Uniform Numerical Method For a Class of Parameterized Singularly Perturbed Problems

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Outline

1. Introduction
   - Singular Perturbation Problem (SPP)
   - Shishkin mesh and Adaptive grid

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- Differential equations where the highest order derivative is multiplied by an arbitrarily small parameter ε known as the singular perturbation parameter.
- Solutions of these problems possess boundary layers which are thin regions in the neighborhood of the boundary of the domain, where the gradients of the solutions steepen as $\varepsilon \longrightarrow 0$.

Motivation

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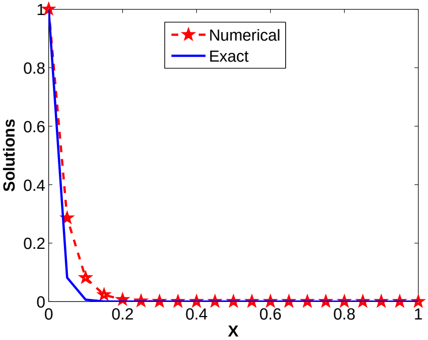
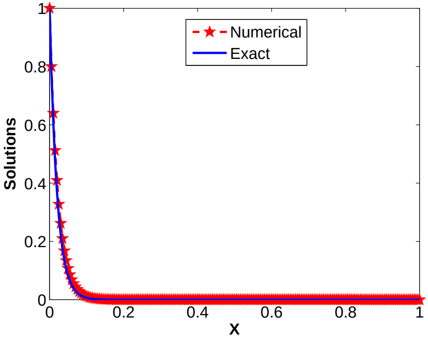


(a) $N = 20$.



(b) $N = 200$.

Figure: *Numerical solution with exact solution for* $\varepsilon = 1e-2$.

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- This is unacceptable due to the vast computational cost.
- This drawback motivates to develop the concept of ε-uniform numerical methods.
- A numerical method is ε-**uniformly convergent**, if

$$\sup_{0<\varepsilon\leq 1} \|u - U^N\|_{\Omega^N} \leq C\, N^{-p}, \quad p > 0,$$

where C is independent of mesh points, mesh size and the parameter ε.

u – Exact solution, U^N – Numerical approximation.

N – No. of mesh elements, p – Rate of convergence.

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- Shishkin mesh

- Adaptive grid

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3. Numerical Experiments
4. Conclusion
5. References

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- Hence, the Shishkin mesh $\Omega_\sigma^N = \{x_i\}_{i=0}^N$, where $x_0 = 0, x_N = 1$ and the mesh width $h_i := x_i - x_{i-1}$ satisfy $h_i = h$ for $i = 1, \cdots N/2$ and $h_i = H$ for $i = N/2 + 1, \cdots N$.

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- The piecewise-uniform mesh is entirely determined by the two chosen parameters N and σ.

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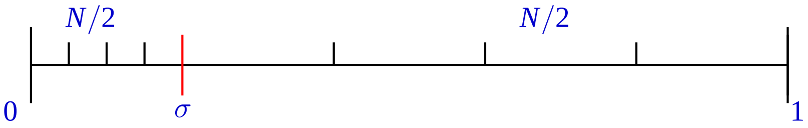

Figure: Shishkin mesh with N=8 for left layer.

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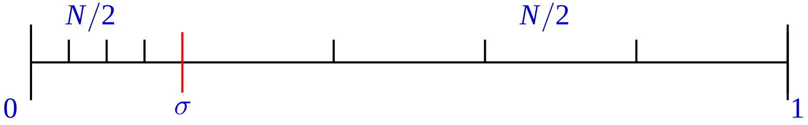

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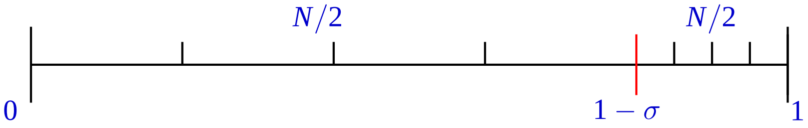

Figure: Shishkin mesh with N=8 for right layer.

Adaptive grid

Adaptive grid

A grid Ω^N is said to be equidistributing, if

$$\int_{x_{j-1}}^{x_j} M(u(s),s)\,ds = \int_{x_j}^{x_{j+1}} M(u(s),s)\,ds, \quad j = 1,\dots,N-1, \quad (1)$$

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Equivalently, (1) can be expressed as

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In practice, the monitor function is often based on a simple function of the derivatives of the solution.

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In other words, we can construct the mesh from (1) as the solution of the following nonlinear system of equations:

$$\begin{cases} (x_{j+1} - x_j)^2 + (U_{j+1}^N - U_j^N)^2 = (x_j - x_{j-1})^2 + (U_j^N - U_{j-1}^N)^2, \\ \qquad\qquad j = 1, \ldots, N-1, \\ x_0 = 0, \quad x_N = 1. \end{cases} \tag{4}$$

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The discrete problem and (4) are solved simultaneously to obtain the solution U_j^N and the grids x_j.

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$$l_i^{(k)} = h_i^{(k)} \sqrt{1 + (D^- u_i^{(k)})^2} = \sqrt{(u_i^{(k)} - u_{i-1}^{(k)})^2 + (h_i^{(k)})^2}$$

be the arc-length between the points $(x_{i-1}^{(k)}, u_{i-1}^{(k)})$ and $(x_i^{(k)}, u_i^{(k)})$ in the piecewise continuous solution $u^{(k)}$. Now the total length is $L^{(k)} := \sum_{i=1}^{N} l_i^{(k)}$.

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4. Test mesh-Choose a constant $C_0 > 1$ to be user-chosen constant. Stopping criteria is if
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holds true, then STOP. Otherwise, continue to step-5.

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5. New mesh-Choose points $\{0 = x_0^{(k+1)} < x_1^{(k+1)} < x_2^{(k+1)} < \dots x_N^{(k+1)} = 1\}$ such that for each i, the distance from $(x_{i-1}^{(k+1)}, u_{i-1}^{(k+1)})$ and $(x_i^{(k+1)}, u_i^{(k+1)})$, measured along the polygonal solution curve $u^{(k)}(x)$, equals $L^{(k)}/N$. Return to step-2.

Movement of the mesh towards left

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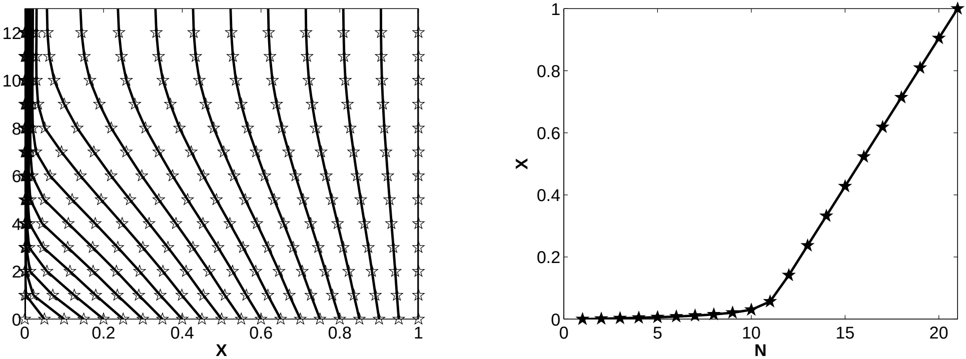
$$\begin{cases} -\varepsilon u_\varepsilon''(x) - u_\varepsilon'(x) = 0, & x \in (0,1), \\ u_\varepsilon(0) = 1, \quad u_\varepsilon(1) = 0, \end{cases}$$



(a) Mesh movement toward left. *(b) Final computed mesh.*

Figure: *for* $\varepsilon = 10^{-2}$ *and* $N = 20$.

Model problem

Consider the following singularly perturbed Parameterized BVP:

$$\begin{cases} Lu(x) \equiv \varepsilon u'(x) + f(x,u,\lambda) = 0, \quad x \in \Omega = (0,1), \\ u(0) = s_0, \quad u(1) = s_1, \end{cases} \tag{5}$$

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- The functions $f(x,u,\lambda)$ is sufficiently smooth such that

$$\begin{cases} f(x,u,\lambda) \in C^3([0,1 \times R^2]), \\ 0 < \alpha \leq \dfrac{\partial f}{\partial u} \leq \alpha^* < \infty \qquad (x,u,\lambda) \in [0,1] \times R^2, \\ 0 < m \leq \left|\dfrac{\partial f}{\partial \lambda}\right| \leq M < \infty \qquad (x,u,\lambda) \in [0,1] \times R^2. \end{cases} \tag{6}$$

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- s_0, s_1 are given constants.
- The solution $u(x)$ exhibits a boundary layer of width $\mathcal{O}(\varepsilon)$ at $x = 0$.

A brief background

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- Whether the adaptive grid approach can be applied to the BVP (5)?
- Whether we can get more efficient and accurate $\varepsilon-$ uniform method using the adaptive grid for the BVP (5) ?

Discrete problem

The upwind finite difference scheme for (5) takes the form

$$\begin{cases} L^N U_j \equiv -\varepsilon D^- U_j + f(x_j, U_j, \lambda^n) = 0\,, \quad 1 \leq j \leq N-1, \\ U_0 = s_0, \quad U_N = s_1\,. \end{cases} \tag{7}$$

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Lemma

The solution $\{u(x), \lambda\}$ *of (5) satisfies the following inequalities:*

$$|\lambda| \leq C, \quad |u^k(x)| \leq C\{1 + \varepsilon^{-k} \exp\left(-\frac{\alpha x}{\varepsilon}\right), \; x \in \overline{\Omega}, \quad k = 0, 1, 2, 3$$

Main Result

We solve the nonlinear problem (7) using the following iteration technique:

$$\lambda^n = \lambda^{n-1} - \frac{(s_1 - u_{N-1}^{n-1})\rho_N^{-1} + f(1, s_1, \lambda^{n-1})}{\partial f/\partial\lambda(1, s_1, \lambda^{n-1})},$$

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Theorem

Let $\{u(x),\ \lambda\}$ and $\{U_j^N,\ \lambda^N\}$ be the exact solution and discrete solution on grids defined above respectively. Then, there exists a constant C independent of N and ε such that

$$\max_j |u(x_j) - U_j^N| < CN^{-1}, \quad |\lambda - \lambda^N| < CN^{-1}. \tag{8}$$

Numerical Example

Example

$$\begin{cases} \varepsilon u'(x) + 2u - \exp(-u) + x^2 + \lambda + \tanh(\lambda + x) = 0, \\ \quad x \in \Omega = (0,1)\,, \\ u(0) = 1, \quad u(1) = 0. \end{cases} \tag{9}$$

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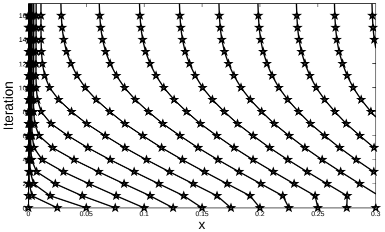
- Define

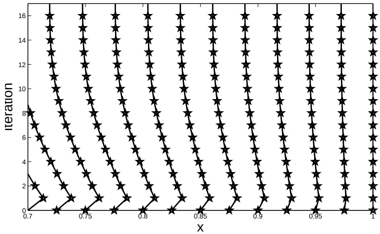
$$E^N_{\varepsilon,u} = \max_j |U^N_j - \overline{U}^{2N}_j|, \quad E^N_{\varepsilon,\lambda} = |\lambda^N - \overline{\lambda}^{2N}|$$

$$r^N_{\varepsilon,u} = \log_2\left(\frac{E^N_{\varepsilon,u}}{E^{2N}_{\varepsilon,u}}\right), \quad r^N_{\varepsilon,\lambda} = \log_2\left(\frac{E^N_{\varepsilon,\lambda}}{E^{2N}_{\varepsilon,\lambda}}\right).$$

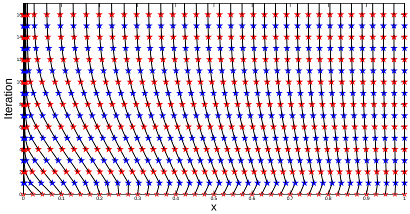
Graphs



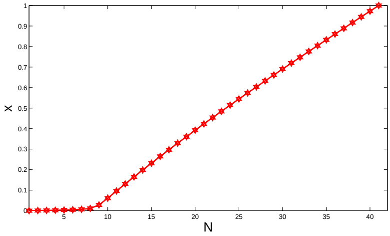
(a) *Mesh movement in the layer region.*



(b) *Mesh movement in the regular region.*



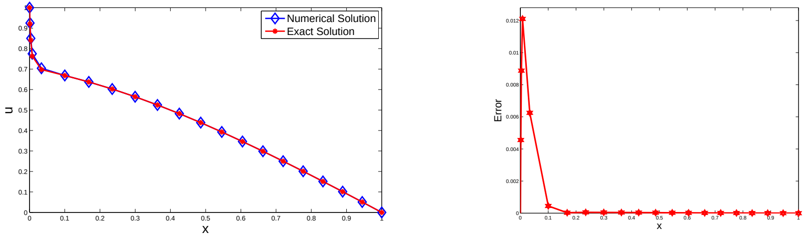
(a) Mesh movement.



(b) Final computed mesh.

Figure: *Mesh movement for* $\varepsilon = 1e - 2$, *and* $N = 40$.

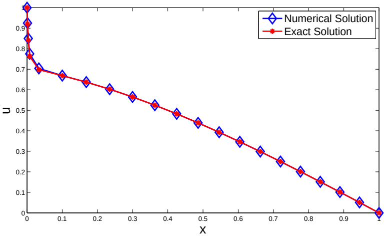
Graphs and Tables



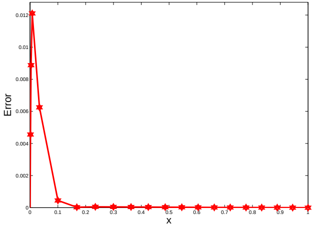
(a) *Solutions.* (b) *Error.*

Figure: *Solutions and the error for* $\varepsilon = 1e - 2$, *and* $N = 20$.

Graphs and Tables



(a) Solutions.



(b) Error.

Figure: *Solutions and the error for $\varepsilon = 1e-2$, and $N = 20$.*

Table: *Maximum point-wise errors $E^N_{\varepsilon,u}$ and the rate of convergence $r^N_{\varepsilon,u}$.*

ε	Number of intervals N						
	16	32	64	128	256	512	1024
$1e-4$	1.369e-02	8.567e-03	4.763e-03	2.537e-03	1.336e-03	6.903e-04	3.526e-04
	0.67	0.85	0.91	0.93	0.95	0.97	
$1e-8$	1.3705e-2	8.5717e-3	4.7648e-3	2.5386e-3	1.337e-03	6.910e-04	3.532e-04
	0.68	0.85	0.91	0.92	0.95	0.97	

Maximum point-wise Error with rate of convergence

Table: *Maximum point-wise errors* $E^N_{\varepsilon,\lambda}$ *and the rate of convergence* $r^N_{\varepsilon,\lambda}$.

ε	Number of intervals N						
	16	32	64	128	256	512	1024
$1e-4$	1.548e-07 1.10	7.206e-08 1.07	3.427e-08 1.21	1.478e-08 1.09	6.936e-9 1.06	3.335e-09 1.17	1.482e-09
$1e-8$	1.549e-11 1.10	7.220e-12 1.05	3.485e-12 1.02	1.713e-12 1.01	8.501e-13 1.00	4.231e-13 1.05	2.051e-13

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Table: *Comparison of numerical results* .

N		$\varepsilon = 1e-4$		$\varepsilon = 1e-6$	
		Result in Amiraliyev(2006)	**Our result**	**Result in Amiraliyev(2006)**	**Our result**
16	$E^N_{\varepsilon,\lambda}$ $r^N_{\varepsilon,\lambda}$	3.550e-06 1.01	1.548e-07 1.10	6.000e-08 1.00	1.549e-09 1.10
32	$E^N_{\varepsilon,\lambda}$ $r^N_{\varepsilon,\lambda}$	1.760e-06 1.01	7.206e-08 1.07	3.000e-08 1.00	7.220e-10 1.05

Extension to the mixed kind BVP

Example

$$\begin{cases} \varepsilon u'(x) + 2u(x) - \exp(-u(x)) + \lambda = 0, \quad x \in \Omega = (0,1), \\ u(0) + \varepsilon u'(0) = 1, \quad u(1) = 0. \end{cases} \tag{10}$$

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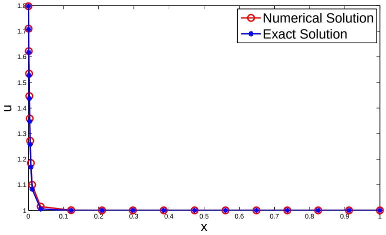
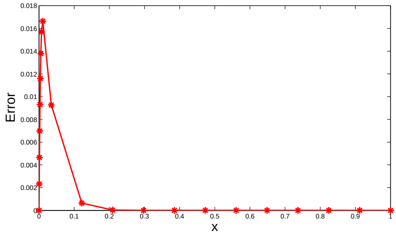


(a)Solution.



(b) Error.

Figure: *Solution and the error for $\varepsilon = 1e - 2$ and $N = 20$.*

Maximum point-wise Error and rate of convergence

Table: *Maximum point-wise errors* $E^N_{\varepsilon,u}$ *and rate of convergence* $r^N_{\varepsilon,u}$.

ε	Number of intervals N						
	16	32	64	128	256	512	1024
$1e-4$	2.059e-02	1.132e-02	5.995e-03	3.153e-03	1.622e-03	8.313e-04	4.238e-04
	0.86	0.92	0.93	0.95	0.96	0.97	
$1e-8$	2.059e-02	1.132e-02	5.997e-03	3.157e-03	1.625e-03	8.317e-04	4.243e-04
	0.86	0.92	0.93	0.95	0.96	0.9703	

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Table: *Comparison of numerical results .*

ε		$N=64$		$N=128$	
		Shishkin mesh	**Adaptive grid**	**Shishkin mesh**	**Adaptive grid**
$1e-4$	$E^N_{\varepsilon,u}$ $r^N_{\varepsilon,u}$	9.858e-03 0.7271	5.995e-03 0.9272	5.956e-03 0.7763	3.153e-03 0.9587
$1e-6$	$E^N_{\varepsilon,u}$ $r^N_{\varepsilon,u}$	9.858e-03 0.7271	5.997e-03 0.9249	5.956e-03 0.7763	3.158e-03 0.9610

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Thank You.