Simultaneous Estimation of Quantiles of Two Normal Populations with a Common Mean

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Abstract. Let $\underline{X} = (X_1, X_2, \ldots, X_m)$ and $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples drawn from two normal populations with a common unknown mean μ and possibly unknown different variances σ_1^2 and σ_2^2 respectively. The problem of simultaneous estimation of the p^{th} $(0 < p < 1)$ quantiles, $\theta_i = \mu + \eta\sigma_i, i = 1, 2$ of the two normal populations is considered with respect to a sum of quadratic loss functions. Here $\eta = \Phi^{-1}(p)$ and $\Phi(.)$ the cumulative distribution function of a standard normal random variable. A general result has been proved for improving the basic estimator for the quantiles. Using this result improved estimators for quantiles have been constructed. A sufficient condition for improving estimators in certain classes of affine and location equivariant estimators are obtained, as a result two complete class theorems have been proved. A massive simulation study has been carried out to compare numerically various proposed efficient estimators for the quantiles. Some practical examples have been discussed to show the applicability of our model.

AMS Subject Classification 62C15, 62F10, 62C20

Keywords: Affine equivariant estimator; complete class; common mean; inadmissibility; location equivariant estimator; risk comparisons; simultaneous estimation of quantiles.

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Outline

1. Introduction
2. A General Result and Construction of Improved Estimators
3. Inadmissibility Conditions for Equivariant Estimators
4. Numerical Comparisons
5. Concluding Remarks and Illustrative Examples

Introduction I

- Let $\underline{X} = (X_1, X_2, \ldots, X_m)$ and $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples drawn from two normal populations $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ respectively.
- Here the common mean μ, and the variances σ_1^2, σ_2^2 are unknown. The p^{th} quantile of the first and second population are $\theta_1 = \mu + \eta\sigma_1$ and $\theta_2 = \mu + \eta\sigma_2$ respectively where $\eta = \Phi^{-1}(p)$; $0 < p < 1$. Here $\Phi(.)$ denotes the cumulative distribution function of a standard normal random variable.
- The problem is to estimate the quantile vector $\underline{\theta}$ with respect to the sum of the quadratic losses given by,

$$L(\underline{d}, \underline{\theta}) = \sum_{i=1}^{2} \left(\frac{d_i - \theta_i}{\sigma_i} \right)^2, \tag{1}$$

where $\underline{d} = (d_1, d_2)$ is an estimator for $\underline{\theta}$.

Introduction II

- The problem of estimation of quantiles has attracted several researchers in the recent past due to its real life applications. For example, quantiles of exponential populations are widely used in the study of reliability, life testing, survival analysis and related areas. Some applications of exponential quantiles has been discussed in [Saleh(1981)] and [Keating and Tripathi(1985)].
- The estimation of quantiles of exponential population has been considered by several authors in the recent past. For some decision theoretic results on estimation of quantiles of exponential population one may refer to the results of [Rukhin(1986)], [Kumar and Sharma(1996)] and the references there in.
- On contrary to the exponential case, a very less attention has been paid on parametric estimation of quantiles of normal population. Probably [Zidek(1969)] was the first to consider this problem with respect to a quadratic loss.

Introduction III

- [Zidek(1969), Zidek(1971)] obtained some inadmissibility results for estimating quantile $\theta = \mu + \eta\sigma$. He proved that the best affine equivariant estimator of θ is inadmissible if $|\eta|$ is chosen very large. [Rukhin(1983)] derived a class of minimax estimators for quantile θ, each of which improves upon the best equivariant estimator.
- Recently, the problem of estimation of quantile θ_1 of first population has been investigated by [Kumar and Tripathy(2011)] when samples from two normal populations are available with a common mean. Exploiting the information available for common mean, they could able to obtain improved estimators for quantiles θ_1. They also derived some inadmissibility conditions for estimators belonging to equivariant classes.

Introduction IV

- The problem of estimating common mean of normal populations is an age old problem in the history of literature. The problem has its origin in the study of recovery of inter-block information in balance incomplete block designs. In literature, this problem is also refer as Meta-Analysis, where samples (data) from multiple sources are combined with a common objective. One may refer to [Vazquez et al.(2007)] for application of Meta-Analysis in clinical trials.
- Probably, [Graybill and Deal(1959)] were the first to study this problem under the assumption of normality. They established that the individual sample mean(s) can be improved by an estimator which is the convex combination of sample means with weights as the functions of sample variances under certain conditions on the sample sizes.

Introduction V

- After then, a lot of attention has been paid by researchers and alternative estimators for common mean has been proposed. For a detailed review, one may refer to [Pal and Sinha(1996)], [Chang and Pal(2008)], [Lin and Lee(2005)] and the references there in.
- In this study, we consider the simultaneous estimation of quantiles that is the vector $\underline{\theta}$. For some decision theoretic results on simultaneous estimation of parameters of a distribution function, one may refer to [Berger(1980)], [Shinozaki(1984)] and [Gupta(1986)].

A General Result and Construction of Improved Estimators I

- Suppose $\underline{X} = (X_1, X_2, \ldots, X_m)$ and $\underline{Y} = (Y_1, Y_2, \ldots, Y_n)$ are independent random samples taken from two normal populations $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ respectively. Here the parameters μ, σ_1^2 and σ_2^2 are unknown.
- Our aim is to estimate the vector $\underline{\theta} = (\theta_1, \theta_2)$, where $\theta_i = \mu + \eta\sigma_i$, $(\eta \neq 0$ and $i = 1, 2)$ with respect to the loss function (1).
- A minimal sufficient statistic for this problem is $(\bar{X}, \bar{Y}, S_1^2, S_2^2)$ where

$$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i,\ \bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j,\ S_1^2 = \sum_{i=1}^{m}(X_i - \bar{X})^2 \text{ and } S_2^2 = \sum_{j=1}^{n}(Y_j - \bar{Y})^2$$

A General Result and Construction of Improved Estimators II

- It is well known that the maximum likelihood estimator (MLE) for the common mean μ, is not obtainable in a closed form (see [Pal et al.(2007)]). Also the minimal sufficient statistics for this problem are not complete, hence the question of finding uniformly minimum variance unbiased estimator (UMVUE) using the standard method is difficult. Consequently it is hard to derive any classical estimators for the quantile, $\theta_1 = \mu + \eta\sigma_1$ and hence for $\underline{\theta}$.
- When we have both the populations $\underline{X}$ and $\underline{Y}$ the problem of estimating the first component θ_1 has been considered by [Kumar and Tripathy(2011)].

A General Result and Construction of Improved Estimators III

- A natural way to construct improved estimators for $\underline{\theta}$ is to combine the improved estimators for the common mean and the improved estimators for the respective standard deviations. Hence we first propose a basic estimator for $\underline{\theta}$ as,

$$\underline{d} = (d_1, d_2), \quad \text{where } d_i = \bar{X} + cS_i, \;\; i = 1, 2.$$

- Let us define

$$c_{m+n} = \frac{\eta\sqrt{2}}{m+n-2}\left[\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m-1}{2})} + \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right]. \qquad (2)$$

A General Result and Construction of Improved Estimators IV

Theorem 1.

If we estimate the quantiles, $\underline{\theta}$ by $\underline{d} = (\bar{X} + cS_1, \bar{X} + cS_2)$ with respect to the loss function (1), then the value of c for which the risk value is minimum is found to be c_{m+n}.

- Let us denote $\underline{d}_{MX} = (\bar{X} + c_{m+n}S_1, \bar{X} + c_{m+n}S_2)$. Next, we give a general result which in parallel to Theorem 2.1 of [Kumar and Tripathy(2011)] that valid for estimating only θ_1.

A General Result and Construction of Improved Estimators V

Theorem 2.

Suppose $\underline{d}_M = (d_M, d_M)$ *be an estimator for* $\underline{\mu} = (\mu, \mu)$, *and* $\underline{d}_S = (d_{S_1}, d_{S_2})$ *be an estimator for* $\underline{\sigma} = (\sigma_1, \sigma_2)$. *Consider* $\underline{d}_Q = (d_{Q1}, d_{Q2}) = \underline{d}_M + \eta \underline{d}_S$ *as an estimator for* $\underline{\theta}$. *Further, assume that given* d_{S_1}, *and* d_{S_2}, d_M *is conditionally unbiased for* μ, *that is*

$$E(d_M | d_{S_1}) = E(d_M | d_{S_2}) = \mu, \tag{3}$$

then,

$$\begin{aligned} E(d_{Q1} - \theta_1)^2 + E(d_{Q2} - \theta_2)^2 &= 2E(d_M - \mu)^2 + \eta^2 \{ E(d_{S_1} - \sigma_1)^2 \\ &\quad + E(d_{S_2} - \sigma_2)^2 \}. \end{aligned} \tag{4}$$

A General Result and Construction of Improved Estimators VI

Theorem 3.

Let $d_\phi = \phi(S_1, S_2)\bar{X} + (1 - \phi(S_1, S_2))\bar{Y}$ be an estimator for the common mean μ. Consider the estimator $\underline{d}_\phi(c) = (d_\phi + cS_1, d_\phi + cS_2)$ for estimating quantile vector $\underline{\theta}$. Then $\underline{d}_\phi(c)$ has smaller risk than $\underline{d}$ with respect to the sum of quadratic loss (1) if and only if d_ϕ has smaller risk than $\bar{X}$. Further, $\underline{d}_\phi(c)$ has minimum risk with respect to the loss (1) when $c = c_{m+n}$.

Remark.

Following Theorem 3, one can easily construct good estimators for $\underline{\theta}$ by replacing $\bar{X}$ in $\underline{d}_{MX}$ by any improved estimator of the form d_ϕ for common mean μ.

A General Result and Construction of Improved Estimators VII

- we propose some estimators for $\underline{\theta}$ which have smaller risk than $\underline{d}_{MX}$ under certain conditions on the sample sizes. We propose the following estimators for quantile vector $\underline{\theta} = (\theta_1, \theta_2)$.

$$
\begin{aligned}
\underline{d}_{GM} &= (\hat{\mu}_{GM} + c_{m+n}S_1, \hat{\mu}_{GM} + c_{m+n}S_2), \\
\underline{d}_{GD} &= (\hat{\mu}_{GD} + c_{m+n}S_1, \hat{\mu}_{GD} + c_{m+n}S_2), \\
\underline{d}_{KS} &= (\hat{\mu}_{KS} + c_{m+n}S_1, \hat{\mu}_{KS} + c_{m+n}S_2), \\
\underline{d}_{CS} &= (\hat{\mu}_{CS} + c_{m+n}S_1, \hat{\mu}_{CS} + c_{m+n}S_2), \\
\underline{d}_{MK} &= (\hat{\mu}_{MK} + c_{m+n}S_1, \hat{\mu}_{MK} + c_{m+n}S_2), \\
\underline{d}_{TK} &= (\hat{\mu}_{TK} + c_{m+n}S_1, \hat{\mu}_{TK} + c_{m+n}S_2), \\
\underline{d}_{BC1} &= (\hat{\mu}_{BC1} + c_{m+n}S_1, \hat{\mu}_{BC1} + c_{m+n}S_2), \\
\underline{d}_{BC2} &= (\hat{\mu}_{BC2} + c_{m+n}S_1, \hat{\mu}_{BC2} + c_{m+n}S_2).
\end{aligned}
$$

A General Result and Construction of Improved Estimators VIII

- Here we denote $\hat{\mu}_{GM} = \frac{m\bar{X}+n\bar{Y}}{m+n}$, $\hat{\mu}_{TK} = \frac{\sqrt{m}\ b_{n-1}S_2\bar{X}+\sqrt{n}\ b_{m-1}S_1\bar{Y}}{\sqrt{m}\ b_{n-1}S_2+\sqrt{n}\ b_{m-1}S_1}$, and $\hat{\mu}_{GD}$, $\hat{\mu}_{KS}$, $\hat{\mu}_{BC1}$, $\hat{\mu}_{BC2}$, $\hat{\mu}_{CS}$, $\hat{\mu}_{MK}$, are estimators for the common mean μ and $b_\nu = \frac{\sqrt{2}\Gamma(\nu/2)}{\Gamma(\frac{\nu+1}{2})}$.

A General Result and Construction of Improved Estimators IX

Theorem 4.

Let the estimators $\underline{d}_{MX}$, $\underline{d}_{GD}$, $\underline{d}_{KS}$, $\underline{d}_{BC1}$, $\underline{d}_{BC2}$, and $\underline{d}_{CS}$ as defined above for estimating $\underline{\theta}$. The loss function be the sum of the quadratic losses as in (1).

(i) *The estimator $\underline{d}_{GD}$ performs better than $\underline{d}_{MX}$ if and only if $m, n \geq 11$.*

(ii) *The estimator $\underline{d}_{KS}$ performs better than $\underline{d}_{MX}$ if and only if $(m-7)(n-7) \geq 16$.*

(iii) *The estimator $\underline{d}_{BC1}$ performs better than $\underline{d}_{MX}$ if and only if $m \geq 2$, $n \geq 3$ and for $0 < b_1 < b_{\max}(m, n)$.*

(iv) *The estimator $\underline{d}_{BC2}$ performs better than $\underline{d}_{MX}$ if and only if $m \geq 2$, $n \geq 6$ and for $0 < b_2 < b_{\max}(m, n-3)$.*

(v) *The estimator $\underline{d}_{CS}$ performs better than $\underline{d}_{MX}$ if $m = n \geq 7$.*

Here b_1, b_2 and $b_{\max}(m, n)$ are as defined in [Kumar and Tripathy(2011)].

Inadmissibility Conditions for Equivariant Estimators I

- Consider the group $G_A = \{g_{a,b} : g_{a,b}(x) = ax + b, \underline{a} > 0, \underline{b} \in R\}$ of affine transformations. Under the transformation, $\bar{X} \to a\bar{X} + b$, $\bar{Y} \to a\bar{Y} + b$, $S_i^2 \to a^2 S_i^2$, $\mu \to a\mu + b$, $\sigma_i^2 \to a^2\sigma_i^2$, and $\underline{\theta} \to a\underline{\theta} + b\underline{e}$, where $e = (1,1)$ and $\underline{\theta} = (\theta_1, \theta_2)$, $\theta_i = \mu + \eta\sigma_i, i = 1, 2$.
- The problem considered is invariant if we choose the loss function as the sum of affine invariant loss functions (1).
- The loss invariance condition is

$$\begin{aligned} L(\bar{g}_{a,b}(\underline{\theta}), \underline{\tilde{d}}) &= \left(\frac{a\theta_1 + b - d_1}{a\sigma_1}\right)^2 + \left(\frac{a\theta_2 + b - d_2}{a\sigma_2}\right)^2 \\ &= L(\underline{\theta}, \underline{d}), \end{aligned}$$

which is satisfied if $\underline{\tilde{d}} = \tilde{g}_{a,b}(\underline{d}) = a\underline{d} + b\underline{e}$.

Inadmissibility Conditions for Equivariant Estimators II

- Therefore the decision rule must satisfy,

$$
\begin{aligned}
&(d_1(a\bar{X}+b, a\bar{Y}+b, a^2S_1^2, a^2S_2^2), d_2(a\bar{X}+b, a\bar{Y}+b, a^2S_1^2, a^2S_2^2)) \\
&\qquad = (ad_1(\bar{X}, \bar{Y}, S_1^2, S_2^2)+b, ad_2(\bar{X}, \bar{Y}, S_1^2, S_2^2)+b).
\end{aligned}
$$

- Taking $b=-a\bar{X}$ where $a=\frac{1}{S_1}$, we obtain the form of an affine equivariant estimator as,

$$
\begin{aligned}
(d_1(\bar{X}, \bar{Y}, S_1^2, S_2^2), d_2(\bar{X}, \bar{Y}, S_1^2, S_2^2)) &= (\bar{X}+S_1\Psi_1(T_1, T_2), \\
&\quad \bar{X}+S_1\Psi_2(T_1, T_2)) \\
&= (d_{\Psi_1}, d_{\Psi_2}) \\
&= \underline{d}_{\underline{\Psi}} \text{ say,} \qquad (5)
\end{aligned}
$$

- where we denote $T_1=\frac{\bar{Y}-\bar{X}}{S_1}$ and $T_2=\frac{S_2}{S_1}$.
- Let us denote $M_1=\min(t_1, 0)$ and $M_2=\max(t_1, 0)$.

Inadmissibility Conditions for Equivariant Estimators III

- Let us define the following functions for any affine equivariant estimator $\underline{d}_{\underline{\Psi}}$.

$$\underline{\Psi}^0 = (\min(\max(\Psi_1, M_1), M_2), \min(\max(\Psi_2, M_1), M_2)), \tag{6}$$

$$\underline{\Psi}^1 = (\max\{M_1 + c_{m+n}, \Psi_1\}, \max\{M_1 + c_{m+n}\sqrt{t_2}, \Psi_2\}), \tag{7}$$

$$\underline{\Psi}^2 = (\min\{M_2 + c_{m+n}, \Psi_1\}, \min\{M_2 + c_{m+n}\sqrt{t_2}, \Psi_2\}). \tag{8}$$

Inadmissibility Conditions for Equivariant Estimators IV

- Next we prove the following result for affine equivariant estimators.

Theorem 5.

Let $\underline{d}_{\underline{\Psi}}$ be an affine equivariant estimator of the form (5) of a quantile vector $\underline{\theta}$, and the loss function be the sum of quadratic loss (1) or the sum of squared errors. Let the functions $\underline{\Psi}^0$, $\underline{\Psi}^1$ and $\underline{\Psi}^2$ be defined as in (6), (7) and (8) respectively. Let $\underline{\alpha} = (\mu, \sigma_1^2, \sigma_2^2)$.

(i) *When $\eta = 0$, the estimator $\underline{d}_{\underline{\Psi}}$ is improved by $\underline{d}_{\underline{\Psi}^0}$ if $P_{\underline{\alpha}}(\underline{\Psi}^0 \neq \underline{\Psi}) > 0$ for some choices of $\underline{\alpha}$.*

(ii) *When $\eta > 0$, the estimator $\underline{d}_{\underline{\Psi}}$ is improved by $\underline{d}_{\underline{\Psi}^1}$ if $P_{\underline{\alpha}}(\underline{\Psi}^1 \neq \underline{\Psi}) > 0$ for some choices of $\underline{\alpha}$.*

(iii) *When $\eta < 0$, the estimator $\underline{d}_{\underline{\Psi}}$ is improved by $\underline{d}_{\underline{\Psi}^2}$ if $(\underline{\Psi}^2 \neq \underline{\Psi}) > 0$ for some choices of $\underline{\alpha}$.*

Inadmissibility Conditions for Equivariant Estimators V

Remark.

The above theorem is basically a complete class result. It tells that for an equivariant estimator of the form (5),

(i) *if* $P_{\underline{\alpha}}(\{\Psi_1 \in [\min(T_1, 0), \max(T_1, 0)]^c\} \bigcup \{\Psi_2 \in [\min(T_1, 0), \max(T_1, 0)]^c\}) > 0$*, then the estimator* $\underline{d}_{\Psi}$ *is improved by* $\underline{d}_{\Psi^0}$*, when* $\eta = 0$.

(ii) *if* $P(\{\Psi_1 < \min(T_1, 0) + \eta b_{m+n}\} \bigcup \{\Psi_2 < \min(T_1, 0) + \eta b_{m+n}\sqrt{T_2}\}) > 0$*, then the estimator* $\underline{d}_{\Psi^1}$ *will improve upon* $\underline{d}_{\Psi}$*, when* $\eta > 0$*,*

(iii) *if* $P(\{\Psi_1 > \max(T_1, 0) + \eta b_{m+n}\} \bigcup \{\Psi_2 > \max(T_1, 0) + \eta b_{m+n}\sqrt{T_2}\}) > 0$*, then the estimator* $\underline{d}_{\Psi^2}$ *will improve upon* $\underline{d}_{\Psi}$ *when* $\eta < 0$.

Numerical Results I

- For simulation purpose, we have generated 20,000 random samples $\underline{X}$ of sizes m and 20,000 random samples $\underline{Y}$ of sizes n from normal populations with equal mean and different variances.
- As the sample sizes increases the risk values of all the estimators decrease for fixed η. Further, the risk values increase as η increases for fixed values of τ and sample sizes.
- If we choose the value of b_1 and b_2 near 0 the estimators $\underline{d}_{BC1}$ and $\underline{d}_{BC2}$ tends to $\underline{d}_{MX}$. If we choose the value of b_2 near 1 the estimator $\underline{d}_{BC2}$ tends to $\underline{d}_{GD}$. So for numerical comparison a convenient choice would be an intermediate value which we take as $\frac{1}{2}b_{max}$. The value of $b_{max}(m, n)$ have been taken from the Table given by Brown and Cohen (Brown and Cohen(1974)).

Numerical Results II

- **Case 1:m $=$ n**
  - (i) It has been observed that the risk values of the estimators $\underline{d}_{MX}$, $\underline{d}_{BC1}$, $\underline{d}_{BC2}$ and $\underline{d}_{CS}$ decreasing as τ increases from 0 to ∞. The estimator $\underline{d}_{GD}$ first increases and attains maximum value then decreases. The estimators $\underline{d}_{GM}$ and $\underline{d}_{MK}$ first decrease attains minimum (in the neighborhood of $\tau = 1$) then increases.
  - (ii) For choices of τ close to 0, the estimator $\underline{d}_{GD}$ is preferred. For $\tau = 1$, $\underline{d}_{GM}$ is strongly recommended. But when $\tau \neq 1$ and τ is near to 1 $\underline{d}_{MK}$ is a good competitor of $\underline{d}_{GM}$. When τ value is large the estimator $\underline{d}_{BC1}$ performs the best.
- **Case 2:(m $<$ n)**
  - (i) The risk values of the estimators $\underline{d}_{MX}$, is decreasing as τ increases. The risk values of $\underline{d}_{GD}$, $\underline{d}_{KS}$ increase and attains maximum then decrease as τ increases. The risk values of estimators $\underline{d}_{GM}$, $\underline{d}_{BC1}$, $\underline{d}_{BC2}$, $\underline{d}_{MK}$ and $\underline{d}_{TK}$ first decrease attains minimum value then increase.

Numerical Results III

(ii) When the sample sizes are small and τ is close to 0 the estimator $\underline{d}_{KS}$ is recommended. For $\tau = 1.0$ the $\underline{d}_{GM}$ performs the best. When the value of τ is close to 1 but $\neq 1$ the estimators $\underline{d}_{TK}$ and $\underline{d}_{GM}$ are good competitor of each other. When the value of τ is large $\underline{d}_{BC1}$ is recommended.

(iii) When the sample sizes are are moderate to large and τ value is close to 0, the estimator $\underline{d}_{KS}$ is recommended. For $\tau = 1.0$ the estimator $\underline{d}_{GM}$ is recommended. When $\tau > 1$ and is close to 1 the estimators $\underline{d}_{KS}$ and $\underline{d}_{MK}$ are good competitor of each other and each one dominated by $\underline{d}_{GD}$. When values of τ are large the estimator $\underline{d}_{BC1}$ is strongly recommended.

- **Case-3:m $>$ n**

  (i) The risk value of$\underline{d}_{MX}$ is decreasing as τ increases. The risk values of $\underline{d}_{GD}$, $\underline{d}_{KS}$, $\underline{d}_{BC1}$ and $\underline{BC2}$ decrease as τ increases. The risk values of estimators $\underline{d}_{GM}$, $\underline{d}_{MK}$, and $\underline{d}_{TK}$ first decrease attains minimum then increase with respect to τ.

Numerical Results IV

(ii) When the sample sizes are small and τ is near to 0 the estimator $\underline{d}_{GD}$ is recommended. When $\tau = 1.0$ the estimator $\underline{d}_{GM}$ is good. When the value of τ is close to 1 but $\neq 1$ the estimators $\underline{d}_{BC1}$ and $\underline{d}_{KS}$ are good competitor of each other. When the value of τ is very large the estimator $\underline{d}_{BC1}$ is strongly recommended.

(iii) When the sample sizes are moderate to large, and τ is close to 0 the estimator $\underline{d}_{GD}$ is recommended. For values of $\tau = 1.0$, the estimator $\underline{d}_{GM}$ is strongly recommended. When τ is close to 1 and $\neq 1$ the estimators $\underline{d}_{KS}$ and $\underline{d}_{MK}$ are good competitor of each other. When τ is very large the estimator $\underline{d}_{BC1}$ is strongly recommended.

Conclusion I

- The problem of simultaneous estimation of quantiles have been considered, for the case of two normal populations, with respect to the sum of the quadratic loss functions.
- In this study, we have considered the estimation of quantile vector (θ_1, θ_2), with respect to the sum of the quadratic loss functions given in (1).The results obtained in this paper, are quite similar to the results obtained by [Kumar and Tripathy(2011)].
- We have implemented the Brewster-Zidek technique ([Brewster and Zidek(1974)]) to the case of estimation of a vector parameter having two components.
- First, we derived sufficient conditions for improving equivariant estimators and in the process some complete class results obtained. We have constructed some improved estimators using one of our result obtained in Section 2.

Conclusion II

- However, the analytical comparison of these estimators is not possible. We have conducted a detailed simulation study to numerically compare these estimators which can be used in practice.

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Thank You