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NUMERICAL SOLUTION OF FRACTIONAL ORDER DIFFERENTIAL-ALGEBRAIC EQUATIONS USING GENERALIZED TRIANGULAR FUNCTION OPERATIONAL MATRICES

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Abstract. This article introduces a new application of piecewise linear orthogonal triangular functions to solve fractional order differential-algebraic equations. The generalized triangular function operational matrices for approximating Riemann-Liouville fractional order integral in the triangular function (TF) domain are derived. Error analysis is carried out to estimate the upper bound of absolute error between the exact Riemann-Liouville fractional order integral and its TF approximation. Using the proposed generalized operational matrices, linear and nonlinear fractional order differential-algebraic equations are solved. The results show that the TF estimate of Riemann-Liouville fractional order integral is accurate and effective.

1. Introduction

Many physical problems in mechatronics, chemical kinetics, optimal control, electric circuit design, chemical process control, molecular dynamics, incompressible fluids, power systems and industrial production processes etc. are successfully modelled by the use of differential-algebraic equations [1, 2].
Generally the differential-algebraic equations are expressed in a fully implicit form given by

$$F\left(t, Y, Y^{'}\right) = 0 \tag{1}$$

where Y is a vector of dependent variables, F is a vector of functions of time t, dependent variables and their derivatives and $\partial F/\partial Y^{'}$ may be singular. The special case of differential-algebraic equations in 1, which is frequently appearing in practical applications, is semi-explicit differential-algebraic equation or ordinary differential equation with algebraic constraints defined as follows.

$$\begin{aligned} Y^{'} &= F\left(t, Y, Z\right) \\ 0 &= g\left(t, Y, Z\right) \end{aligned} \tag{2}$$

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The mathematical theory and properties of differential-algebraic equations can be found in Brenan et al. [3], Campbell et al. [4] and Gear [5]. The complexity in solving differential-algebraic equation is usually indicated by differential index. The differential index is the minimum number of differentiations required to convert a given differential-algebraic equation into a system of ordinary differential equations. The differential index of semi-explicit differential-algebraic equation in 2 is 1 if $\partial g/\partial Z$ is non-singular. The high index (index greater than 1) differential-algebraic models are difficult to solve so that an alternative is to utilize index reducing method which transforms high index problems into low index (or to one) problems by successive differentiation of algebraic constraints [6].

Several real world phenomena, which exhibit fractional dynamics, in biology, electrochemistry, electromagnetism, acoustics, material science, control theory, physics etc. are well represented by fractional order derivative than the classical one [7, 8, 9, 10, 11]. The non-local nature of fractional order derivative made it capable of explaining the inherent fractional order description of such physical processes. Electrochemical processes, non-integer order optimal controller design, complex biochemical processes may find application of fractional differential algebraic equations.

In this article, we consider the following fractional order differential-algebraic equations.

$$\begin{aligned} D_*^{\alpha_i} x_i\left(t\right) &= f_i\left(t, x_1, x_2, \ldots, x_n\right) \\ 0 &= g\left(t, x_1, x_2, \ldots, x_n\right) \end{aligned} \tag{3}$$

with initial conditions $x_i\left(0\right) = a_i, i = 1, 2, \ldots, n.$

Here $D_*^{\alpha_i}$ is Caputo fractional derivative of order α_i satisfying the relation $m-1 < \alpha_i \leq m, m \in N$.

In general, most fractional order differential-algebraic equations do not have exact solutions. Therefore, development of effective numerical techniques, which offer precise approximate solutions, has become an active research area. In this regard, Ibis and Bayram [12], Ibis et al. [13], Zurigat et al. [14] and Ding and Jiang [15] extended the application of Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Fractional Differential Transform Method (FDTM), Homotopy Analysis Method (HAM) and waveform relaxation method to solve fractional order differential-algebraic equations.

The idea of Fourier to use arbitrary functions, even the one defined by different equations in its adjacent segments of its range, which a single analytical function cannot represent actually paved the way of using family of orthogonal functions. The processes having jump, delay and discontinuity may be better represented by orthogonal functions. The orthogonal triangular function sets developed by Deb et al. [16] are a complementary pair of piecewise linear polynomial function sets evolved from a simple dissection of block pulse function (BPF) set [17]. The authors have derived a complementary pair of operational matrices for first order integration in the TF domain and demonstrated that the TF domain technique for dynamical systems analysis is computationally more effective than the BPF domain technique. Besides system analysis, the orthogonal TFs also find applications in system identification, optimal controller design and numerical analysis of classical integral and differential equations [18, 19, 20, 21, 22]. Those successful applications have made us strongly believe that the TFs having enough potential to be applicable in fractional order systems. To the best of our knowledge, there is no literature until date in fractional calculus that reported the use of orthogonal TFs for solving fractional order differential-algebraic equations. These facts motivated us to extend the application of orthogonal TFs to solve fractional order differential-algebraic equations shown in 3. To accomplish our goal, we have proposed the generalized triangular function operational matrices for estimating the Riemann-Liouville fractional order integral in the TF domain. The rest of the paper is prepared as follows. Useful definitions of fractional calculus are provided in section 2. Generation of complementary pair of TF sets from BPF set is discussed in section 3. Section 4 presents the basic properties of orthogonal TF sets. The method of estimating classical and fractional integration in the TF domain is explained in section 5. An upper bound of absolute error between the exact Riemann-Liouville fractional order integral and its TF estimate is computed in section 6. Section 7 implements the proposed operational matrices on illustrative examples. Finally, the paper is concluded in section 8.

2. Basic definitions of fractional calculus

In this section, we provide widely used definitions of fractional calculus [23].
Definition 2.1 A real function $f(t)\,,\ t>0$, is said to be in the space $C_\mu,\ \mu\in R$ if there exists a real number $p\,(>\mu)$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C\,[0,\infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)}\in C_\mu,\ m\in N$.
Definition 2.2 The Riemann-Liouville fractional order integral of $\alpha\,(>0)$ of function $f(t)\in C_\mu,\ \mu>-1$ is defined as

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau \tag{4}$$

Definition 2.3 The Riemann-Liouville fractional order derivative (RL) of function $f(t)$ is defined as

$$D_t^\alpha f(t)=D^m J^{m-\alpha} f(t)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,d\tau, t>0 \tag{5}$$

where α is a non-integer satisfying the relation $m-1<\alpha\le m$, $m\in N$.
Definition 2.4 The fractional order derivative of $f(t)$ in Caputo sense is defined as

$$D_*^\alpha f(t)=J^{m-\alpha}D^m f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\,d\tau,\ t>0 \tag{6}$$

3. Triangular functions

In this section, firstly, we review block pulse functions in brief and then we introduce the method of dissecting the block pulse function set to formulate a complementary pair of orthogonal TF sets.

3.1 Review of block pulse functions

Let us consider a square integrable function $f(t)$ of Lebesgue measure, which is continuous in the interval $[0,T)$. Divide the interval into m subintervals of constant width $h=T/m$ as $[t_i,t_{i+1})\,,i=0,1,2,\ldots,m-1$.

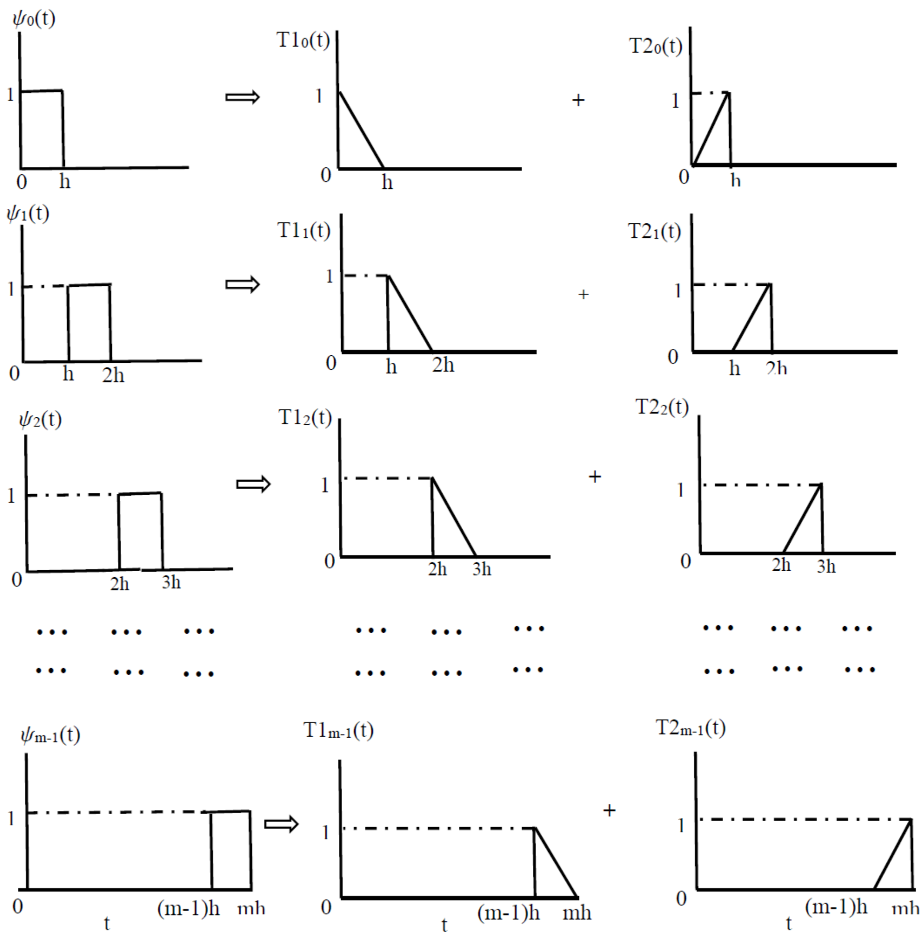

FIGURE 1. Generation of TFs from BPFs

Let $\psi_m(t)$ be a set of block pulse functions containing m component functions in the interval $[0, T)$.

$$\psi_m(t) = [\psi_0(t), \psi_1(t), \psi_2(t), \ldots, \psi_{m-1}(t)]^T_{1\times m} \tag{7}$$

where $[\cdots]^T$ signifies transpose.
The i^{th} component of the BPF vector $\psi_m(t)$ is defined as

$$\psi_i(t) = \begin{cases} 1 & ih \le t < (i+1)h \\ 0 & otherwise \end{cases} \quad i = 0, 1, 2, \ldots, m-1 \tag{8}$$

The square integrable function $f(t)$ can be approximated by BPFs as

$$f(t) \cong \sum_{i=0}^{m-1} f_i \psi_i(t) = [f_0, f_1, \ldots, f_{m-1}] \psi_m(t) = F^T \psi_m(t) \tag{9}$$

where the constant coefficients f_i are defined as $f_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(t)\,dt$.
The BPF estimate for first order integration of function $f(t)$ can be derived as [24]

$$Jf(t) = \int_0^t f(s)\,ds \cong \int_0^t F^T\psi_m(s)\,ds = F^T\int_0^t \psi_m(s)\,ds = F^T P\psi_m(t) \tag{10}$$

where P is the operational matrix for first order integration in the BPF domain:

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & \cdots & 2 \\ 0 & 1 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 0 & 1 & \ddots & 2 \\ \vdots & \cdots & \cdots & \ddots & \ddots & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{m\times m}$$

3.2 Complementary pair of TF sets

Let us divide the first component of BPF vector $\psi_m(t)$ into a complementary pair of linear polynomial functions as shown in figure 1.

$$\psi_0(t) = T1_0(t) + T2_0(t) \tag{11}$$

where $T1_0(t) = \left(1 - \frac{t}{h}\right)$ and $T2_0(t) = \left(\frac{t}{h}\right)$
and the second component $\psi_1(t)$,

$$\psi_1(t) = T1_1(t) + T2_1(t) \tag{12}$$

where $T1_1(t) = 1 - \left(\frac{t-h}{h}\right)$ and $T2_1(t) = \left(\frac{t-h}{h}\right)$.
In the same fashion, we can divide the remaining components of $\psi_m(t)$ into respective complementary pairs of linear polynomial functions. Thus, for the whole set of BPFs, we now have two sets of linear polynomial functions, namely, $T1_m(t)$ and $T2_m(t)$ each contains m component functions in the interval $[0, T)$.

$$\psi_m(t) = T1_m(t) + T2_m(t) \tag{13}$$

where $T1_m(t) = [T1_0(t), \ldots, T1_{m-1}(t)]^T, T2_m(t) = [T2_0(t), \ldots, T2_{m-1}(t)]^T$
The triangular function vectors; $T1_m(t)$ and $T2_m(t)$ together form the entire set of BPFs, hence, $T1_m(t)$ and $T2_m(t)$ are complement to each other as far as BPF set is considered. We recognize from figure 1 that the shapes of $T1_i$ and $T2_i$ are left-handed and right-handed triangles, respectively. So, we name these two sets as left-handed triangular functions vector (LHTF) and right-handed triangular functions vector (RHTF), respectively.
Now we define the i^{th} component of the LHTF vector $T1_m(t)$ as

$$T1_i(t) = \begin{cases} 1 - \left(\frac{t-ih}{h}\right) & ih \le t < (i+1)h \\ 0 & otherwise \end{cases} \quad i = 0, 1, 2, \ldots, m-1 \tag{14}$$

and the i^{th} component of the RHTF vector $T2_m(t)$ as

$$T2_i(t) = \begin{cases} \frac{t-ih}{h} & ih \le t < (i+1)h \\ 0 & otherwise \end{cases} \quad i = 0, 1, 2, \ldots, m-1 \tag{15}$$

Like BPFs, TFs can also be employed for the approximation of the square integrable function $f(t)$ in the interval $[0, T)$.

$$\left.\begin{aligned} f(t) &\cong [c_0, c_1, \ldots, c_{m-1}]\, T1_m(t) + [d_0, d_1, \ldots, d_{m-1}]\, T2_m(t) \\ &= C^T T1_m(t) + D^T T2_m(t) \end{aligned}\right\} \tag{16}$$

where the constant coefficients c_i and d_i can be computed as $c_i = f(ih)$, $d_i = f((i+1)h)$.
The expressions for the coefficients c_i and d_i emphasis that the function evaluations at the equidistant nodes, $t_i, i = 0, 1, 2, \ldots, m-1$ are enough to find their numerical values. Whereas the coefficients in BPF series representation in equation 9 demand the integration of $f(t)$. Thus, the function approximation in TF domain is computationally more effective compared to that in BPF domain.

4. Basic properties of triangular functions

This section introduces orthogonal and a few operational properties of LHTF set $T1_m(t)$ and RHTF set $T2_m(t)$. The proofs of theorems 4.1 to 4.5 are published elsewhere [17].

Theorem 4.1. *If $T1_i(t), T1_j(t) \in T1_m(t)$, $i, j \le m$, the condition of orthogonality of LHTF set $T1_m(t)$ is* $\int_0^T T1_i(t)\, T1_j(t)\, dt = \begin{cases} \frac{h}{3} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Theorem 4.2. *If $T2_i(t), T2_j(t) \in T2_m(t)$, $i, j \le m$, the condition of orthogonality of RHTF set $T2_m(t)$ is* $\int_0^T T2_i(t)\, T2_j(t)\, dt = \begin{cases} \frac{h}{3} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Theorem 4.3. *If $T1_i(t), T1_j(t) \in T1_m(t)$, $i, j \le m$, then*
$T1_i(t)\, T1_j(t) = \begin{cases} 0 & \text{if } i \ne j \\ T1_i(t) & \text{if } i = j \end{cases}$

Theorem 4.4. *If $T2_i(t), T2_j(t) \in T2_m(t)$, $i, j \le m$, then*
$T2_i(t)\, T2_j(t) = \begin{cases} 0 & \text{if } i \ne j \\ T2_i(t) & \text{if } i = j \end{cases}$

Theorem 4.5. *If $T1_i(t) \in T1_m(t)$ and $T2_j(t) \in T2_m(t)$, then $T1_i(t)\, T2_j(t) = 0$, $\forall i, j \le m$*

Theorem 4.6. *If $g(t)$ is a square integrable function of Lebesgue measure and continuous in the interval $[0, T)$, then the TF estimate of n^{th} power of $g(t)$ is $(g(t))^n \cong \left[c_0^n, c_1^n, \ldots, c_{m-1}^n\right] T1_m(t) + \left[d_0^n, d_1^n, \ldots, d_{m-1}^n\right] T2_m(t)$, $n \in N$.*

Proof. Using equation 16, expanding the function $g(t)$ in the orthogonal TF domain as

$$g(t) \cong C^T T1_m(t) + D^T T2_m(t) \tag{17}$$

The product $g(t) \times g(t)$ can be expressed in terms of complementary pair of TF sets as

$$\left.\begin{aligned} g(t) \times g(t) \cong C^T .* C^T\left(T1_m(t)\, T1_m(t)\right) + C^T .* D^T\left(T1_m(t)\, T2_m(t)\right) + \\ D^T .* C^T\left(T2_m(t)\, T1_m(t)\right) + D^T .* D^T\left(T2_m(t)\, T2_m(t)\right) \end{aligned}\right\} \tag{18}$$

Here the operator $(.*)$ denotes element-by-element product.
Employing theorems 4.3 to 4.5, the above equation can be simplified into the following form.

$$g(t) \times g(t) \cong \left[c_0^2, c_1^2, \ldots, c_{m-1}^2\right] T1_m(t) + \left[d_0^2, d_1^2, \ldots, d_{m-1}^2\right] T2_m(t) \tag{19}$$

In the same manner, n^{th} power of $g(t)$ can be estimated in the TF domain as shown in equation 20.

$$(g(t))^n \cong \left[c_0^n, c_1^n, \ldots, c_{m-1}^n\right] T1_m(t) + \left[d_0^n, d_1^n, \ldots, d_{m-1}^n\right] T2_m(t) \tag{20}$$

□

Theorem 4.7. *Suppose the square integrable functions; $h_1(t)$,$h_2(t)$, $h_3(t)$,..., $h_n(t)$ are approximated via TFs. Then their product can be expressed as*

$$\left.\begin{array}{r} h_1(t)\, h_2(t) \cdots h_n(t) \cong \left[c_{1_0} c_{2_0} \cdots c_{n_0}, \ldots, c_{1_{m-1}} c_{2_{m-1}} \cdots c_{n_{m-1}}\right] T1_m(t) + \\ \left[d_{1_0} d_{2_0} \cdots d_{n_0}, \ldots, d_{1_{m-1}} d_{2_{m-1}} \cdots d_{n_{m-1}}\right] T2_m(t) \end{array}\right\}$$

Proof. The piecewise linear approximation of $h_1(t)$, $h_2(t)$ and $h_3(t)$ by TFs are

$$\left.\begin{array}{l} h_1(t) \cong C_1^T T1_m(t) + D_1^T T2_m(t) \\ h_2(t) \cong C_2^T T1_m(t) + D_2^T T2_m(t) \\ h_3(t) \cong C_3^T T1_m(t) + D_3^T T2_m(t) \end{array}\right\} \tag{21}$$

Employing theorems 4.3 to 4.5, the product $h_1(t)\, h_2(t)$ is estimated as

$$\left.\begin{array}{rl} h_1(t)\, h_2(t) \cong & \left(C_1^T .* C_2^T\right) T1_m(t) + \left(D_1^T .* D_2^T\right) T2_m(t) \\ = & \left[c_{1_0} c_{2_0}, c_{1_1} c_{2_1}, \ldots, c_{1_{m-1}} c_{2_{m-1}}\right] T1_m(t) + \\ & \left[d_{1_0} d_{2_0}, d_{1_1} d_{2_1}, \ldots, d_{1_{m-1}} d_{2_{m-1}}\right] T2_m(t) \end{array}\right\} \tag{22}$$

The product $h_1(t)\, h_2(t)\, h_3(t)$ in the TF domain is expressed as

$$\left.\begin{array}{r} h_1(t)\, h_2(t)\, h_3(t) \cong \left[c_{1_0} c_{2_0} c_{3_0}, c_{1_1} c_{2_1} c_{3_1}, \ldots, c_{1_{m-1}} c_{2_{m-1}} c_{3_{m-1}}\right] T1_m(t) + \\ \left[d_{1_0} d_{2_0} d_{3_0}, d_{1_1} d_{2_1} d_{3_1}, \ldots, d_{1_{m-1}} d_{2_{m-1}} d_{3_{m-1}}\right] T2_m(t) \end{array}\right\} \tag{23}$$

Likewise, the product of n square integrable functions can be expressed as

$$\left.\begin{array}{r} h_1(t)\, h_2(t) \cdots h_n(t) \cong \left[c_{1_0} c_{2_0} \cdots c_{n_0}, \ldots, c_{1_{m-1}} c_{2_{m-1}} \cdots c_{n_{m-1}}\right] T1_m(t) + \\ \left[d_{1_0} d_{2_0} \cdots d_{n_0}, \ldots, d_{1_{m-1}} d_{2_{m-1}} \cdots d_{n_{m-1}}\right] T2_m(t) \end{array}\right\} \tag{24}$$

□

5. Triangular function operational matrices for classical and fractional integration

In the following subsections, we establish the method of approximating classical as well as fractional integration in the TF domain.

5.1 The TF estimate of first order integral of function $f(t)$

One-fold integration of square integrable function $f(t)$ is

$$Jf(t) = \int_0^t f(s)\, ds \tag{25}$$

Substituting the TF estimate of $f(t)$ in equation 25 leads to

$$\left.\begin{aligned}\int_0^t f(s)\,ds &\cong C^T\int_0^t T1_m(s)\,ds + D^T\int_0^t T2_m(s)\,ds\\ &= C^T\left[\int_0^t T1_0(s)\,ds,\ldots,\int_0^t T1_{m-1}(s)\,ds\right]^T T1_m(t)+\\ &\quad D^T\left[\int_0^t T2_0(s)\,ds,\ldots,\int_0^t T2_{m-1}(s)\,ds\right]^T T2_m(t)\end{aligned}\right\} \tag{26}$$

Therefore, the integration of function $f(t)$ is now changed to the integration of LHTF set and RHTF set. Since the function $f(t)$ is square integrable, its estimate is also square integrable.
The graph of $T1_i(t)$ versus t and $T2_i(t)$ versus t depicted in figure 1 can be expressed mathematically as

$$T1_i(t) = u(t-ih) - \frac{t-ih}{h}u(t-ih) + \frac{t-(i+1)h}{h}u(t-(i+1)h) \tag{27}$$

$$T2_i(t) = \frac{t-ih}{h}u(t-ih) - \frac{t-(i+1)h}{h}u(t-(i+1)h) - u(t-(i+1)h) \tag{28}$$

We now integrate each component of LHTF set $T1_m(t)$ using equation 27 and express the result in terms of LHTF set $T1_m(t)$ and RHTF set $T2_m(t)$.

$$\left.\begin{aligned}\int_0^t T1_0(s)\,ds &= \int_0^t\left(1-\tfrac{s}{h}\right)u(s)\,ds + \int_h^t\left(\tfrac{s-h}{h}\right)u(s-h)\,ds\\ &= \tfrac{h}{2}\begin{bmatrix}0 & 1 & \ldots & 1 & 1\end{bmatrix}T1_m(t) + \tfrac{h}{2}\begin{bmatrix}1 & 1 & \ldots & 1 & 1\end{bmatrix}T2_m(t)\end{aligned}\right\} \tag{29}$$

$$\int_0^t T1_1(s)\,ds = \frac{h}{2}\begin{bmatrix}0 & 0 & 1 & \ldots & 1 & 1\end{bmatrix}T1_m(t) + \frac{h}{2}\begin{bmatrix}0 & 1 & 1\ldots & 1 & 1\end{bmatrix}T2_m(t) \tag{30}$$

$$\vdots$$

$$\vdots$$

$$\int_0^t T1_{m-1}(s)\,ds = \frac{h}{2}\begin{bmatrix}0 & 0 & \ldots & 0 & 0\end{bmatrix}T1_m(t) + \frac{h}{2}\begin{bmatrix}0 & 0 & \ldots & 0 & 1\end{bmatrix}T2_m(t) \tag{31}$$

Therefore, the first order integration of LHTF set $T1_m(t)$ is

$$\int_0^t T1_m(s)\,ds = P_1T1_m(t) + P_2T2_m(t) \tag{32}$$

where

$$P_1 = \frac{h}{2}\begin{bmatrix}0 & 1 & 1 & \cdots & \cdots & 1\\ 0 & 0 & 1 & 1 & \cdots & 1\\ 0 & 0 & 0 & 1 & \cdots & 1\\ 0 & 0 & 0 & 0 & \ddots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \ddots & 1\\ 0 & 0 & 0 & 0 & \cdots & 0\end{bmatrix}_{m\times m},\quad P_2 = \frac{h}{2}\begin{bmatrix}1 & 1 & 1 & \cdots & \cdots & 1\\ 0 & 1 & 1 & 1 & \cdots & 1\\ 0 & 0 & 1 & 1 & \cdots & 1\\ 0 & 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\end{bmatrix}_{m\times m}$$

Following the same procedure, the first order integration of RHTF set $T2_m(t)$ using equation 28 is

$$\int_0^t T2_m(s)\,ds = P_1T1_m(t) + P_2T2_m(t) = \int_0^t T1_m(s)\,ds \tag{33}$$

Equation 25 becomes

$$\int_0^t f(s)\,ds \cong \left(C^T P_1 + D^T P_1\right) T1_m(t) + \left(C^T P_2 + D^T P_2\right) T2_m(t) \tag{34}$$

Here P_1 and P_2 are complement to each other as far as P is considered. This complementary pair is acting as a first order integrator in the TF domain.

5.2 The TF estimate of Riemann-Liouville fractional order integral of function $f(t)$

Replacing $f(t)$ with its TF estimate in equation 4,

$$\left.\begin{aligned} J^\alpha f(t) &\cong C^T\left(J^\alpha T1_m(t)\right) + D^T\left(J^\alpha T2_m(t)\right) \\ &= C^T \begin{bmatrix} J^\alpha T1_0(t) \\ J^\alpha T1_1(t) \\ \vdots \\ J^\alpha T1_{m-1}(t) \end{bmatrix} + D^T \begin{bmatrix} J^\alpha T2_0(t) \\ J^\alpha T2_1(t) \\ \vdots \\ J^\alpha T2_{m-1}(t) \end{bmatrix} \end{aligned}\right\} \tag{35}$$

Similar to equation 29, we compute the α - order Riemann-Liouville fractional integral of $T1_0(t)$ and express the result by means of complementary pair of TF sets.

$$\left.\begin{aligned} J^\alpha T1_0(t) = \tfrac{h^\alpha}{\Gamma(\alpha+2)} &\begin{bmatrix} 0 & \varsigma_1 & \varsigma_2 & \cdots & \cdots & \varsigma_{m-1} \end{bmatrix} T1_m(t) + \\ \tfrac{h^\alpha}{\Gamma(\alpha+2)} &\begin{bmatrix} \varsigma_1 & \varsigma_2 & \varsigma_3 & \cdots & \cdots & \varsigma_m \end{bmatrix} T2_m(t) \end{aligned}\right\} \tag{36}$$

where $\varsigma_j = \left(j^\alpha(1+\alpha-j) + (j-1)^{\alpha+1}\right)$, $j = 1, 2, \ldots, m$.

$$\left.\begin{aligned} J^\alpha T1_1(t) = \tfrac{h^\alpha}{\Gamma(\alpha+2)} &\begin{bmatrix} 0 & 0 & \varsigma_1 & \cdots & \varsigma_{m-2} \end{bmatrix} T1_m(t) + \\ \tfrac{h^\alpha}{\Gamma(\alpha+2)} &\begin{bmatrix} 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-1} \end{bmatrix} T2_m(t) \end{aligned}\right\} \tag{37}$$

$$\vdots$$

$$\vdots$$

$$\left.\begin{aligned} J^\alpha T1_{m-1}(t) = \tfrac{h^\alpha}{\Gamma(\alpha+2)} &\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \end{bmatrix} T1_m(t) + \\ \tfrac{h^\alpha}{\Gamma(\alpha+2)} &\begin{bmatrix} 0 & 0 & 0 & \cdots & \varsigma_1 \end{bmatrix} T2_m(t) \end{aligned}\right\} \tag{38}$$

Therefore, the Riemann-Liouville fractional integral of order α of LHTF set $T1_m(t)$ is

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T1_m(\tau)\,d\tau = P_1^\alpha T1_m(t) + P_2^\alpha T2_m(t) \tag{39}$$

where

$$P_1^\alpha = \tfrac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & \varsigma_1 & \varsigma_2 & \varsigma_3 & \cdots & \varsigma_{m-1} \\ 0 & 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-2} \\ 0 & 0 & 0 & \varsigma_1 & \cdots & \varsigma_{m-3} \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \varsigma_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, P_2^\alpha = \tfrac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} \varsigma_1 & \varsigma_2 & \varsigma_3 & \cdots & \cdots & \varsigma_m \\ 0 & \varsigma_1 & \varsigma_2 & \varsigma_3 & \cdots & \varsigma_{m-1} \\ 0 & 0 & \varsigma_1 & \varsigma_2 & \cdots & \varsigma_{m-2} \\ 0 & 0 & 0 & \varsigma_1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \varsigma_1 \end{bmatrix}$$

Following the same procedure as we applied for LHTF set, the Riemann-Liouville fractional order integral of RHTF set using equation 28 is derived as

$$J^{\alpha}T2_m(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left((t-\tau)^{\alpha-1}T2_m(\tau)\right)d\tau = P_3^{\alpha}T1_m(t) + P_4^{\alpha}T2_m(t) \quad (40)$$

where

$$P_3^{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\begin{bmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \xi_1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \xi_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, P_4^{\alpha} = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \cdots & \cdots & \xi_m \\ 0 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 0 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 0 & \xi_1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \xi_1 \end{bmatrix}$$

$\xi_j = j^{\alpha+1} - (j+\alpha)(j-1)^{\alpha}$

From equations 35 , 39 and 40,

$$\left.\begin{aligned} J_{TF}^{\alpha}f(t) &= \tfrac{1}{\Gamma(\alpha)}\textstyle\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau \\ &\cong \left(C^TP_1^{\alpha} + D^TP_3^{\alpha}\right)T1_m(t) + \left(C^TP_2^{\alpha} + D^TP_4^{\alpha}\right)T2_m(t) \end{aligned}\right\} \quad (41)$$

For the special case of $\alpha = 1$,

$$P_1^{\alpha} = P_3^{\alpha} = P_1, P_2^{\alpha} = P_4^{\alpha} = P_2 \quad (42)$$

The TF estimate of fractional order integral is reduced to the TF estimate of first order integral when $\alpha = 1$. Theorefore, the classical complementary pair of operational matrices is a particular case of generalized operational matrices.

6. Error analysis

Let us denote the TF estimate of function $f(t)$ as

$$f_{TF}(t) = C^TT1_m(t) + D^TT2_m(t) \quad (43)$$

From equations 14 to 16, we can approximate $f(t)$ in the interval $[ih,(i+1)h)$ as

$$\left.\begin{aligned} f_{TF}(t) &= f(ih)T1_i(t) + f((i+1)h)T2_i(t) \\ &= f(ih)\left(1 - \tfrac{t-ih}{h}\right) + f((i+1)h)\left(\tfrac{t-ih}{h}\right) \\ &= f(ih) + f((i+1)h)\left(\tfrac{t-ih}{h}\right) - f(ih)\left(\tfrac{t-ih}{h}\right) \\ &= f(ih) + \left(\tfrac{f((i+1)h)-f(ih)}{h}\right)(t-ih) \\ &\cong f(ih) + f^{'}(ih)(t-ih) \qquad \text{as} \quad h \to 0 \end{aligned}\right\} \quad (44)$$

Expanding the exact function $f(t)$ by Taylor series with the center ih as

$$f(t) = f(ih) + (t-ih)f^{'}(ih) + \frac{(t-ih)^2}{2}f^{''}(ih) + \sum_{k=3}^{\infty}\frac{(t-ih)^k}{k!}f^{(k)}(ih) \quad (45)$$

From equations 44 and 45, the absolute error between the function and its TF estimate can be determined as

$$|f(t) - f_{TF}(t)| = \frac{(t-ih)^2}{2}\left|f^{''}(ih)\right| + O(t-ih)^3 \quad (46)$$

Because $(t-ih) < h$ and $mh = T$, the above equation becomes

$$|f(t) - f_{TF}(t)| \le \frac{T^2}{2m^2}\left|f^{''}(ih)\right| + O\left(\frac{1}{m^3}\right) \quad (47)$$

We replace $f(t)$ with $f_{TF}(t)$ in equation 4 and we call the resulting integral the m^{th} approximate of the α - order Riemann-Liouville fractional integral of $f(t)$.

$$J_{TF}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f_{TF}(\tau)\, d\tau \tag{48}$$

The absolute error between the exact fractional integral $J^{\alpha} f(t)$ and the m^{th} approximate $J_{TF}^{\alpha} f(t)$ is

$$\left.\begin{aligned}
\epsilon_m &= \left|J^{\alpha} f(t) - J_{TF}^{\alpha} f(t)\right| \\
&= \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (t-\tau)^{\alpha-1} \left|f(\tau) - f_{TF}(\tau)\right| d\tau \\
&= \tfrac{1}{\Gamma(\alpha)} \left[\textstyle\sum_{r=0}^{r=i-1} \int_{rh}^{(r+1)h} (t-\tau)^{\alpha-1} \left|f(\tau) - f_{TF}(\tau)\right| d\tau\right] + \\
&\qquad \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{ih}^{t} (t-\tau)^{\alpha-1} \left|f(\tau) - f_{TF}(\tau)\right| d\tau \\
&\le \tfrac{1}{\Gamma(\alpha)} \left[\textstyle\sum_{r=0}^{r=i-1} \int_{rh}^{(r+1)h} (t-\tau)^{\alpha-1} \left(\tfrac{T^2}{2m^2} \left|f''(ih)\right| + O\left(\tfrac{1}{m^3}\right)\right) d\tau\right] + \\
&\qquad \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{ih}^{t} (t-\tau)^{\alpha-1} \left(\tfrac{T^2}{2m^2} \left|f''(ih)\right| + O\left(\tfrac{1}{m^3}\right)\right) d\tau \\
&\le \tfrac{1}{\Gamma(\alpha)} \left(\tfrac{T^2}{2m^2} \left|f''(ih)\right| + O\left(\tfrac{1}{m^3}\right)\right) \textstyle\int_0^t (t-\tau)^{\alpha-1} d\tau \\
&\le \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} \tfrac{T^2}{2m^2} \left|f''(ih)\right| + O\left(\tfrac{1}{m^3}\right)
\end{aligned}\right\} \tag{49}$$

We now consider the following assumption.

$$Max \left|f''(ih)\right| \le M, \quad i = 0, 1, 2, \ldots, m-1 \tag{50}$$

where M is finite positive value.
From equations 49 and 50, the upper bound of absolute error between $J^{\alpha} f(t)$ and $J_{TF}^{\alpha} f(t)$ can be estimated as

$$\epsilon_m \le \frac{MT^{2+\alpha}}{2m^2 \Gamma(\alpha+1)} + O\left(\frac{1}{m^3}\right) \tag{51}$$

To confirm whether the maximal absolute error caused by TFs is smaller than or equal to the theoretical upper bound derived in equation 51, we consider the function $f(t) = t$ in the interval $[0, 1]$, which is divided into five equal subintervals $m = 5$. We select the value of fractional order as 0.5.
The exact fractional integral of function $f(t)$ is

$$J^{0.5} t = \frac{\Gamma(2)}{\Gamma(2.5)} t^{1.5} \tag{52}$$

Using equation 41, the TF estimate is obtained as

$$J_{TF}^{0.5} \cong \begin{bmatrix} 0 \\ 0.0672835339205376 \\ 0.190306572389629 \\ 0.349615497789465 \\ 0.538268271364301 \end{bmatrix}^T T1_m(t) + \begin{bmatrix} 0.0672835339205376 \\ 0.190306572389629 \\ 0.349615497789465 \\ 0.538268271364301 \\ 0.752252778063675 \end{bmatrix}^T T2_m(t) \tag{53}$$

In the BPF domain [25],

$$J_{BPF}^{0.5} \cong \begin{bmatrix} 0.0336417669602688 \\ 0.128795053155083 \\ 0.269961035089547 \\ 0.443941884576883 \\ 0.645260524713988 \\ 0.870557367621520 \end{bmatrix}^T \psi_m(t) \tag{54}$$

Table 1 presents the absolute error given by TF domain analysis and BPF domain analysis. It can be noticed that the TF solution and the samples of exact fractional integral are precisely equal that demonstrates the accuracy and efficiency of the proposed TF approximation of Riemann-Liouville fractional order integral. For the selected step size, BPFs could not offer approximate solution with resonable accuracy due to its piecewise constant nature.

7. Applications and results

In this section, we solve linear and nonlinear fractional order differential-algebraic equations by employing the basic properties of orthgonal TFs and the derived gneralized triagular function operational matrices. In the following examples, the Matlab built-in function 'fsolve' is used for solving system of algbraic equations.

Example 1

Let us consider the following linear fractional order differential-algebraic equation [15]

$$\begin{cases} D_*^{1/2} x_1(t) + 2x_1(t) - \dfrac{\Gamma(7/2)}{\Gamma(3)} x_2(t) + x_3(t) = f_1(t), \\ D_*^{1/2} x_2(t) + x_2(t) + x_3(t) = f_2(t), \\ 2x_1(t) + x_2(t) - x_3(t) = f_3(t), \qquad t \in [0,1] \end{cases} \tag{55}$$

with initial conditions $x_1(0) = x_2(0) = x_3(0) = 0$.
where $f_1(t) = 2t^{5/2} + \sin t$, $f_2(t) = \frac{\Gamma(3)}{\Gamma(5/2)} t^{3/2} + t^2 + \sin t$ and $f_3(t) = 2t^{5/2} + t^2 - \sin t$.
The exact solution is $x_1(t) = t^{5/2}, x_2(t) = t^2, x_3(t) = \sin t$.

Solution:

Performing fractional integration on both sides of equation 55 results in the following form.

$$\left. \begin{aligned} x_1(t) &= J^{1/2}(f_1(t)) - J^{1/2}(2x_1(t)) + \frac{\Gamma(7/2)}{\Gamma(3)} J^{1/2}(x_2(t)) - J^{1/2}(x_3(t)) \\ x_2(t) &= J^{1/2}(f_2(t)) - J^{1/2}(x_2(t)) - J^{1/2}(x_3(t)) \\ x_3(t) &= -f_3(t) + 2x_1(t) + x_2(t) \end{aligned} \right\} \tag{56}$$

Expanding the known functions; $f_1(t)$, $f_2(t)$ and $f_3(t)$ and the unknowns functions; $x_1(t)$, $x_2(t)$ and $x_3(t)$ by means of TFs,

$$\left. \begin{aligned} f_1(t) &\cong C_{01}^T T1_m(t) + D_{01}^T T2_m(t), f_2(t) \cong C_{02}^T T1_m(t) + D_{02}^T T2_m(t) \\ f_3(t) &\cong C_{03}^T T1_m(t) + D_{03}^T T2_m(t), x_1(t) \cong C_1^T T1_m(t) + D_1^T T2_m(t) \\ x_2(t) &\cong C_2^T T1_m(t) + D_2^T T2_m(t), x_3(t) \cong C_3^T T1_m(t) + D_3^T T2_m(t) \end{aligned} \right\} \tag{57}$$

By substituting equations 41 and 57 in 56 and comparing the coefficients of LHTF and RHTF set of resulting equation, we have the following expressions.

$$\left.\begin{array}{ll} C_1^T = & \left(C_{01}^T P_1^{1/2} + D_{01}^T P_3^{1/2}\right) - 2\left(C_1^T P_1^{1/2} + D_1^T P_3^{1/2}\right) + \frac{\Gamma(7/2)}{\Gamma(3)} C_2^T P_1^{1/2} + \\ & \frac{\Gamma(7/2)}{\Gamma(3)} D_2^T P_3^{1/2} - \left(C_3^T P_1^{1/2} + D_3^T P_3^{1/2}\right) \\ D_1^T = & \left(C_{01}^T P_2^{1/2} + D_{01}^T P_4^{1/2}\right) - 2\left(C_1^T P_2^{1/2} + D_1^T P_4^{1/2}\right) + \frac{\Gamma(7/2)}{\Gamma(3)} C_2^T P_2^{1/2} + \\ & \frac{\Gamma(7/2)}{\Gamma(3)} D_2^T P_4^{1/2} - \left(C_3^T P_2^{1/2} + D_3^T P_4^{1/2}\right) \\ C_2^T = & \left(C_{02}^T P_1^{1/2} + D_{02}^T P_3^{1/2}\right) - \left(C_2^T P_1^{1/2} + D_2^T P_3^{1/2}\right) - \left(C_3^T P_1^{1/2} + D_3^T P_3^{1/2}\right) \\ D_2^T = & \left(C_{02}^T P_2^{1/2} + D_{02}^T P_4^{1/2}\right) - \left(C_2^T P_2^{1/2} + D_2^T P_4^{1/2}\right) - \left(C_3^T P_2^{1/2} + D_3^T P_4^{1/2}\right) \\ C_3^T = & -C_{03}^T + 2C_1^T + C_2^T \\ D_3^T = & -D_{03}^T + 2D_1^T + D_2^T \end{array}\right\} \tag{58}$$

Upon solving the above system of linear algebraic equations, the approximate solution of $x_1(t)$, $x_2(t)$ and $x_3(t)$ are obtained. The maximal absolute errors produced by TFs for various values of h are calculated and tabulated in table 2. The time taken by TFs for computing approximate solutions in each run is recorded and shown in table 2. As the number of subintervals is increasing, accuracy is increasing, hence, the approximate solutions are converging to the exact solutions.

Table 1. Absolute error using TFs and BPFs

t	$\lvert J^\alpha f(t) - J_{TF}^\alpha f(t)\rvert$	$\lvert J^\alpha f(t) - J_{BPF}^\alpha f(t)\rvert$
0	0	0.0336417669602
0.2	0	0.0615115192345
0.4	0	0.0796544626999
0.6	0	0.0943263867874
0.8	0	0.106992253349
1	0	0.118304589557

Table 2. Absolute error using TFs for example 1

h	Maximal absolute error			CPU time (seconds)
	$x_1(t)$	$x_2(t)$	$x_3(t)$	
0.1	5.7540940e-04	7.42963900e-04	0.00167301	0.4212
0.02	40528065e-05	4.13615672e-05	6.63586143e-05	1.2480
0.01	6.05908342e-06	1.11465507e-05	1.70625007e-05	3.4632
0.002	6.66389858e-07	7.44211153e-07	1.76015764e-06	127.296

Example 2

Consider the nonlinear fractional order differential-algebraic equation

$$\begin{cases} D_*^\alpha x(t) - x(t) + z(t)x(t) = 1, \\ D_*^\alpha z(t) - y(t) + x^2(t) + z(t) = 0, \\ y(t) - x^2(t) = 0, t \in [0,1],\ 0 < \alpha \le 1 \end{cases} \tag{59}$$

TABLE 3. Approximate solution of $x(t)$ for $\alpha = 1$

t	FDTM	ADM	VIM	HAM	TF	Exact
0	1	1	1	1	1	1
0.1	1.1051709	1.1051709	1.1051709	1.1051709	1.1051709	1.1051709
0.2	1.2214028	1.2214027	1.2214027	1.2214027	1.2214028	1.2214027
0.3	1.3498588	1.3498588	1.3498588	1.3498588	1.3498589	1.3498588
0.4	1.4918247	1.4918246	1.4918246	1.4918246	1.4918249	1.4918246
0.5	1.6487213	1.6487212	1.6487212	1.6487212	1.6487215	1.6487212
0.6	1.8221188	1.8221188	1.8221188	1.8221188	1.8221191	1.8221188
0.7	2.0137527	2.0137527	2.0137527	2.0137527	2.0137531	2.0137527
0.8	2.2255409	2.2255409	2.2255409	2.2255409	2.2255415	2.2255409
0.9	2.4596031	2.4596031	2.4596031	2.4596031	2.4596038	2.4596031
1	2.7182818	2.7182818	2.7182818	2.7182818	2.7182827	2.7182818

subject to initial conditions $x(0) = y(0) = z(0) = 1$.
For the special case of $\alpha = 1$, the given problem has analytical solution; $x(t) = e^t, y(t) = e^{2t}, z(t) = e^{-t}$.

Solution:

The given problem can be simplified as

$$\left.\begin{aligned} D_*^\alpha x(t) &= 1 + x(t) - z(t)\,x(t) \\ D_*^\alpha z(t) &= -z(t) \end{aligned}\right\} \tag{60}$$

Multiplying with fractional integrator J^α on both sides of above equation gives

$$\left.\begin{aligned} &x(t) = x(0) + J^\alpha(1) + J^\alpha(x(t)) - J^\alpha(z(t)\,x(t)) \\ &z(t) = z(0) - J^\alpha(z(t)) \end{aligned}\right\} \tag{61}$$

In the TF domain, the approximations for $x(0)$, $x(t)$ and $z(t)$ are attained as follows.

$$\left.\begin{aligned} 1 &\cong C_0^T T1_m(t) + D_0^T T2_m(t) \\ x(t) &\cong C_1^T T1_m(t) + D_1^T T2_m(t) \\ z(t) &\cong C_2^T T1_m(t) + D_2^T T2_m(t) \end{aligned}\right\} \tag{62}$$

From theorem 4.7 and equations 41, 61 and 62, we get the following system of nonlinear algebraic equations.

$$\left.\begin{aligned} C_1^T &= C_0^T + \left(C_0^T P_1^\alpha + D_0^T P_3^\alpha\right) + \left(C_1^T P_1^\alpha + D_1^T P_3^\alpha\right) - \left(C_1^T .* C_2^T\right) P_1^\alpha - \\ &\qquad \left(D_1^T .* D_2^T\right) P_3^\alpha \\ D_1^T &= D_0^T + \left(C_0^T P_2^\alpha + D_0^T P_4^\alpha\right) + \left(C_1^T P_2^\alpha + D_1^T P_4^\alpha\right) - \left(C_1^T .* C_2^T\right) P_2^\alpha - \\ &\qquad \left(D_1^T .* D_2^T\right) P_4^\alpha \\ C_2^T &= C_0^T - \left(C_2^T P_1^\alpha + D_2^T P_3^\alpha\right) \\ D_2^T &= D_0^T - \left(C_2^T P_2^\alpha + D_2^T P_4^\alpha\right) \end{aligned}\right\} \tag{63}$$

The above system of nonlinear algebraic equations are solved using step size of 0.002 and the approximate solutions of $x(t)$ and $z(t)$ for different values of α are given in tables 3 to 8. The TF solutions are in good agreement with the solutions obtained by FDTM [13], ADM [12], VIM [12] and HAM [13].

Example 3

Consider the following nonlinear fractional order differential-algebraic equation

$$\begin{cases} D_*^\alpha y(t) = y(t) - z(t)\,w(t) + \sin t + t\cos t, \\ D_*^\alpha z(t) = tw(t) + y^2(t) + \sec^2 t - t^2\left(\cos t + \sin^2 t\right), \\ \qquad 0 = y(t) - w(t) + t(\cos t - \sin t), t \in [0,1], 0 < \alpha \leq 1 \end{cases} \tag{64}$$

with initial conditions $y(0) = z(0) = w(0) = 0$.
In case of $\alpha = 1$, we have analytical solution; $y(t) = t\sin t, z(t) = \tan t, w(t) = t\cos t$.

Solution:

Simplifying equation 64,

$$\begin{cases} D_*^\alpha y(t) = y(t) - z(t)\,y(t) - z(t)\,f_1(t) + f_2(t) \\ D_*^\alpha z(t) = ty(t) + y^2(t) + f_3(t) \end{cases} \tag{65}$$

where $f_1(t) = t\cos t - t\sin t$, $f_2(t) = \sin t + t\cos t$ and $f_3(t) = \sec^2 t - t^2 \sin t - t^2 \sin^2 t$.
Rewriting equation 65,

$$\left.\begin{aligned} y(t) &= J^\alpha(y(t)) - J^\alpha(z(t)\,y(t)) - J^\alpha(z(t)\,f_1(t)) + J^\alpha(f_2(t)) \\ z(t) &= J^\alpha(ty(t)) + J^\alpha\left(y^2(t)\right) + J^\alpha(f_3(t)) \end{aligned}\right\} \tag{66}$$

TABLE 4. Approximate solution of $x(t)$ for $\alpha = 0.75$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	1.2187069	1.2187068	1.2187068	1.2187068	1.2187043
0.2	1.4000280	1.4000278	1.4000278	1.4000279	1.4000262
0.3	1.5841270	1.5841268	1.5841268	1.5841270	1.5841259
0.4	1.7769089	1.7769086	1.7769086	1.7769089	1.7769081
0.5	1.9813870	1.9813865	1.9813865	1.9813869	1.9813866
0.6	2.1997453	2.1997446	2.1997447	2.1997452	2.1997453
0.7	2.4338838	2.4338829	2.4338829	2.4338838	2.4338842
0.8	2.6856249	2.6856238	2.6856234	2.6856248	2.6856256
0.9	2.9568125	2.9568110	2.9568095	2.9568130	2.9568136
1	3.2493684	3.2493666	3.2493623	3.2493749	3.2493700

The following equation gives the TF estimates of $f_1(t)$, $f_2(t)$, $f_3(t)$, t, $y(t)$ and $z(t)$.

$$\left.\begin{aligned} f_1(t) \cong C_{10}^T T1_m(t) + D_{10}^T T2_m(t), f_2(t) \cong C_{20}^T T1_m(t) + D_{20}^T T2_m(t) \\ f_3(t) \cong C_{30}^T T1_m(t) + D_{30}^T T2_m(t), t \cong C_{40}^T T1_m(t) + D_{40}^T T2_m(t) \\ y(t) \cong C_1^T T1_m(t) + D_1^T T2_m(t), z(t) \cong C_2^T T1_m(t) + D_2^T T2_m(t) \end{aligned}\right\} \tag{67}$$

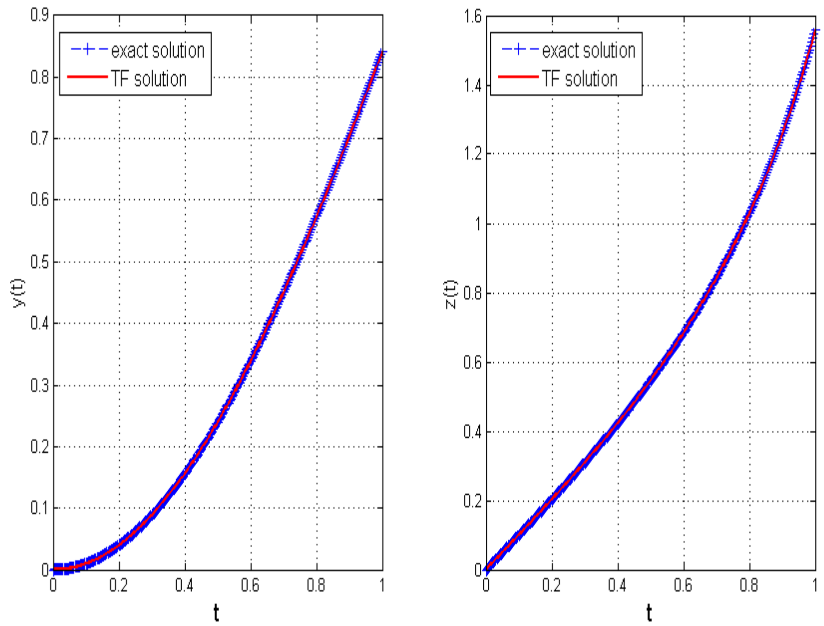

FIGURE 2. Comparison between TF solution and exact solution for $\alpha = 1$

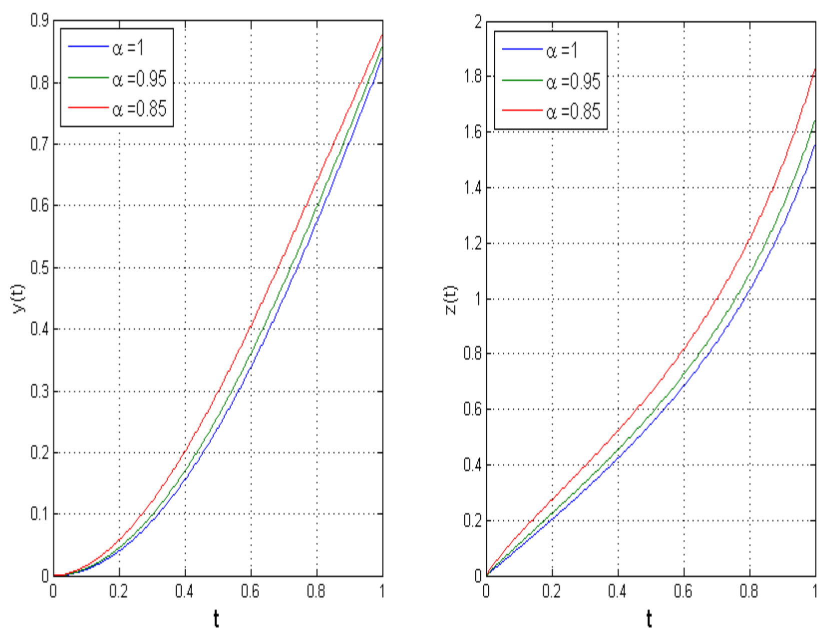

FIGURE 3. TF solutions for example 3

Table 5. Approximate solution of $x(t)$ for $\alpha = 0.5$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	1.4678849	1.4678849	1.4678849	1.4678849	1.4678635
0.2	1.7411322	1.7411321	1.7411320	1.7411321	1.7411215
0.3	1.9927891	1.9927891	1.9927879	1.9927891	1.9927834
0.4	2.2392557	2.2392557	2.2392487	2.2392557	2.2392532
0.5	2.4871415	2.4871415	2.4871134	2.4871415	2.4871415
0.6	2.7401183	2.7401183	2.7400292	2.7401183	2.7401203
0.7	3.0006469	3.0006469	3.0004093	3.0006469	3.0006508
0.8	3.2706054	3.2706053	3.2700447	3.2706053	3.2706109
0.9	3.5515666	3.5515664	3.5503632	3.5515665	3.5515737
1	3.8450346	3.8449407	3.8455419	3.8450350	3.8449494

Table 6. Approximate solution of $z(t)$ for $\alpha = 1$

t	FDTM	ADM	VIM	HAM	TF	Exact
0	1	1	1	1	1	1
0.1	0.9048374	0.9048374	0.9048374	0.9048374	0.9048373	0.9048374
0.2	0.8187308	0.8187307	0.8187307	0.8187307	0.8187306	0.8187307
0.3	0.7408182	0.7408182	0.7408182	0.7408182	0.7408181	0.7408182
0.4	0.6703201	0.6703200	0.6703200	0.6703200	0.6703199	0.6703200
0.5	0.6065307	0.6065306	0.6065306	0.6065306	0.6065305	0.6065306
0.6	0.5488116	0.5488116	0.5488116	0.5488116	0.5488115	0.5488116
0.7	0.4965853	0.4965853	0.4965853	0.4965853	0.4965851	0.4965853
0.8	0.4493290	0.4493289	0.4493289	0.4493289	0.4493288	0.4493289
0.9	0.4065697	0.4065696	0.4065696	0.4065696	0.4065695	0.4065696
1	0.3678794	0.3678794	0.3678794	0.3678794	0.3678793	0.3678794

Table 7. Approximate solution of $z(t)$ for $\alpha = 0.75$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	0.8282505	0.8282506	0.8282505	0.8282505	0.8282476
0.2	0.7325847	0.7325847	0.7325847	0.7325847	0.7325825
0.3	0.6603375	0.6603375	0.6603375	0.6603374	0.6603357
0.4	0.6021211	0.6021211	0.6021211	0.6021210	0.6021196
0.5	0.5536026	0.5536026	0.5536026	0.5536025	0.5536012
0.6	0.5122851	0.5122852	0.5122853	0.5122850	0.5122839
0.7	0.4765549	0.4765551	0.4765557	0.4765549	0.4765539
0.8	0.4452924	0.4452926	0.4452945	0.4452924	0.4452915
0.9	0.4176821	0.4176823	0.4452945	0.4176820	0.4176812
1	0.3931083	0.3931086	0.3931206	0.3931083	0.3931076

We now substitute equations 41 and 67 in 66 and equate the coefficients of LHTF and RHTF sets on both sides of resulting equation.

$$\left.\begin{aligned}
C_1^T &= \left(C_{20}^T P_1^\alpha + D_{20}^T P_3^\alpha\right) + \left(C_1^T P_1^\alpha + D_1^T P_3^\alpha\right) - \left(C_2^T .* C_1^T\right) P_1^\alpha - \\
&\quad \left(D_2^T .* D_1^T\right) P_3^\alpha - \left(\left(C_2^T .* C_{10}^T\right) P_1^\alpha + \left(D_2^T .* D_{10}^T\right) P_3^\alpha\right) \\
D_1^T &= \left(C_{20}^T P_2^\alpha + D_{20}^T P_4^\alpha\right) + \left(C_1^T P_2^\alpha + D_1^T P_4^\alpha\right) - \left(C_2^T .* C_1^T\right) P_2^\alpha - \\
&\quad \left(D_2^T .* D_1^T\right) P_4^\alpha - \left(\left(C_2^T .* C_{10}^T\right) P_2^\alpha + \left(D_2^T .* D_{10}^T\right) P_4^\alpha\right) \\
C_2^T &= \left(C_{30}^T P_1^\alpha + D_{30}^T P_3^\alpha\right) + \left(\left(C_{40}^T .* C_1^T\right) P_1^\alpha + \left(D_{40}^T .* D_1^T\right) P_3^\alpha\right) + \\
&\quad \left(\left(C_1^T .* C_1^T\right) P_1^\alpha + \left(D_1^T .* D_1^T\right) P_3^\alpha\right) \\
D_2^T &= \left(C_{30}^T P_2^\alpha + D_{30}^T P_4^\alpha\right) + \left(\left(C_{40}^T .* C_1^T\right) P_2^\alpha + \left(D_{40}^T .* D_1^T\right) P_4^\alpha\right) + \\
&\quad \left(\left(C_1^T .* C_1^T\right) P_2^\alpha + \left(D_1^T .* D_1^T\right) P_4^\alpha\right)
\end{aligned}\right\} \tag{68}$$

TABLE 8. Approximate solution of $z(t)$ for $\alpha = 0.5$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	0.7235784	0.7235784	0.7235784	0.7235784	0.7235556
0.2	0.6437883	0.6437882	0.6437886	0.6437882	0.6437750
0.3	0.5920184	0.5920184	0.5920222	0.5920184	0.5920090
0.4	0.5536063	0.5536062	0.5536241	0.5536062	0.5535990
0.5	0.5231566	0.5231565	0.5232163	0.5231565	0.5231507
0.6	0.4980246	0.4980245	0.4981839	0.4980245	0.4980196
0.7	0.4767027	0.4767027	0.4770678	0.4767027	0.4766985
0.8	0.4582460	0.4582460	0.4589938	0.4582460	0.4582423
0.9	0.4420214	0.4420214	0.4424279	0.4420214	0.4420181
1	0.4275836	0.4275835	0.4270570	0.4275835	0.4275806

TABLE 9. Approximate solution of $x(t)$ for $\alpha = 1$

t	FDTM	ADM	VIM	HAM	TF	Exact
0	1	1	1	1	1	1
0.1	0.9148208	0.9148207	0.9148207	0.9148208	0.9148207	0.9148208
0.2	0.8584646	0.8584646	0.8584646	0.8584646	0.8584645	0.8584646
0.3	0.8294743	0.8294742	0.8294742	0.8294743	0.8294741	0.8294743
0.4	0.8260874	0.8260873	0.8260873	0.8260874	0.8260872	0.8260874
0.5	0.8462434	0.8462434	0.8462434	0.8462434	0.8462431	0.8462434
0.6	0.8875971	0.8875971	0.8875971	0.8875971	0.8875968	0.8875971
0.7	0.9475377	0.9475376	0.9475376	0.9475377	0.9475373	0.9475377
0.8	1.0232138	1.0232138	1.0232138	1.0232138	1.0232134	1.0232138
0.9	1.1115639	1.1115638	1.1115638	1.1115639	1.1115633	1.1115639
1	1.2093504	1.2093504	1.2093504	1.2093505	1.2093498	1.2093504

TABLE 10. Approximate solution of $x(t)$ for $\alpha = 0.75$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	0.8492996	0.8492996	0.8492996	0.8492995	0.8492966
0.2	0.8016697	0.8016697	0.8016697	0.8016696	0.8016674
0.3	0.7978999	0.7978998	0.7978998	0.7979000	0.7978979
0.4	0.8250871	0.8250871	0.8250871	0.8250873	0.8250855
0.5	0.8760144	0.8760143	0.8760143	0.8760146	0.8760129
0.6	0.9454582	0.9454581	0.9454581	0.9454582	0.9454568
0.7	1.0290757	1.0290756	1.0290756	1.0290755	1.0290744
0.8	1.1229595	1.1229594	1.1229594	1.1229592	1.1229582
0.9	1.2234368	1.2234365	1.2234365	1.2234363	1.2234354
1	1.3269767	1.3269759	1.32697591	1.3269757	1.3269747

The approximate solution of $y(t)$ and $z(t)$ for $\alpha = 1$ are obtained by solving 68 with step size of 0.003 and compared with analtyical solutions in figure 2. Figure 3 shows TF solutions for different values of α.

TABLE 11. Approximate solution of $x(t)$ for $\alpha = 0.5$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	0.7642925	0.7642924	0.7642924	0.7642925	0.7642694
0.2	0.7545096	0.7545096	0.7545096	0.7545097	0.7544962
0.3	0.7903162	0.7903161	0.7903161	0.7903162	0.7903066
0.4	0.8524950	0.8524950	0.8524950	0.8524950	0.8524876
0.5	0.9323247	0.9323246	0.9323246	0.9323247	0.9323186
0.6	1.0242052	1.0242051	1.0242051	1.0242052	1.0242000
0.7	1.1237906	1.1237905	1.1237905	1.1237906	1.1237861
0.8	1.2273291	1.2273291	1.2273291	1.2273291	1.2273251
0.9	1.3313915	1.3313915	1.3313915	1.3313916	1.3313879
1	1.4327552	1.4327552	1.4327552	1.4327552	1.4327519

TABLE 12. Approximate solution of $x(t)$ for $\alpha = 1$

t	FDTM	ADM	VIM	HAM	TF	Exact
0	1	1	1	1	1	1
0.1	0.9048374	0.90483742	0.90483742	0.9048374	0.9048373	0.9048374
0.2	0.8187308	0.81873076	0.81873076	0.8187308	0.8187307	0.8187308
0.3	0.7408182	0.74081822	0.74081822	0.7408182	0.7408181	0.7408182
0.4	0.6703201	0.67032005	0.67032005	0.6703201	0.6703199	0.6703201
0.5	0.6065307	0.60653066	0.60653066	0.6065307	0.6065305	0.6065307
0.6	0.5488116	0.54881164	0.54881164	0.5488116	0.5488115	0.5488116
0.7	0.4965853	0.49658531	0.49658531	0.4965853	0.4965852	0.4965853
0.8	0.4493290	0.44932897	0.44932896	0.449329	0.4493288	0.449329
0.9	0.4065697	0.40656966	0.40656966	0.4065697	0.4065695	0.4065697
1	0.3678795	0.36787944	0.36787944	0.3678794	0.3678793	0.3678794

Example 4

Consider the following linear fractional order differential-algebraic equation

$$\begin{cases} D_*^\alpha x(t) - ty'(t) + x(t) - (t+1)\,y(t) = 0, \\ y(t) - \sin t = 0, \qquad t \in [0,1]\,, 0 < \alpha \le 1 \end{cases} \tag{69}$$

with initial conditions $x(0) = 1$, $y(0) = 0$.
For $\alpha = 1$, the given problem has exact solution; $x(t) = e^{-t} + t\sin t$ and $y(t) = \sin t$.

Solution:

Equation 69 can be reduced to the following form.

$$D_*^\alpha x(t) = t\cos t + (1+t)\sin t - x(t) \tag{70}$$

Peforming fractional integration on both sides of above equation,

$$x(t) = x(0) + J^\alpha(f_1(t)) - J^\alpha(x(t)) \tag{71}$$

where $f_1(t) = t\cos t + (1+t)\sin t$.
Approximating the functions; $x(0)$, $f_1(t)$ and $x(t)$ in the TF domain,

$$\left.\begin{array}{r} x(0) \cong C_{10}^T T1_m(t) + D_{10}^T T2_m(t) \\ f_1(t) \cong C_{20}^T T1_m(t) + D_{20}^T T2_m(t) \\ x(t) \cong C_1^T T1_m(t) + D_1^T T2_m(t) \end{array}\right\} \tag{72}$$

From equaions 41, 71 and 72, we get the following expressions for C_1^T and D_1^T.

$$\left.\begin{array}{l} C_1^T = C_{10} + (C_{20}P_1^\alpha + D_{20}P_3^\alpha) - (C_1P_1^\alpha + D_1P_3^\alpha) \\ D_1^T = D_{10} + (C_{20}P_2^\alpha + D_{20}P_4^\alpha) - (C_1P_2^\alpha + D_1P_4^\alpha) \end{array}\right\} \tag{73}$$

As shown in tables 9 to 11, the TF solutions are in good agreement with the solutions obatined by FDTM [13], ADM [12], VIM [12] and HAM [13].

Table 13. Approximate solution of $x(t)$ for $\alpha = 0.75$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	0.8373931	0.8373929	0.837393	0.8373931	0.8373884
0.2	0.7494391	0.7494390	0.7494389	0.7494391	0.7494360
0.3	0.6816129	0.6816128	0.6816129	0.6816129	0.6816107
0.4	0.6250322	0.6250320	0.6250321	0.6250322	0.6250306
0.5	0.5760122	0.5760123	0.5760121	0.5760122	0.5760109
0.6	0.5326238	0.5326236	0.5326237	0.5326238	0.5326228
0.7	0.4937128	0.4937126	0.4937126	0.4937128	0.4937119
0.8	0.4585198	0.4585196	0.4585196	0.4585197	0.4585190
0.9	0.4265076	0.4265077	0.4265076	0.4265076	0.4265070
1	0.3972738	0.3972732	0.3972735	0.3972736	0.3972731

Table 14. Approximate solution of $x(t)$ for $\alpha = 0.5$

t	FDTM	ADM	VIM	HAM	TF
0	1	1	1	1	1
0.1	0.760891	0.7608910	0.7608910	0.760891	0.7608617
0.2	0.6909262	0.6909261	0.6909260	0.6909262	0.6909113
0.3	0.6396502	0.6396501	0.6396501	0.6396505	0.6396406
0.4	0.5970877	0.5970876	0.5970877	0.5970878	0.5970808
0.5	0.5599926	0.5599927	0.5599928	0.5599926	0.5599872
0.6	0.5268894	0.5268890	0.5268894	0.5268894	0.5268850
0.7	0.496964	0.4969651	0.4969653	0.4969640	0.4969604
0.8	0.4697024	0.4697065	0.4697067	0.4697022	0.4696991
0.9	0.4447444	0.4447740	0.4447576	0.4447448	0.4447420
1	0.4218206	0.4219688	0.4218510	0.4218207	0.4218184

Example 5

The linear fractional order differential-algebraic equation

$$\begin{cases} x(t) + y(t) = e^{-t} + \sin t, \quad t \in [0,1], 0 < \alpha \le 1 \\ D_*^\alpha x(t) + x(t) - y(t) + \sin t = 0, \end{cases} \tag{74}$$

Table 15. Computational times for Examples 2-5

Example	h	CPU time (seconds)
2	0.002	117.516
3	0.003	114.130
4	0.002	15.802
5	0.002	13.182

subject to initial conditions $x(0) = 1$, $y(0) = 0$.
For the special case of $\alpha = 1$, we have analytical solution; $x(t) = e^{-t}$ and $y(t) = \sin t$.

Solution:
Equation 74 can be written as

$$x(t) = x(0) + J^{\alpha}\left(e^{-t}\right) - J^{\alpha}(2x(t)) \tag{75}$$

In the TF domain,

$$\left.\begin{array}{r} x(0) \cong C_{10}^T T1_m(t) + D_{10}^T T2_m(t) \\ e^{-t}(t) \cong C_{20}^T T1_m(t) + D_{20}^T T2_m(t) \\ x(t) \cong C_1^T T1_m(t) + D_1^T T2_m(t) \end{array}\right\} \tag{76}$$

From equaions 41, 75 and 76, we get the following expressions for C_1^T and D_1^T.

$$\left.\begin{array}{l} C_1^T = C_{10} + (C_{20}P_1^{\alpha} + D_{20}P_3^{\alpha}) - 2 * (C_1 P_1^{\alpha} + D_1 P_3^{\alpha}) \\ D_1^T = D_{10} + (C_{20}P_2^{\alpha} + D_{20}P_4^{\alpha}) - 2 * (C_1 P_2^{\alpha} + D_1 P_4^{\alpha}) \end{array}\right\} \tag{77}$$

The approximate numerical solution of $x(t)$ for different values of α are obtained by solving equation 77 and compared with the solutions by FDTM [13], ADM [12], VIM [12] and HAM [13] in tables 12 to 14.

8. Conclusions

Present work proposes the application of orthogonal triangular function for solving differential algebraic equations (DAEs); the fractional DAEs in specific. The proposed generalized TF operational matrices are successfully implemented on set of test problems consisting of linear and non-linear fractional order differential-algebraic equations. The implementation of the TF estimate for Riemann-Liouville fractional order integral appears to be complicated and time consuming. However the solutions in examples 1 to 5 (tables 2 and 15) ultimately utilizes algebraic equation solvers and do not require much CPU time to yield approximate numerical solutions with good accuracy. The proposed generalized TF operational matrices can be deployed for the analysis of real process dynamics; including non-linear one consisting of fractional DAEs.

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