A CATEGORICAL CONSTRUCTION OF MINIMAL MODEL

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Abstract - Under a reasonable assumption, the minimal model of a 1-connected d.g.a. can be expressed as the Adams cocompletion of the d.g.a. with respect to a chosen set of d.g.a.-maps.

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I. INTRODUCTION

It is to be emphasized that many algebraic and geometrical constructions in Algebraic Topology, Differential Topology, Differentiable Manifolds, Algebra, Analysis, Topology, etc., can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms.

The notion of generalized completion (Adams completion) arose from a categorical completion process suggested by Adams [1, 2]. Originally this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [4], where an arbitrary category and an arbitrary set of morphisms of the category are considered; moreover they have also suggested the dual notion, namely the cocompletion (Adams cocompletion) of an object in a category.

The central idea of this note is to investigate a case showing how such an algebraic geometrical construction is characterized in terms of Adams cocompletion.

II. ADAMS COCOMPLETION

We recall the definitions of Grothendieck universe, category of fractions, calculus of right fractions, Adams cocompletion and some characterizations of Adams cocompletion.

2.1 Definition. ([11], p. 266) A *Grothendeick universe* (or simply *universe*) is a collection $\mathcal{U}$ of sets such that the following axioms are satisfied:

U(1) : If $\{X_i : i \in I\}$ is a family of sets belonging to $\mathcal{U}$, then $\bigcup_{i\in I} X_i$ is an element of $\mathcal{U}$.
U(2) : If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$.
U(3) : If $x \in X$ and $X \in \mathcal{U}$ then $x \in \mathcal{U}$.
U(4) : If X is a set belonging to $\mathcal{U}$, then $P(X)$, the power set of X, is an element of $\mathcal{U}$.
U(5) : If X and Y are elements of $\mathcal{U}$, then $\{X, Y\}$, the ordered pair (X, Y) and $X \times Y$ are elements of $\mathcal{U}$.

We fix a universe $\mathcal{U}$ that contains $\mathbb{N}$ the set of natural numbers (and so $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$).

2.2 Definition. ([11], p. 267) A category $\mathcal{C}$ is said to be a *small $\mathcal{U}$-category*, $\mathcal{U}$ being a fixed Grothendeick universe, if the following conditions hold:

S(1) : The objects of $\mathcal{C}$ form a set which is an element of $\mathcal{U}$.
S(2) : For each pair (X, Y) of objects of $\mathcal{C}$, the set $\mathrm{Hom}(X, Y)$ is an element of $\mathcal{U}$.

2.3 Definition. ([11], p. 269) Let $\mathcal{C}$ be any arbitrary category and S a set of morphisms of $\mathcal{C}$. A *category of fractions* of $\mathcal{C}$ with respect to S is a category denoted by $\mathcal{C}[S^{-1}]$ together with a functor

$$F_S : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$$

having the following properties:

CF(1) : For each $s \in S$, $F_S(s)$ is an isomorphism in $\mathcal{C}[S^{-1}]$.
CF(2) : F_S is universal with respect to this property: If $G : \mathcal{C} \rightarrow \mathcal{D}$ is a functor such that $G(s)$ is an isomorphism in $\mathcal{D}$, for each $s \in S$, then there exists a unique functor

$$H : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$$

such that $G = HF_S$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F_S} & \mathcal{C}[S^{-1}] \\ G \downarrow & \swarrow H & \\ \mathcal{D} & & \end{array}$$

2.4 Note. For the explicit construction of the category $\mathcal{C}[S^{-1}]$, we refer to [11]. We content ourselves merely with the observation that the objects of $\mathcal{C}[S^{-1}]$ are same as those of $\mathcal{C}$ and in the case when S admits a calculus of left (right) fractions, the category $\mathcal{C}[S^{-1}]$ can be described very nicely [7, 11].

2.5 Definition. ([11], p. 267) A family S of morphisms in a category $\mathcal{C}$ is said to admit a *calculus of right fractions* if

(a) any diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow f \\ Z & \xrightarrow[s]{} & Y \end{array}$$

in $\mathcal{C}$ with $s \in S$ can be completed to a diagram

$$\begin{array}{ccc} W & \xrightarrow{t} & X \\ g \downarrow & & \downarrow f \\ Z & \xrightarrow[s]{} & Y \end{array}$$

with $t \in S$ and $ft = sg$,

(b) given

$$W \xrightarrow{t} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Z$$

with $s \in S$ and $sf = sg$, there is a morphism $t : W \rightarrow X$ in S such that $ft = gt$.

A simple characterization for a family S to admit a calculus of right fractions is the following.

2.6 Theorem. ([4], Theorem1.3*, p. 70) *Let S be a closed family of morphisms of $\mathcal{C}$ satisfying*

(a) *if $vu \in S$ and $v \in S$, then $u \in S$,*

(b) *any diagram*

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow f \\ \bullet & \underset{s}{\rightarrow} & \bullet \end{array}$$

in $\mathcal{C}$ with $s \in S$, can be embedded in a weak pull-back diagram

$$\begin{array}{ccc} \bullet & \overset{t}{\rightarrow} & \bullet \\ g \downarrow & & \downarrow f \\ \bullet & \underset{s}{\rightarrow} & \bullet \end{array}$$

with $t \in S$.

Then S admits a calculus of right fractions.

2.7 Remark. There are some set-theoretic difficulties in constructing the category $\mathcal{C}[S^{-1}]$; these difficulties may be overcome by making some mild hypotheses and using Grothendeick universes. Precisely speaking, the main logical difficulty involved in the construction of a category of fractions and its use, arises from the fact that if the category $\mathcal{C}$ belongs to a particular universe, the category $\mathcal{C}[S^{-1}]$ would, in general belongs to a higher universe ([11], Proposition 19.1.2). In most applications, however, it is necessary that we remain within the given initial universe. This logical difficulty can be overcome by making some kind of assumptions which would ensure that the category of fractions remains within the same universe [5]. Also the following theorem shows that if S admits a calculus of left (right) fractions, then the category of fractions $\mathcal{C}[S^{-1}]$ remains within the same universe as to the universe to which the category $\mathcal{C}$ belongs.

2.8 Theorem. ([10], Proposition, p. 425) *Let $\mathcal{C}$ be a small $\mathcal{U}$-category and S a set of morphisms of $\mathcal{C}$ that admits a calculus of left (right) fractions. Then $\mathcal{C}[S^{-1}]$ is a small $\mathcal{U}$-category.*

2.9 Definition. [4] Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect S and

$$F: \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$$

be the canonical functor. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$,

$$\mathcal{C}[S^{-1}](Y, -) : \mathcal{C} \rightarrow \mathcal{S}$$

defines a covariant functor. If this functor is representable by an object Y_S of $\mathcal{C}$, i.e.,

$$\mathcal{C}[S^{-1}](Y, -) \cong \mathcal{C}(Y_S, -)$$

Then Y_S is called the (*generalized*) *Adams cocompletion* of Y with respect to the set of morphisms S or simply the *S-cocompletion of Y*. We shall often refer to Y_S as the cocompletion of Y [4].

We recall some results on the existence of Adams cocompletion. We state Deleanu's theorem [5] that under certain conditions, global Adams cocompletion of an object always exists.

2.10 Theorem. *Let $\mathcal{C}$ be a complete small $\mathcal{U}$-category ($\mathcal{U}$ is a fixed Grothendeick universe) and S a set of morphisms of $\mathcal{C}$ that admits a calculus of right fractions. Suppose that the following compatibility condition with product is satisfied.*

(C) *If each $s_i : X_i \rightarrow Y_i$, $i \in I$, is an element is an element of $\mathcal{U}$, then*

$$\prod_{i \in I} s_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$$

is an element of S.

Then every object X *of* $\mathcal{C}$ *has an Adams cocompletion* X_S *with respect to the set of morphisms* S.

The concept of Adams cocompletion can be characterized in terms of a couniversal property.

2.11 Definition. [4] Given a set S of morphisms of $\mathcal{C}$, we define $\bar{S}$, the *saturation* of S as the set of all morphisms u in $\mathcal{C}$ such that $F(u)$ is an isomorphism in $\mathcal{C}[S^{-1}]$. S is said to be *saturated* if $S = \bar{S}$.

2.12 Proposition. ([4], Proposition 1.1, p. 63) *A family* S *of morphisms of* $\mathcal{C}$ *is saturated if and only if there exists a factor* $F : \mathcal{C} \to \mathcal{D}$ *such that* S *is the collection of morphisms* f *such that* Ff *is invertible.*

Deleanu, Frei and Hilton have shown that if the set of morphisms S is saturated then the Adams cocompletion of a space is characterized by a certain couniversal property.

2.13 Theorem. ([4], Theorem 1.4, p. 68) *Let* S *be a saturated family of morphisms of* $\mathcal{C}$ *admitting a calculus of right fractions. Then an object* Y_S *of* $\mathcal{C}$ *is the* S*-cocompletion of the object* Y *with respect to* S *if and only if there exists a morphism*

$$e : Y_S \to Y$$

in S *which is couniversal with respect to morphisms of* S : *given a morphism*

$$s : Z \to Y$$

in S *there exists a unique morphism*

$$t : Y_S \to Z$$

in S *such that* $st = e$. *In other words, the following diagram is commutative:*

$$\begin{array}{ccc} Y_S & \xrightarrow{e} & Y \\ t \downarrow & \nearrow s & \\ Z & & \end{array}$$

For most of the application it is essential that the morphism $e : Y_S \to Y$ has to be in S; this is the case when S is saturated and the result is as follows:

2.14 Theorem. ([4], dual of Theorem 2.9, p. 76) *Let* S *be a saturated family of morphisms of* $\mathcal{C}$ *and let every object of* $\mathcal{C}$ *admit an* S*-cocompletion. Then the morphism* $e : Y_S \to Y$ *belongs to* S *and is universal for morphisms to* S*-cocomplete objects and couniversal for morphisms in* S.

III. THE CATEGORY $\mathcal{DGA}$

Let $\mathcal{DGA}$ be the category of 1-connected differential graded algebras over $\mathbb{Q}$ (in short d.g.a.'s) and d.g.a.-homomorphisms. Let S denote the set of all d.g.a.-epimorphisms in $\mathcal{DGA}$ which induce homology isomorphisms in all dimensions. The following results will be required in the sequel.

3.1 Proposition. S *is saturated.*

Proof. The proof is evident from Proposition 2.12.

■

3.2 Proposition. S *admits a calculus of right fractions*.

Proof. Clearly, S is a closed family of morphisms of the category $\mathcal{DGA}$. We shall verify conditions (a) and (b) of Theorem 2.6. Let $u, v \in S$. We show that if $vu \in S$ and $v \in S$, then $u \in S$. Clearly u is an epimorphism. We have $(vu)_* = v_* u_*$ and v_* are both homology isomorphisms implying u_* is a homology isomorphism. Thus $u \in S$. Hence condition (a) of Theorem 2.6 holds.

To prove condition (b) of Theorem 2.6 consider the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \underset{s}{\rightarrow} & B \end{array}$$

in $\mathcal{DGA}$ with $s \in S$. We assert that the above diagram can be completed to a weak pull-back diagram

$$\begin{array}{ccc} D & \overset{t}{\rightarrow} & A \\ g\downarrow & & \downarrow f \\ C & \underset{s}{\rightarrow} & B \end{array}$$

in $\mathcal{DGA}$ with $s \in S$. Since A, B and C are in $\mathcal{DGA}$ we write

$$A = \sum_{n\geq 0} A_n, \quad B = \sum_{n\geq 0} B_n, \quad C = \sum_{n\geq 0} C_n,$$

$$f = \sum_{n\geq 0} f_n, s = \sum_{n\geq 0} s_n$$

and

$$f_n : A_n \rightarrow B_n, \qquad s_n : C_n \rightarrow B_n,$$

are d.g.a.-homomorphisms. Let

$$D_n = \{(a,c) \in A_n \times C_n : f_n(a) = s_n(c)\}$$

$$\subset A_n \times C_n.$$

We have to show that

$$D = \sum_{n\geq 0} D_n$$

is a differential graded algebra. Let

$$t_n : D_n \rightarrow A_n$$

and

$$g_n : D_n \rightarrow C_n$$

be the usual projections. Let

$$t = \sum_{n\geq 0} t_n$$

and

$$g = \sum_{n\geq 0} g_n.$$

Clearly the above diagram is commutative. It is required to show that D is a d.g.a.. Define a multiplication in D in following way:

$$(a,c)\cdot(a',c') = (aa',cc'),$$

where $(a,c) \in D_n$, $(a',c') \in D_m$. Let

$$d^A = \sum_{n\geq 0} d_n^A, \qquad d_n^A : A_n \rightarrow A_{n+1}$$

and

$$d^C = \sum_{n\geq 0} d_n^C, \qquad d_n^C : C_n \rightarrow C_{n+1}.$$

Define

$$d_n^D : D_n \rightarrow D_{n+1}$$

by the rule

$$d_n^D(a,c) = (d_n^A(a), d_n^C(c)), \ (a,c) \in D_n.$$

Let

$$d^D = \sum_{n\geq 0} d_n^D.$$

Since

$$d^D d^D(a,c) = \left(d^A d^A(a), d^C d^C(c)\right) = (0,0)$$

for all $(a,c) \in D$ we have that d^D is a differential. Next we show that d^D is a derivation: For $(a_1,c_1) \in D_n$ and $(a_2,c_2) \in D_m$,

$$d^D((a_1,c_1)\cdot(a_2,c_2))$$

$= \; d^D(a_1a_2, c_1c_2)$

$= \; (d^A(a_1a_2), d^C(c_1c_2))$

$= \; (d^A(a_1)\cdot(a_2) + (-1)^n(a_1)\cdot d^A(a_2),$
$\quad d^C(c_1)\cdot(c_2) + (-1)^n(c_1)\cdot d^C(c_2))$

$= \; (d^A(a_1)\cdot a_2\,, d^C(c_1)\cdot c_2)$
$\quad +((-1)^n a_1\cdot d^A(a_2), (-1)^n c_1\cdot d^C(c_2))$

$= \; (d^A(a_1), d^C(c_1))\cdot(a_2,c_2) +$
$\quad ((-1)^n a_1, (-1)^n c_1)\cdot(d^A(a_2), d^C(c_2))$

$= \; d^D(a_1,c_1)\cdot(a_2,c_2) +$
$\quad (-1)^n(a_1,c_1)\cdot d^D(a_2,c_2).$

Thus D becomes a d.g.a..

We show that D is 1-connected, i.e., $H_0(D) \cong \mathbb{Q}$ and $H_1(D) \cong 0$. We have

$H_0(D)$

$= \; Z_0(D)/B_0(D)$

$= \; Z_0(D)$

$= \; \{(a,c) \in Z_0(A)\times Z_0(C) : f_0(a) = s_0(c)\}.$

Let $1_A \in A$ and $1_C \in C$. Then

$$d^D(1_A,1_C) = (d^A(1_A), d^C(1_C)) = 0$$

implies that $(1_A,1_C) \in Z_0(D)$.

$$H_0(A) = Z_0(A) \cong \mathbb{Q}$$

implies that

$$Z_0(A) = \mathbb{Q}1_A.$$

Similarly,

$$H_0(C) = Z_0(C) \cong \mathbb{Q}$$

implies that

$$Z_0(C) = \mathbb{Q}1_C.$$

Thus

$$(a,c) \in H_0(D) = Z_0(D) \subset Z_0(A)\times Z_0(C)$$

if and only if $a = r1_A$ and $c = r1_C$ for some $r \in \mathbb{Q}$. Thus $H_0(D) \cong \mathbb{Q}$.

In order to show $H_1(D) \cong 0$, let $(a,c) \in Z_1(D)$. This implies that $a \in Z_1(A)$, $c \in Z_1(C)$ and $f_1(a) = s_1(c)$. Since A is 1-connected we have $H_1(A) \cong 0$, i.e., $Z_1(A)/B_1(A) = B_1(A)$; hence $a = d_0^A(a')$, $a' \in A_0$. Similarly since C is 1-connected we have $H_1(C) \cong 0$, i.e., $Z_1(C)/B_1(C) = B_1(C)$; hence $c = d_0^C(c')$, $c' \in C_0$. Now

$$f_1(a) = s_1(c),$$

i.e.,

$$f_1(d_0^A(a')) = s_1(d_0^C(c')).$$

This gives

$$d_0^B f_0(a') = d_0^B s_0(c'),$$

i.e.,

$$d_0^B(f_0(a') - s_0(c')) = 0.$$

Thus

$$f_0(a') - s_0(c') \in Z_0(B).$$

But $s \in S$. Hence $s_* : H_0(C) \to H_0(B)$ is an isomorphism, i.e., $s_0 : Z_0(C) \to Z_0(B)$ is an isomorphism. Hence there exists an element $\tilde{c} \in Z_0(C)$ such that

$$s_0(\tilde{c}) = f_0(a') - s_0(c').$$

Moreover

$$d_0^D(a', \tilde{c} + c') = (d_0^A(a'),\ d_0^C(\tilde{c}) + d_0^C(c'))$$

$$= (d_0^A(a'),\ 0 + d_0^C(c'))$$

$$= (d_0^A(a'),\ d_0^C(c'))$$

$$= (a, c)$$

showing that $(a, c) \in B_1(D)$. Thus $H_1(D) \cong 0$.

Clearly t is a d.g.a.-epimorphism. We show that $t_* : H_*(D) \to H_*(A)$ is an isomorphism. First we show that $t_* : H_*(D) \to H_*(A)$ is a monomorphism. The hollowing commutative diagram will be used in the sequel.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xleftarrow{s_{n-2}} & C_{n-2} \\
d_{n-2}^A \downarrow & & d_{n-2}^B \downarrow & & \downarrow d_{n-2}^C \\
A_{n-1} & \underset{f_{n-1}}{\longrightarrow} & B_{n-1} & \underset{s_{n-1}}{\longleftarrow} & C_{n-1} \\
d_{n-1}^A \downarrow & & d_{n-1}^B \downarrow & & \downarrow d_{n-1}^C \\
A_n & \underset{f_n}{\rightarrow} & B_n & \underset{s_n}{\leftarrow} & C_n \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

Since $t_n : D_n \to A_n$ is the usual projection we have $t_n(a, c) = a$ for every $(a, c) \in D_n$. Hence the algebra homomorphism

$$t_* : H_n(D) \to H_n(A)$$

is given by

$$t_*[(a, c)] = [t_n(a, c)] = [a]$$

for $[(a, c)] \in H_n(D)$. We note that

$$H_n(D)$$

$$= Z_n(D) / B_n(D)$$

$$\subset (Z_n(A) \times Z_n(C))/(B_n(A) \times B_n(C)).$$

Hence

$$H_n(D)$$

$$= (Z_n(\bar{A}) \times Z_n(\bar{C}))/(B_n(\bar{A}) \times B_n(\bar{C}))$$

for some $\bar{A}_n \subset A_n$ and $\bar{C}_n \subset C_n$. For any $[(a, c)] \in H_n(D)$ we have

$$[(a, c)] = (a, c) + B_n(D)$$

$$= (a, c) + (B_n(\bar{A}) \times B_n(\bar{C})),$$

$(a, c) \in Z_n(D) \subset D_n$. Then

$$(a, c) + d_{n-1}^D(a', c') \in (a, c) + B_n(D),$$

for every $d_{n-1}^D(a', c') \in B_n(D)$ where $(a', c') \in D_{n-1} \subset D_n$, i.e.,

$$(a, c) + d_{n-1}^D(a', c')$$

$$= (a, c) + (d_{n-1}^A(a'),\ d_{n-1}^C(c'))$$

$$\in (a, c) + (B_n(\bar{A}) \times B_n(\bar{C})),$$

for every

$$d_{n-1}^{D}(a',c')$$
$$= (d_{n-1}^{A}(a'), d_{n-1}^{A}(c'))$$
$$\in B_n(\overline{A}) \times B_n(\overline{C}).$$

Thus

$$(a + d_{n-1}^{A}(a'), c + d_{n-1}^{C}(c'))$$
$$\in (a,c) + (B_n(\bar{A}) \times B_n(\bar{C})),$$

i.e.,

$$[(a + d_{n-1}^{A}(a'), c + d_{n-1}^{C}(c')]$$
$$= [(a,c)] \in H_n(D).$$

We note that

$$[a] = [a + d_{n-1}^{A}(a')]$$

and

$$[c] = [c + d_{n-1}^{C}(c')].$$

Now let

$$[(a_1,c_1)], [(a_2,c_2)] \in H_n(D)$$

and assume that

$$t_*[(a_1,c_1)] = t_*[(a_2,c_2)];$$

this gives

$$[a_1] = [a_2],$$

i.e.,

$$[a_1 + d_{n-1}^{A}(a')] = [a_2 + d_{n-1}^{A}(a')].$$

Since

$$(a_1,c_1), (a_2,c_2), (d_{n-1}^{A}(a'), d_{n-1}^{C}(c')) \in D_n$$

We have

$$f_n(a_1) = s_n(c_1),$$
$$f_n(a_2) = s_n(c_2)$$

and

$$f_n d_{n-1}^{A}(a') = s_n\, d_{n-1}^{C}(c').$$

So

$$f_n(a_1 + d_{n-1}^{A}(a')) = s_n(c_1 + d_{n-1}^{C}(c'))$$

and

$$f_n(a_2 + d_{n-1}^{A}(a')) = s_n(c_2 + d_{n-1}^{C}(c')).$$

Therefore, from the above

$$t_*[(a_1,c_1)] = t_*[(a_2,c_2)]$$

gives

$$f_*[a_1 + d_{n-1}^{A}(a')] = f_*[a_1 + d_{n-1}^{A}(a')]$$

i.e.,

$$[f_n(a_1 + d_{n-1}^{A}(a'))] = [f_n(a_2 + d_{n-1}^{A}(a'))];$$

this gives

$$[s_n(c_1 + d_{n-1}^{C}(c'))] = [s_n(c_2 + d_{n-1}^{C}(c'))]$$

i.e.,

$$s_*[c_1 + d_{n-1}^{C}(c')] = s_*[c_2 + d_{n-1}^{C}(c')].$$

Since s_* is an isomorphism we have

$$[c_1 + d_{n-1}^{C}(c')] = [c_2 + d_{n-1}^{C}(c')].$$

Hence we have

$$([a_1 + d_{n-1}^{A}(a')], [c_1 + d_{n-1}^{C}(c')])$$
$$= ([a_2 + d_{n-1}^{A}(a')], [c_2 + d_{n-1}^{C}(c')]);$$

we apply the isomorphism

$$\alpha_* : (Z_n(\bar{A}) / B_n(\bar{A})) \times (Z_n(\bar{C})/B_n(\bar{C})) \to (Z_n(\bar{A}) \times Z_n(\bar{C}))/(B_n(\bar{A}) \times B_n(\bar{C}))$$

to the above to get

$$\alpha_*([a_1 + d_{n-1}^{A}(a')], [c_1 + d_{n-1}^{C}(c')])$$
$$= \alpha_*([a_2 + d_{n-1}^{A}(a')], [c_2 + d_{n-1}^{C}(c')]),$$

i.e.,

$$[(a_1 + d^A_{n-1}(a'), c_1 + d^C_{n-1}(c'))]$$
$$= [(a_2 + d^A_{n-1}(a'), c_2 + d^C_{n-1}(c'))].$$

Thus

$$[(a_1, c_1)] = [(a_2, c_2)],$$

showing that

$$t_* : H_*(D) \to H_*(A)$$

is a monomorphism.

Next we show that $t_* : H_*(D) \to H_*(A)$ is an epimorphism. Let $[a] \in H_*(A)$ be arbitrary. Then $f_n(a) \in B_n$. Since s is an epimorphism $f_n(a) = s_n(c)$ for some $c \in C_n$. Hence $(a, c) \in D_n$. Then

$$t_*[(a, c)] = [t_n(a, c)] = [a]$$

showing t_* is an epimorphism. Since t is an epimorphism and t_* is an isomorphism we conclude that $t \in S$.

Next for any d.g.a.

$$E = \sum_{n \geq 0} E_n$$

and d.g.a.-homomorphisms

$$u = \{u_n : E_n \to A_n\}$$

and

$$v = \{v_n : E_n \to C_n\}$$

in $\mathcal{DGA}$ let the following diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & A \\ v \downarrow & & \downarrow f \\ C & \xrightarrow[s]{} & B \end{array}$$

commute, i.e., $fu = sv$. Consider the diagram

$$\begin{array}{ccccc} E & & & \searrow u & \\ & \searrow h & & & \\ v \searrow & & D & \xrightarrow{t} & A \\ & & g \downarrow & & \downarrow f \\ & & C & \xrightarrow[s]{} & B \end{array}$$

Define $h = \{h_n : E_n \to D_n\}$
by the rule

$$h(x) = (u(x), v(x))$$

for $x \in E$. Clearly h is well defined and is a d.g.a. homomorhism. Now for any $x \in E$,

$$th(x) = t\big(u(x), v(x)\big) = u(x)$$

and

$$gh(x) = g\big(u(x), v(x)\big) = v(x),$$

i.e., $th = u$ and $gh = v$. This completes the proof of Proposition 3.2. ■

3.3 Proposition. *If each* $s_i : A_i \to B_i, i \in I$, *is an element of* S, *where the index set* I *is an element of* $\mathcal{U}$, *then*

$$\prod_{i \in I} s_i : \prod_{i \in I} A_i \to \prod_{i \in I} B_i$$

is an element of S.

Proof. The proof is trivial. ■

The following result can be obtained from the above discussion.

3.4 Proposition. *The category* $\mathcal{DGA}$ *is complete*.

From Propositions 3.3 - 3.4, it follows that the conditions of Theorem 2.10 are fulfilled and by the use of Theorem 2.13, we obtain the following result.

3.5 Theorem. *Every object* A *of the category* $\mathcal{DGA}$ *has an Adams cocompletion* A_S *with respect to the set of morphisms* S *and there exists a morphism*

$$e : A_S \to A$$

in S *which is couniversal with respect to the morphisms in* S*, that is, given a morphism*

$$s : B \to A$$

in S *there exists a unique morphism*

$$t : A_S \to B$$

such that $st = e$*. In other words the following diagram is commutative*:

$$\begin{array}{ccc} A_S & \xrightarrow{e} & A \\ t \downarrow & \nearrow s & \\ B & & \end{array}$$

IV. MINIMAL MODEL

We recall the following algebraic preliminaries.

4.1 Definition. [6, 12] A d.g.a. M is called a *minimal algebra* if it satisfies the following properties:

(a) M is free as a graded algebra.
(b) M has decomposable differentials.
(c) $M_0 = \mathbb{Q}$, $M_1 = 0$.
(d) M has homology of finite type, i.e., for each n, $H_n(M)$ is a finite dimensional vector space.

Let $\mathcal{M}$ be the full subcategory of the category $\mathcal{DGA}$ consisting of all minimal algebras and all d.g.a.-maps between them.

4.2 Definition. [6, 12] Let A be a simply connected d.g.a.. A d.g.a. $M = M_A$ is called a *minimal model* of A if the following conditions hold:

(i) $M \in \mathcal{M}$.
(ii) There is a d.g.a.-map $\rho : M_A \to A$ which induces an isomorphism on homology, i.e.,

$$\rho_* : H_*(M) \xrightarrow{\cong} H_*(A).$$

Henceforth we assume that the d.g.a.-map $\rho : M_A \to A$ is a d.g.a.-epimorphism.

4.3 Theorem. (Theorem 2.13, p. 48 [6]; Lemma 2, p. 38 and Theorem 2, p. 45 [12]) *Let* A *be a simply connected d.g.a. and* M_A *be its minimal model. The map*

$$\rho : M_A \to A$$

has couniversal property, i.e., for any d.g.a. Z *and d.g.a.-map*

$$\varphi : Z \to A,$$

there exists a d.g.a.-map

$$\theta : M_A \to Z$$

such that $\rho \simeq \varphi\theta$*; furthermore if the d.g.a.-map* $\varphi : Z \to A$ *is an epimorphism then* $\rho = \varphi\theta$*,* i.e., *the following diagram is commutative*:

$$\begin{array}{ccc} M_A & \stackrel{\rho}{\rightarrow} & A \\ \theta \downarrow & \nearrow \varphi & \\ Z & & \end{array}$$

V. THE RESULT

We show that under a reasonable assumption, the minimal model of a 1-connected d.g.a. can be expressed as the Adams cocompletion of the d.g.a. with respect to the chosen set of d.g.a.-maps.

5.1 Theorem. $M_A \cong A_S$.

Proof. Let $e : A_S \to A$ be the map as in Theorem 3.5 and $\rho : M_A \to A$ be the d.g.a.-map as in Theorem 4.3. Since the d.g.a.-map $\rho : M_A \to A$ is a d.g.a.-epimorphism, by the couniversal property of e there exists a d.g.a.-map $\theta : A_S \to M_A$ such that $e = \rho\theta$.

$$\begin{array}{ccc} A_S & \stackrel{e}{\rightarrow} & A \\ \theta \downarrow & \nearrow \rho & \\ M_A & & \end{array}$$

By the couniversal property of ρ there exists a d.g.a.-map $\varphi : M_A \to A_S$ such that $e\varphi = \rho$.

$$\begin{array}{ccc} M_A & \stackrel{\rho}{\rightarrow} & A \\ \varphi \downarrow & \nearrow e & \\ A_S & & \end{array}$$

Consider the following diagram

$$\begin{array}{cccc} & A_S & \stackrel{e}{\rightarrow} & A \\ & \theta \downarrow & & \\ 1_{A_S} \downarrow & M_A & \nearrow e & \\ & \varphi \downarrow & & \\ & A_S & & \end{array}$$

Thus we have $e\varphi\theta = \rho\theta = e$. By the uniqueness condition of the couniversal property of e (Theorem 3.5), we conclude that $\varphi\theta = 1_{A_S}$.

Next consider the diagram

$$M_A \overset{\rho}{\rightarrow} \qquad\qquad A$$

$$\varphi \downarrow$$

$$1_{M_A} \downarrow \quad A_S \qquad\qquad \nearrow \rho$$

$$\theta \downarrow$$

$$M_A$$

Thus we have $\rho\theta\varphi = e\varphi = \rho$. By the couniversal oroperty of ρ (Theorem 4.3), we conclude that $\theta\varphi = 1_{M_A}$. Thus $M_A \cong A_S$. This completes the proof of Theorem 5.1. ■

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