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Finite element based vibration analysis of a nonprismatic Timoshenko beam with transverse open crack

R.K.Behera [1*], A.R.Biswal[2], T.Roy[3]
[1*] Associate Professor, NIT Rourkela, Rourkela, India, 769008.
[2] Ph.d scholar, NIT Rourkela, Rourkela, India, 769008.
[3] Assisstant Professor, NIT Rourkela, Rourkela, India, 769008.

ABSTRACT

The present day structures and machineries are designed based on optimizing of multi-objectives such as maximum strength, maximum life, minimum weight and minimum cost. Due to this they are flexible and allow having a very high level of stresses. This leads to development of cracks in their elements. Many engineering structures may have structural defects such as cracks due to long-term service. So it is very much essential to know the property of structures and response of such structures in various cases. The present article deals with finite element based vibration analysis of a cracked beam. The beam is modeled using the Timoshenko beam theory. The governing equation of motion has been derived by the Hamilton's principle. In order to solve the governing equation two noded beam element with two degrees of freedom (DOF) per node has been considered. In this work the effect of structural damping has also been incorporated in the finite element model. The analysis is carried out by using state space model in time domain.

Keywords: Finite element method, Nonprismatic beam, Timoshenko beam, State space.

1. INTRODUCTION

Presence of crack in a structural member is a serious threat to the performance of the structure. The effects of crack on the dynamic behaviour of the structural elements have been the subject of several investigations for the last few decades. Due to the existence of such cracks the frequencies of natural vibration, amplitudes of forced vibration, and areas of dynamic stability change. In order to identify the magnitude and location of the crack, analysis of these changes is essential. The information from the analysis enables one to determine the degree of sustainability of the structural element and the whole structure.

Beams are one of the most commonly used elements in structures and machines, and fatigue cracks are the main cause of beams failure. The introduction of local flexibility due to presence of transverse crack in a structural member whose dimension depends on the number of degrees of freedom considered [1]. It has been observed that the local flexibility matrix is

*Further author information:
E-mail: rabi57@rediffmail.com, Telephone: 06612462504
E-mail: alokbiswal82@gmail.com
E-mail: tarapada@nitrkl.ac.in, Telephone: 06612462507

mainly appropriate for the analysis of a cracked beam if one employs an analytical method by solving the differential equations piece wisely [2]. One way to detect cracks on structures is to employ modal testing in which changes in modal parameters such as variations in frequencies and mode shapes are used to detect damage. The detection of structural damage through changes in frequencies was discussed [3]. Moreover, the damage identification basically based on changes in the nodes of mode shapes [4]. It was demonstrated that appropriate use of resonances and anti-resonances can be used to avoid the non-uniqueness of damage location for symmetrical beams.

Identification of cracks in beam structures using Timoshenko and Euler beam formulation has been studied [5]. The Timoshenko and Euler beam formulations have been used to estimate the influence of crack size and location on natural frequencies of cracked beam. Frequency contour method has been used to identify the crack size and location properly. The free vibration analysis of cantilever beam was discussed. It has been observed that the presence of crack in the beam, will affect the natural frequency. The magnitude for the change of natural frequency depends on the change of (number, depth and location) for the crack.Also the change of dynamic property will effect on stiffness and dynamic behaviour.

A method was used to find the lowest four natural frequencies of the cracked structure by finite element method [6]. It has been obtained the approximate crack location by using Armon's Rank-ordering method that uses the above four natural frequencies. A method for shaft crack detection have proposed [7] which is based on combination of wave-let based elements and genetic algorithm. The experimental investigations of the effects of cracks and damages on the structures have been reported [8]. The reduction of Eigen frequencies and sensitivity analysis to localize a crack in a non-rotating shaft coupled to an elastic foundation have been studied [9]. The shaft was modelled by the finite element method and coupled to an experimentally identified foundation model. The different damage scenarios by reducing the local thickness of the selected elements at different locations along with finite element model (FEM) for quantification and localization of damage in beam-like structures is investigated [10]. An analytical as well as experimental approach to the crack detection in cantilever beams by vibration analysis was discussed [11].

The finite element method for static and dynamic analysis of a cracked prismatic beam on the basis of Hamilton's principle was discussed [12]. The crack section was modelled as an elastic hinge by considering fracture mechanics theory. The component mode synthesis technique along with finite element method for free vibration analysis of uniform and stepped cracked beam with circular cross section was discussed [13] . The finite element analysis of a cracked cantilever beam and the relation between the modal natural frequencies with crack depth, modal natural frequency with crack location has been studied [14]. Only single crack at different depth and at different location are evaluated. The analysis reveals a relationship between crack depth and modal natural frequency. An overall flexibility matrix instead of local flexibility matrix in order to find out the total flexibility matrix and the stiffness matrix of the cracked beam is considered [15]. It has been observed that the consideration of 'overall additional flexibility matrix', due to the presence of the crack, can indeed give more accurate results than those obtained from using the local flexibility matrix. The overall additional flexibility matrix parameters are computed by 128-point (1D) and 128 ×128(2D) Gauss quadrature and then further best fitted using the least-squares method. The best-fitted formulas agree very well with the numerical integration results [16]. After getting the stiffness matrix of a cracked beam element standard FEM procedure can be followed, which will lead to a generalized eigenvalue problem and thus the natural frequencies can be obtained.

This article exclusively focused on vibration analysis of a cracked nonprismatic Timoshenko cantilever beam by using finite element analysis. The governing equation of motion has been derived by using Hamilton's principle. In order to solve the governing equation two noded beam element with two degrees of freedom (DOF) per node has been considered. The effect of structural damping has also been incorporated in the finite element model.

2. MATHEMATICAL FORMULATION

This mathematical formulation dealt with finite element modeling of uncracked beam and cracked beam which are discussed in the following sections.

2.1 Finite Element Modeling of Uncracked Timoshenko Beam

In order for modeling the cross section of the beam the shape function profile can be represented as

$$A(x) = A_0\left(1 - c\frac{x}{L_b}\right) \tag{1}$$

Where $A(x)$ is represented as the area at any position x of the beam.A_0 is the cross section area near the clamped end of the beam. Lb is the length of the beam.c is the taper values which vary from 0 to 1.

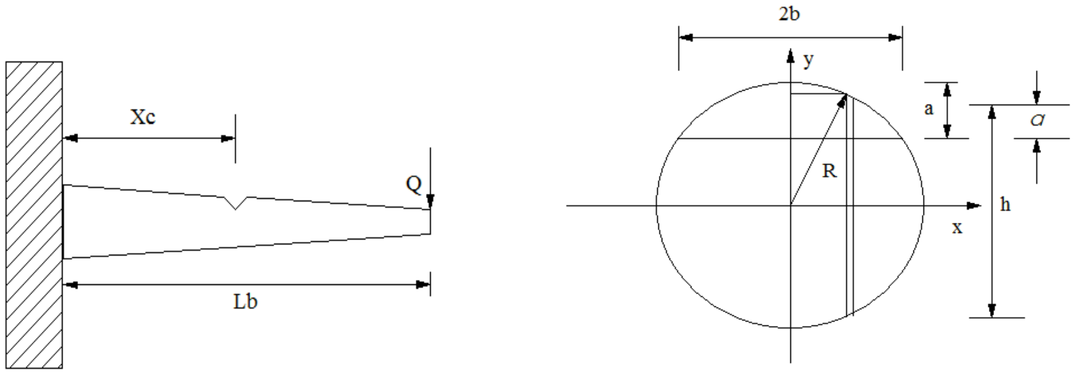

Figure 1. A cantilever beam with crack subjected to shear force and bending moment.

Figure 1 shows a cantilever beam of circular cross-section having diameter '*D*' with a single transverse crack with constant depth '*a*'. The crack is at a distance of 'X_c' from the clamped end of the beam. The beam is divided into number of equal number of finite elements having length 'L_e'. The dynamic equations of motion for system are derived using Hamilton'sprinciple as

$$\partial\Psi = \int_{t_1}^{t_2} \left[\partial\left(KE - PE + Wp \right) \right] dt = 0 \quad . \tag{2}$$

Where'∂' is the variation, t_1 and t_2 are the starting and finish time, *KE* is the total kinetic energy, *PE* is the total potential energy and *Wp* is the total work done by the external mechanical force. The sum of (*KE*−*PE*+*Wp*) is called the Lagrangian *L*a.

$$PE = \frac{1}{2} \int_{V_b} S^T T dV_b \ . \tag{3}$$

$$KE = \frac{1}{2} \int_{V_b} \rho_b \dot{q}^T \dot{q} dV_b \ . \tag{4}$$

$$W_p = \sum_{i=1}^{n_f} \partial q\left(x_i\right) . Q_i\left(x_i\right) . \tag{5}$$

V is the volume, *q* is the displacement, *x* is the position along the beam, *ρ* is the density and the subscripts *b* represents the beam material. Now using Eqs. (3), (4) and (5) and putting it in in Eq. (2) we get

$$\partial\psi = \int_{t_1}^{t_2}\left[\int_{v_b}\rho_b \delta\dot{q}^T \dot{q}dv_b - \int_{v_b}\delta S^T c^s S dv_b + \sum_{i=1}^{n_f}\delta q(x_i)Q(x_i)\right] \tag{6}$$

This equation can now be used to solve for the equations of motion of dynamical mechanical system. By using finite element formulations the displacement field in terms of shape functions can be represented as

$$q(x,t) = [N_w]\{w\},$$
$$q^{'}(x,t) = [N_\theta]\{w\}. \tag{7}$$

Where $[N_w]$ and $[N_\theta]$ are the shape functions for displacement and rotation and $\{w\}$ is the nodal displacements. Using Eqs. (7), we can simplify the variational indicator to include terms that represent physical parameters. By doing this the equations describing the system become more recognizable when compared to those of a typical system and help give physical meaning to the parameters in the equations of motion. The mass matrix for the system can be written as

$$[M_b] = \int_0^{L_b}[N_w]^T \rho_b A_b(x)[N_w]dx. \tag{8}$$

The stiffness matrix can be written as:

$$[K_b] = \int_0^{L_b}\left[\frac{\partial[N_\theta]}{\partial x}\right]^T E_b I_b(x)\left[\frac{\partial[N_\theta]}{\partial x}\right]dx \tag{9}$$

Now considering all the equations in Eq. (6), the equation will become

$$\partial\psi = \int_{t_1}^{t_2}\left[\left\{\delta\dot{w}^T(t)[M_b]\dot{w}(t)\right\} + \left\{\delta w^T(t)[K_b]w(t)\right\} + \left\{\sum_{i=1}^{n_f}\delta w(t).[N_w]^T Q_i(t)\right\}\right]. \tag{10}$$

Taking the integral of the above equation leaves the dynamic equations.

$$[M_b]\ddot{w}(t) + [K_b]w(t) = \sum_{i=1}^{n_f}[N_w]^T Q_i(t) \tag{11}$$

The Eq. (11) now represents the mechanical system and can be used to determine the motion of the beam. In addition to this, the system should have some type of additional mechanical damping that needs to be accounted for. The amount of mechanical damping added to the model was determined from experimental results. This is done using proportional damping methods and the damping ratio that is predicted from the measured frequency response function. With the damping ratio known, proportional damping can be found as [17].

$$c = \alpha[M_b] + \beta[K_b]. \tag{12}$$

Where α and β are determined from

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \quad \text{i=1, 2, 3…n}$$

Where ζ_i is the damping ratio found from frequency response of the structure.Hence the Eq. (11) will become

$$[M]\ddot{w}(t) + [C]\dot{w}(t) + [K]w(t) = \sum_{i=1}^{nf} [N_w]^T Q_i(t) \tag{13}$$

Where [M], [C] and [K] are the global mass, damping and stiffness matrix of the system .The mass matrix and stiffness matrix have been calculated by numerical integration using Gauss quadrature.

2.2 Finite Element Modeling of Cracked Timoshenko Beam Element

From figure (1) the various geometric dimensions can be obtained as

$$b = \sqrt{Da - a^2}$$

$$h = \sqrt{D^2 - 4x^2}$$

$$\alpha = \frac{1}{2}\left[\sqrt{D^2 - 4x^2} - (D - 2a)\right]. \tag{14}$$

The additional strain energy due to the existence of the crack can be expressed as [18], [19].

$$\Pi_c = \int_{Ac} G dA \tag{15}$$

Where 'G' is the strain energy release rate function. The strain energy release rate function can be expressed as

$$G = \frac{1}{E'}\left[(K_{I2} + K_{I3})^2 + K_{II2}{}^2\right] \tag{16}$$

Where $E' = E$ for plane stress problem, $E' = E/1 - \mu^2$ for plane strain problem. K_{I2}, K_{I3}, K_{II2} are the stress intensity factors. The values of stress intensity factor can be can be expressed as

$$K_{ni} = \sigma_i \sqrt{\pi\alpha} F_n\left(\frac{\alpha}{h}\right). \tag{17}$$

Using Paris equation

$$\delta_i = \frac{\partial \Pi_c}{\partial P_i} \qquad (i = 2,3). \tag{18}$$

The overall additional flexibility matrix C_{ij} can be obtained as [19]

$$C_{ij} = \frac{\partial \delta_i}{\partial P_j} = \frac{\partial^2 \Pi_c}{\partial P_i \partial P_j} \quad (i = 2,3) \quad . \tag{19}$$

By combining the equations we get

$$C_{ij} = \frac{1}{E'} \frac{\partial^2}{\partial P_i P_j} \int_{-\sqrt{(Da-a^2)}}^{\sqrt{(Da-a^2)}} \int_0^{\frac{1}{2}\left[\sqrt{(Da-4x^2)}-(D-2a)\right]} \left[\left\{ \frac{32 P_2 L_c h}{\pi D^4} \sqrt{\pi \alpha} F_2\left(\frac{\alpha}{h}\right) + \frac{32 P_3 h}{\pi D^4} \sqrt{\pi \alpha} F_2\left(\frac{\alpha}{h}\right) \right\}^2 + \frac{16 {P_2}^2}{\pi^2 D^4} \pi \alpha {F_{II}}^2\left(\frac{\alpha}{h}\right) \right] d\alpha dx \tag{20}$$

2.2.1 Overall Additional Flexibility Matrix Under Conventional FEM Co-Ordinate System

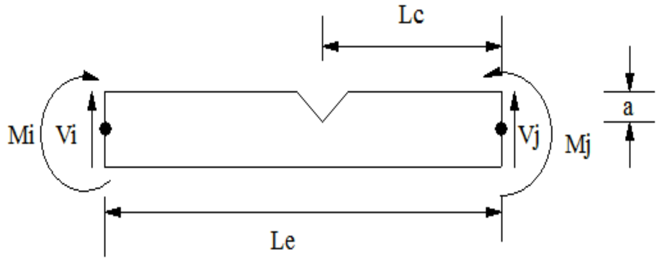

Figure 2. Cracked Timoshenko beam element.

Figure 2 shows a cracked beam element with generated loading. The beam is subjected to shearing force ‘*V*’ and bending moment ‘*M*’ at each node. The corresponding displacements are denoted as ‘*y*’ and ‘*θ*’. ‘L_c’ denotes the distance between the right hand side end node and the crack location. ‘*a*’ denotes the crack depth. The beam element has length ‘L_e’, cross-sectional area ‘*A*’ and flexural rigidity ‘*EI*’.Under the FEM co-ordinate and notation system, the relationship between the displacement and the forces can be expressed as

$$\begin{Bmatrix} y_j - y_i - \theta_i \\ \theta_j - \theta_i \end{Bmatrix} = C_{ovl} \begin{Bmatrix} V_j \\ M_j \end{Bmatrix} . \tag{21}$$

2.2.2 Flexibility Matrix For Intact Timoshenko Beam Element

The flexibility matrix C_{intact} of the intact Timoshenko beam element can be written as

$$\begin{Bmatrix} y_j - y_i - \theta_i \\ \theta_j - \theta_i \end{Bmatrix} = C_{\mathrm{int}act} \begin{Bmatrix} V_j \\ M_j \end{Bmatrix} . \tag{22}$$

2.2.3 Total Flexibility Matrix Of The Cracked Timoshenko Beam Element

The Total flexibility matrix of the cracked Timoshenko beam element is obtained by the combination of over-all additional flexibility matrix and flexibility matrix of an intact beam

$$C_{total} = C_{ovl} + C_{\text{intact}} \tag{23}$$

2.2.4 Stiffness Matrix Of A Cracked Timoshenko Beam Element

Through the equilibrium conditions, the stiffness matrix 'K_c' of a cracked beam element can be obtained as follows [20] [21]

$$K_c = LC_{total}{}^{-1}L^T . \tag{24}$$

Where

$$L = \begin{bmatrix} -1 & 0 \\ -L_e & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.3 State Space Representation

This method is used to derive the uncoupled equations governing the motion of the free vibrations of the system in terms of principal coordinates by introducing a linear transformation between the generalized coordinates *w (t)* and the principal coordinates *η(t)* [22].The displacement vector *r(t)* can be approximated by using a transformation matrix between the generalised coordinates and the modal coordinates as

$$w(t) = \Phi \eta(t) \tag{25}$$

Where ϕ is the modal matrix containing the eigenvectors representing the vibratory modes. Representing Eq. (25) in the state-space form as

$$\begin{aligned} \{\dot{X}\} &= [A]\{X\} + [B]\{u\} \\ \{y\} &= [C]\{X\} \end{aligned} , \tag{26}$$

Where [*A*] is the system matrix, [*B*] is the input matrix, [*C*] is the output matrix, {*X*} is the state vector and {*u*} is the input vector.

3. RESULT AND DISCUSSION

By considering the mathematical formulations discussed, a MATLAB code has been established for analysis of Timoshenko beam. The developed MATLAB code is validated and different results are presented. For the analysis a uniform cantilever beam of circular cross section is considered for validation of present code. The length and diameter of the beam are 1000mm and 20 mm. Material properties for the beam are considered as E = 206 GPA, ρ = 7800 kg/m^3 and μ = 0.3. The beam is subjected to 1 N at the end. The length of beam is divided into finite numbers of small elements. The fundamental frequencies are calculated by using the present code developed and compared with the exact solution obtained [23] in Table 1. From Table 1 it is observed that results obtained from the present code are in good agreement with the exact results [23].

Table 1. Comparison of natural frequencies of cantilever beam

Natural frequency(rad/sec)	Exact [23]	Present code
ω_1	87.19	88.80
ω_2	556.04	556.92
ω_3	1584.69	1560.46

The effect of taper on the natural frequencies is shown in table 2.The value of taper varies from 0.2 to 0.8.From the table it is observed that as the taper value increases the natural frequencies decreases.

Table 2. Effect of taper on natural frequencies

	Natural frequency(rad/sec)		
c	ω_1	ω_2	ω_3
0.2	89.03	557.80	1563.82
0.3	89.04	557.94	1564.56
0.4	88.94	557.40	1563.47
0.5	88.69	555.88	1559.71
0.6	88.20	552.93	1552.02
0.7	87.36	547.84	1538.49
0.8	86.00	539.51	1516.00

Table [3-6] represents the natural frequencies of a cracked Timoshenko beam at various crack positions (such as 0.2, 0.4, 0.6 and 0.8) and relative crack depths (such as 0.2, 0.3, 0.4, and 0.5). The taper value of the beam is taken as 0.5.

Table 3. Natural frequencies of Cracked Timoshenko beam, Xc/L=0.075.

ω (rad/sec)	Xc/L	α/D=0.2	α/D =0.3	α/D =0.4	α/D =0.5
ω_1	0.075	86.77	86.47	86.34	86.28
ω_2	0.075	544.45	542.55	541.70	541.29
ω_3	0.075	1530.05	1524.64	1522.22	1521.07

Table 4. Natural frequencies of Cracked Timoshenko beam, Xc/L=0.275.

ω (rad/sec)	Xc/L	α/D=0.2	α/D =0.3	α/D =0.4	α/D =0.5
ω_1	0.275	86.83	86.52	86.37	86.31
ω_2	0.275	551.71	547.07	544.98	543.97
ω_3	0.275	1552.34	1538.58	1532.34	1529.33

Table 5. Natural frequencies of Cracked Timoshenko beam, Xc/L=0.475.

ω (rad/sec)	Xc/L	α/D=0.2	α/D =0.3	α/D =0.4	α/D =0.5
ω_1	0.475	86.34	86.21	86.16	86.13
ω_2	0.475	554.31	548.85	546.31	545.08
ω_3	0.475	1555.35	1540.66	1533.90	1530.63

Table 6. Natural frequencies of Cracked Timoshenko beam, Xc/L=0.675.

ω (rad/sec)	Xc/L	α/D=0.2	α/D =0.3	α/D =0.4	α/D =0.5
ω_1	0.675	86.06	86.04	86.03	86.02
ω_2	0.675	545.13	543.15	542.20	541.72
ω_3	0.675	1556.73	1542.09	1535.13	1531.71

It is observed from the tables that natural frequencies of cracked beam decreases as crack depth increases at a particular position. This is due to fact that presence of crack reduces the stiffness matrix of the beam hence reduces the natural frequency. Figures 3-5 show the mode shapes for uncracked and cracked Timoshenko beam with relative crack position of 0.275 and relative crack depths of 0.2, 0.3, 0.4 and 0.5.The taper value of the beam is taken as 0.5. It is observed that due to presence of crack with various relative crack depths there are variations in mode shapes.

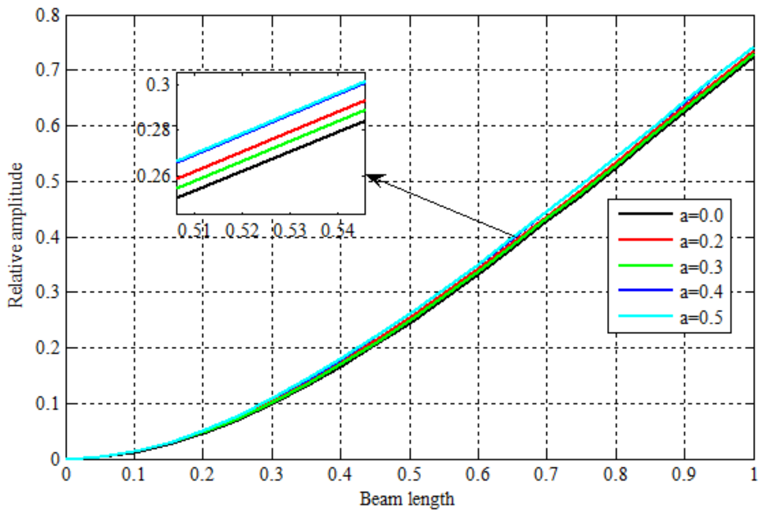

Figure 3. First mode shapes of uncracked (a=0.0) and cracked beam (a=0.2,0.3,0.4,0.5), X_C/L=0.275.

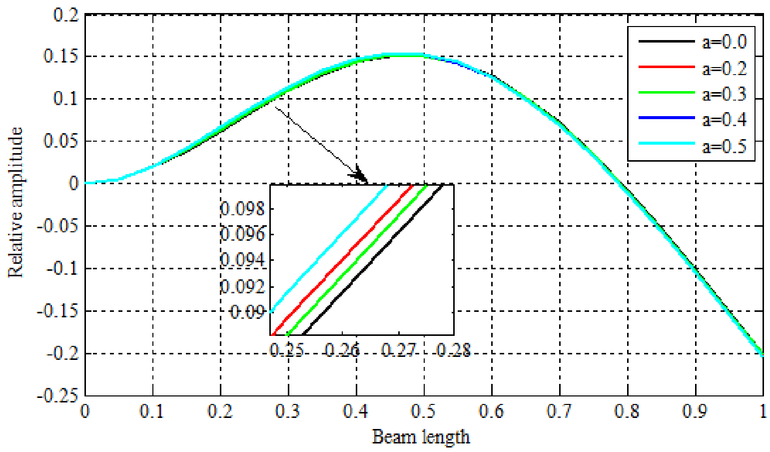

Figure 4. mode shapes of uncracked (a=0.0) and cracked beam (a=0.2,0.3,0.4,0.5), X_C/L=0.275.

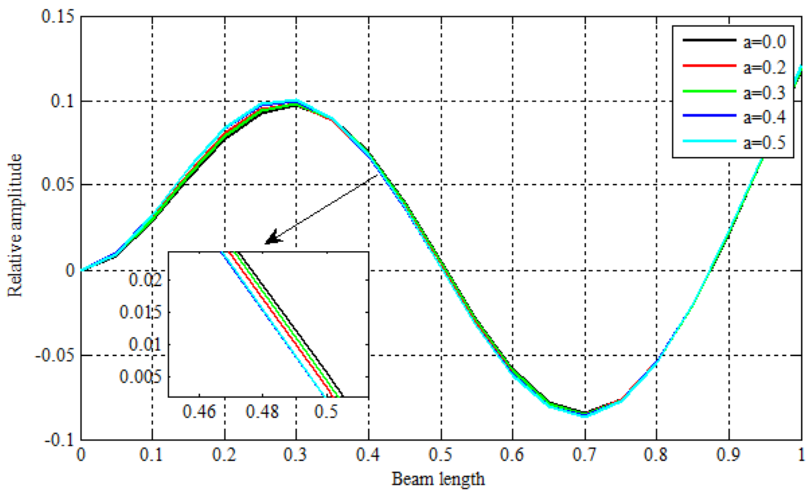

Figure 5. Third mode shapes of uncracked (a=0.0) and cracked beam (a=0.2,0.3,0.4,0.5), X_C/L=0.275.

By using the state space representation the dynamic analysis of the cracked beam has been conducted.The load of 1 N is applied at the end of cantilever beam.The frequency response of uncracked and cracked beam with relative crack depths of 0.2,0.3,0.4 and 0.5 has been shown in figure 6.From the figure it is observed that the amplitude of cracked beam at first natural frequency decreases for all cases as compared to uncracked beam.

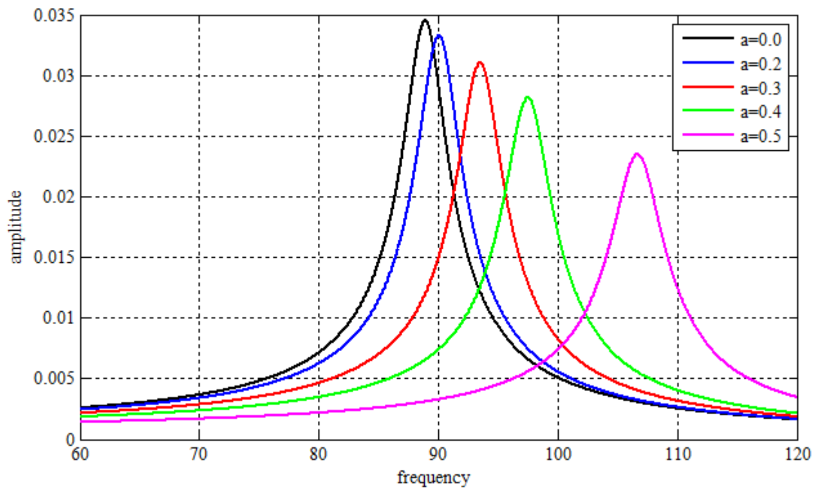

Figure 6. Frequency response of uncracked (a=0.0) and cracked beam (a=0.2,0.3,0.4,0.5), X_C/L=0.275.

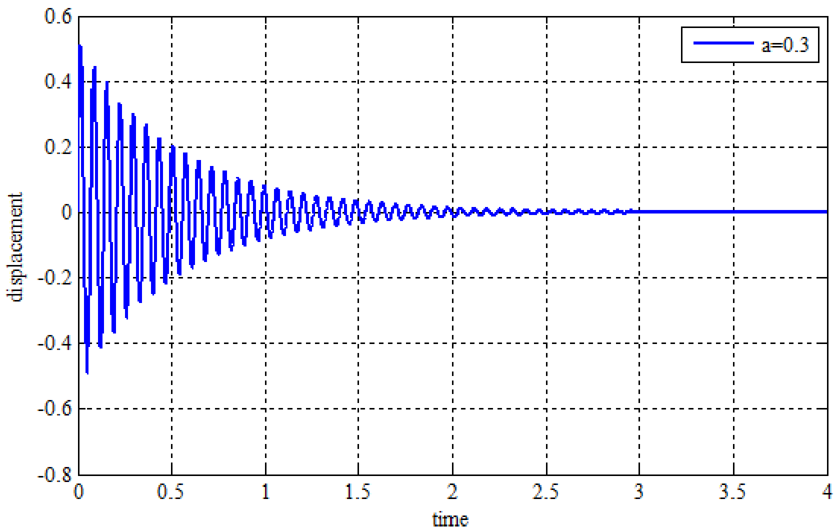

Figure 7. Frequency response of cracked beam (a=0.3), X_C/L=0.275.in time domain

The frequency response of cracked beam in time domain is shown in figure 7. From the figure it is concluded that the frequency response of cracked beam dies out after sometime. This is due to the presence of structural damping which is responsible for reduction in amplitude.

4. CONCLUSION

The present article focused on the vibration analysis of a cracked beam using finite element formulation. The beam is modeled using the Timoshenko beam theory by considering the rotary inertia and shear deformations. Two noded beam elements with two degrees of freedom at each node is considered in order to solve the governing equation. From the analysis it is observed that due to presence of crack in a beam the natural frequencies decreases as the relative crack depth increases. Again from dynamic analysis it is observed that the amplitude of vibration of cracked beam decreases by varying the relative crack depths as compared to uncracked beam.

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