_{2m+4}]$$

$$= \frac{B_{m+1} B_{m+3}}{B_2} (B_{m+2} B_{m+4})$$

which shows the statement is true for $n = m + 1$.

Well-known sum formula for natural numbers:

$$\sum_{k=1}^{n}(-1)^{n-k} \cdot 2(2k-1) = 2n.$$

A sum formula for balancing numbers resembling the above formula

$$\sum_{k=1}^{n}(-1)^{n-k} B_2 B_{2k-1} = B_{2n}.$$

Verification:

$$B_2(B_1 - B_3 + B_5) = 6(1 - 35 + 1189) = \mathbf{6930} = \boldsymbol{B_6}$$

Well-known sum formula for natural numbers:

$$\sum_{k=1}^{n} k \cdot 3k = \frac{n(n+1)(2n+1)}{2}.$$

Sum formula for balancing numbers resembling the above formula (which can be proved by induction)

$$\sum_{k=1}^{n} B_k B_{3k} = \frac{B_n B_{n+1} B_{2n+1}}{B_2}$$

Verification:

$$B_1 B_3 + B_2 B_6 = 1 \cdot 35 + 6.6930 = \mathbf{41615}$$

$$\frac{B_2 B_3 B_5}{B_2} = 35 \cdot 1189 = \mathbf{41615}$$

Well-known sum formula for natural numbers:

$$\sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n - 1}{2^n}.$$

Sum formula for balancing numbers resembling the above formula

$$\sum_{k=1}^{n} \frac{1}{B_{2^k}} = \frac{B_{2^n - 1}}{B_{2^n}}.$$

Proof:

$$\sum_{k=1}^{n} \frac{1}{B_{2^k}} = \sum_{k=1}^{n} \frac{B_{2^k-2^{k-1}}}{B_{2^{k-1}} B_{2^k}}$$

$$= \sum_{k=1}^{n} \frac{B_{2^{k-1}+1} B_{2^k} - B_{2^{k-1}} B_{2^k+1}}{B_{2^{k-1}} B_{2^k}}$$

$$= \sum_{k=1}^{n} \left[\frac{B_{2^{k-1}+1}}{B_{2^{k-1}}} - \frac{B_{2^k+1}}{B_{2^k}} \right]$$

$$= \frac{B_2}{B_1} - \frac{B_{2^n+1}}{B_{2^n}}$$

$$= \frac{B_{2^n-1}}{B_{2^n}}$$

Other sum formulas

$$\sum_{k=1}^{n} B_k B_{k+1} \cdots B_{k+2m} C_{k+m} = \frac{B_n B_{n+1} \cdots B_{n+2m+1}}{2B_{m+1}}$$

Proof: Let

$$l_n = \sum_{k=1}^{n} B_k B_{k+1} \cdots B_{k+2m} C_{k+m}, r_n = \frac{B_n B_{n+1} \dots B_{n+2m+1}}{2B_{m+1}}.$$

Since, $r_n - r_{n-1} = \frac{B_{n+1} \cdots B_{n+2m}}{2B_{m+1}} [B_{n+2m+1} - B_{n-1}]$

$$= \frac{B_{n+1} \cdots B_{n+2m}}{2B_{m+1}} (2B_{m+1} C_{n+m})$$

$$= B_{n+1} \cdots B_{n+2m} C_{n+m}$$

$$= l_n - l_{n-1}$$

Now, $r_1 = \frac{B_1 B_2 \cdots B_{2m+2}}{2B_{m+1}} = B_1 B_2 \cdots B_{2m+1} C_{m+1} = l_1.$

Applying similar technique one can also prove

$$\sum_{k=1}^{n}(-1)^k \cdot B_k B_{k+1} \cdots B_{k+2m} C_{k+m} = (-1)^n \cdot \frac{B_n B_{n+1} \cdots B_{n+2m+1}}{2B_{m+1}}$$

$$\sum_{k=1}^{n} C_k C_{k+1} \cdots C_{k+2m} B_{k+m} = \frac{C_n C_{n+1} \cdots B_{n+2m+1} - C_1 C_2 \cdots C_{2m+1}}{16B_{m+1}}$$

$$\sum_{k=1}^{n}(-1)^k \cdot C_k C_{k+1} \cdots C_{k+2m} C_{k+m}$$

$$= \frac{(-1)^n C_n C_{n+1} \cdots B_{n+2m+1} - C_1 C_2 \cdots C_{2m+1}}{2C_{m+1}}$$

$$\sum_{k=1}^{n} {B_k}^2 \, {B_{k+1}}^2 \cdots {B_{k+2m}}^2 B_{k+2m} = \frac{{B_n}^2 {B_{n+1}}^2 \cdots {B_{n+2m+1}}^2}{B_{2m+2}}$$

$$\sum_{k=1}^{n} {C_k}^2 \, {C_{k+1}}^2 \cdots {C_{k+2m}}^2 B_{k+2m}$$

$$= \frac{{C_n}^2 {C_{n+1}}^2 \cdots {C_{n+2m+1}}^2 - {C_1}^2 {C_2}^2 \cdots {C_{2m+1}}^2}{8B_{2m+2}}$$

Some convolution-type sum formulas for balancing and Lucas balancing numbers

$$\sum_{k=0}^{n} B_{mk}C_{m(n-k)} = \frac{1}{2}(n+1)B_{nm}$$

Proof:

$\sum_{k=0}^{n} B_{mk}C_{m(n-k)}$

$$= \frac{1}{8\sqrt{2}}\sum_{i=0}^{n}(\alpha^{2nm} - \beta^{2nm} + \alpha^{4mk-2nm} - \beta^{4nk-2nm})$$

$$= \frac{1}{2}(n+1)B_{nm} + \frac{1}{2\sqrt{2}}\sum_{i=0}^{n} B_{mi-nm}$$

$$= \frac{1}{2}(n+1)B_{nm}$$

Since $\sum_{i=0}^{n} B_{m(i-n)} = 0$ as $B_0 = 0, B_{-x} = -B_x$

$$\sum_{k=1}^{n} \binom{n}{k} B_{mk} C_{m(n-k)} = 2^{m-1} B_{mn}$$

$$\sum_{k=1}^{n} (-1)^k B_{mk} C_{m(n-k)} = \begin{cases} \dfrac{B_{mn}}{2} & \text{if } n \text{ is even} \\ -\dfrac{B_{m(n+1)}}{2C_m} & \text{if } n \text{ is odd} \end{cases}$$

Verification: For $n = 2$,
$\binom{2}{0} B_0 C_4 + \binom{2}{1} B_2 C_2 + \binom{2}{2} B_4 C_0 = 0 + 2 \cdot 6 \cdot 17 + 1 \cdot 204.1 = 408 = 2 \cdot 204 = 2 \cdot B_4$

$$\sum_{k=0}^{n}(-1)^k C_{mk}\, B_{m(n-k)} = \begin{cases} \dfrac{B_{mn}}{2} & \text{if } n \text{ is even} \\ -\dfrac{B_{m(n+1)}}{2C_m} & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} B_{mk}\, C_{m(n-k)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\left(4\sqrt{2}\right)^{n-1} {B_m}^{n} & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} C_{mk}\, B_{m(n-k)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \left(4\sqrt{2}\right)^{n-1} {B_m}^{n} & \text{if } n \text{ is odd} \end{cases}$$

For $k \geq 6$,

$$\sum_{n=0}^{\infty} \frac{B_n}{k^n} = \frac{k}{k^2 - 6k + 1}.$$

Proof:

$$\sum_{n=0}^{\infty} \frac{B_n}{k^n} = \frac{1}{4\sqrt{2}} \sum_{n=0}^{\infty} \left[\left(\frac{\alpha}{k}\right)^n - \left(\frac{\beta}{k}\right)^n \right]$$

$$= \frac{1}{4\sqrt{2}} \left[\frac{k}{k-\alpha} - \frac{k}{k-\beta} \right]$$

$$= \frac{k}{4\sqrt{2}} \left[\frac{\alpha-\beta}{(k-\alpha)(k-\beta)} \right]$$

$$= \frac{k}{k^2-6k+1}$$

The convergence can be made faster by choosing larger value of k.

For $k \geq 6$

$$\sum_{n=0}^{\infty} \frac{C_n}{k^n} = \frac{k(k-3)}{k^2 - 6k + 1}$$

$$\sum_{n=0}^{\infty} \frac{nB_n}{k^n} = \frac{k(k^2-1)}{(k^2 - 6k + 1)^2}.$$

$$\sum_{n=0}^{\infty} \frac{nC_n}{k^n} = \frac{k(3k^2 - 2k + 3)}{(k^2 - 6k + 1)^2}$$

Under the same condition

$$\sum_{n=0}^{\infty} \frac{n^2 B_n}{k^n} = \frac{k(k^4 + 6k^3 - 6k^2 + 6k + 1)}{(k^2 - 6k + 1)^3}$$

$$\sum_{n=0}^{\infty} \frac{n^2 C_n}{k^n} = \frac{k(3k^4 + 14k^3 - 14k - 3)}{(k^2 - 6k + 1)^3}$$

REFERENCES

1. G. K. Panda and S. S. Rout, A class of recurrent sequences exhibiting some exciting properties of balancing numbers, International Journal of Mathematics and Computer Science 6 (2012), 4-6.

2. A. Behera and G. K. Panda, On the square roots of triangular numbers, Fib. Quart., 37(2) (1999), 98-105.

3. G. K. Panda, Some fascinating properties of balancing numbers, In Proc. of Eleventh Internat. Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium, 194, (2009), 185-189.

4. R. K. Davala and G. K. Panda , On sum and ratio formulas for balancing numbers, Journal of the Indian Math. Soc. 82(2) (2015), 23- 32.

5. G. K. Panda and P. K. Ray, Cobalancing numbers and cobalancers, Internat. J. Math. Math. Sci. 8 (2005), 1189–1200.

6. S. H. Holliday and T. Komatsu, “On the sum of reciprocal generalized Fibonacci numbers,” Integers, vol. 11, no. 4, pp. 441–455, 2011

7. T. Komatsu, “On the nearest integer of the sum of reciprocal Fibonacci numbers, A-portaciones,”Matematicas Investigation, vol. 20, pp. 171–184, 2011.

8. T. Komatsu and V. Laohakosol, “On the sum of reciprocals of numbers satisfying a recurrence relation of order s,” Journal of Integer Sequences, vol. 13, no. 5, pp. 1–9, 2010.