Adaptive Grid for Parameterized Singular Perturbation Problem

D. Shakti , J. Mohapatra

Abstract--- A quasilinear first order singularly perturbed boundary value problem depending on a parameter is considered. The problem is solved by a backward Euler fnitediference operator on an appropriate non-uniform mesh constructed adaptively by equidistributing a monitor function based on the solution. An error bound in the maximum norm is established theoretically whose error constants are shown to be independent of the singular perturbation parameter. The method is first-order convergent. Numerical experiment illustrates in practice the result of convergence proved theoretically.

Keywords--- Singular perturbation, Parameterized problem, Boundary layer, Adaptive mesh, Uniform convergence.

I. INTRODUCTION

In this article, we consider the following singularly perturbed boundary value problem (SPBVP) depending on a parameter:

$$\begin{cases} Lu(x) \equiv \varepsilon u'(x) + f(x,u,\lambda) = 0, \quad x \in \Omega = (0,1), \\ \qquad u(0) = s_0, \ \ u(1) = s_1, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is known as the singular perturbation parameter and s_0, s_1 are given constants. Here the parameter λ is known as the control parameter. The function $f(x,u,\lambda)$ is assumed to be sufficiently smooth. Throughout this paper, we assume that

$$\begin{cases} \qquad f(x,u,\lambda), \epsilon C^3([0,1]\times R^2), \\ 0 < \alpha \le \dfrac{\partial f}{\partial u} \le \alpha^* < \infty \quad (x,u,\lambda)\epsilon[0,1]\times R^2 \\ 0 < m \le \left|\dfrac{\partial f}{\partial \lambda}\right| \le M < \infty \quad (x,u,\lambda)\epsilon[0,1]\times R^2 \end{cases} \tag{1.2}$$

Under these assumption, the problem (1.1) has a unique solution. (refer [1, 7, 8]) and has a boundary layer of width $O(\varepsilon)$ near $x = 0$. For more details on singular perturbation, one can refer thebooks ([3, 5]) and the references therein. These problems arises in physical chemistry and physics, describing distributions, the oscillation of a mass attached by two spring lead to a differentialequation with a parameter. Latter, real problems have been encountered at CERN (European Organization for Nuclear Research) and the main goal is the computation of the control parameterwith high precision.

Parameterized boundary value problem have been considered for many years by many researcher. Amiraliyev et. al. [1] gave a uniform finite difference method on a Shishkin mesh [3]for (1.1) and proved that the method is first order convergent with a logarithmic factor. A hybrid difference scheme which combines a standard central difference on the fine mesh with the mid pointupwind scheme on the coarse mesh was considered by Cen [2] and Xie et. al. [10] used boundarylayer correction technique to solve the parameterized problem of the form (1.1).

In this article, we propose a combination of the classical upwind scheme on a specially constructed nonuniform mesh known as equidistributed grid [4, 6] and show that the method is uniformly convergent of order $O(N^{-1})$. To validate the results, numerical experiment on a nonlinear test problem is carried out and the results are shown in shape of Tables and Figures. Throughout this paper C denote a generic positive constant.

II. ANALYTICAL RESULTS

Lemma 2.1 *The solution {u(x), λ} of (1.1) satisfies the following the inequalities*:

$$|\lambda| \le C, \ \ |u^k(x)| \le C\left\{1 + \varepsilon^{-k} exp\left(-\frac{\alpha x}{\varepsilon}\right)\right\}, x\epsilon\bar{\Omega}\,, \quad k = 0,1\,,2,3\,.$$

Proof. One can find the proof in [1, 2]

III. DISCRETIZATION AND MESH

A. Discrete problem

We will consider difference approximations of (1.1) on a non-uniform partition $\Omega^N = \{0 = x_0 < x_1 < \cdots x_{N-1} < x_N = 1\}$, and denote $hj \ = \ x_j - x_{j-1}; \ \ j \ = \ 1, \ 2, \cdots, N$. For a meshfunction ϕ_j, we define $D^-\phi_j = \frac{\phi_j - \phi_{j-1}}{h_j}$. Now, the backward Euler scheme for(1.1) takes theform,

$$\begin{cases} L^N U_j \equiv \varepsilon D^- U_j + f\left(x_j, U_j, \lambda^n\right) = 0, \ \ 1 \le j \le N-1, \\ U_0 = s_0, \ \ U_N = s_1, \end{cases} \tag{3.1}$$

B. Generation of mesh

Since the solution u(x) of the BVP (1.1) exhibits boundary layer, one has to use layer-adapted nonuniform spatial grids, which are fine inside the boundary layer region and coarse in the outer region. To obtain such a grid, we use the idea of equidistribution of a positive monitor functiongiven depending upon the solution and its derivatives. A grid Ω^N is said to be equidistributing if

$$\int_{x_{j-1}}^{x_j} M(u(s),s)ds = \int_{x_j}^{x_{j+1}} M(u(s),s)ds, \quad j = 1(1)N-1,$$

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(3.2)
where $M(u(s),s)>0$ is called a monitor function. The adaptive grid generation is based on theidea of equidistribution. Here, we used arc length monitor function $M(u(s),s)=\sqrt{1+(u'(x))^2}$ to construct a nonuniform mesh. The grid equidistribution gives arise to a mapping $x(\xi)$, relating the computational coordinate $\xi\epsilon[0,1]$ to the physical coordinate $x\epsilon[0,1]$ defined by

$$\int_0^{x(\xi)} M(u(s),s)ds=\xi\int_0^1 M(u(s),s)ds=\xi l,$$

where$l$ is the length of u over $(0,l)$. So

$$\frac{dx}{d\xi}=\frac{l}{\sqrt{1+(u'(x))^2}} \qquad (3.3)$$

The evenly space grid for ξ will be given by, $\xi=\frac{j}{N}, j=0(1)N$.

More precisely, $x_j=\int_0^{\xi_j}\frac{l}{\sqrt{1+(u'(s))^2}}ds$ and the mesh size is given by $h_j=x_j-x_{j-1}=\int_{\xi_{j-1}}^{\xi_j}\frac{l}{\sqrt{1+(u'(s))^2}}ds$, from (3.3) we have$(1+(u'(x))^2)(dx)^2=(ld\xi)^2$ which results

$$\left[1+\left(\frac{U_{j+1}-U_j}{x_{j+1}-x_j}\right)^2\right](x_{j+1}-x_j)^2=\left(\frac{l}{N}\right)^2, \quad j=0(1)N-1. \qquad (3.4)$$

Now we can construct the mesh as the solution of following system of equations:

$$\begin{cases}\left(x_{j+1}-x_j\right)^2+\left(U_{j+1}-U_j\right)^2=\left(x_j-x_{j-1}\right)^2+\left(U_j-U_{j-1}\right)^2, \\ \qquad j=1(1)N-1 \\ x_0=0, \quad x_N=1.\end{cases} \qquad (3.5)$$

The solution of (3.1) and (3.5) produces the numerical approximation to the solution of (1.1).

We solve the nonlinear problem (3.1) using the following iteration technique:

$$\lambda^n=\lambda^{n-1}-\frac{(s_1-u_{N-1}^{n-1})\rho_N^{-1}+f(1,s_1,\lambda^{n-1})}{\partial f/_{\partial\lambda}(1,s_1,\lambda^{n-1})},$$

$$u^n=u_i^{n-1}-\frac{(u_i^{n-1}-u_{i-1}^n)\rho_i^{-1}+f(x_i,\ u_i^{n-1},\ \lambda^n)}{\partial f/_{\partial u}(x_i,\ u_i^{n-1},\ \lambda^n)+\rho_i^{-1}}$$

where $\rho_i=h_i/\varepsilon$ and λ^0, u_i^0 are the initial iterations given.

IV. MAIN RESULT

Theorem 4.1*Let* $\{u(x),\lambda\}$ *and* $\{U_j^N,\ \lambda^N\}$ *be the exact solution and discrete solution on grids defined in (3.5) respectively. Then, there exists a constant* C *independent of* N *and* ε *such that*

$$\underbrace{max}_{j}\left|u(x_j)-U_j^N\right|<CN^{-1}, \quad |\lambda-\lambda^N|<CN^{-1}. \qquad (4.1)$$

Proof. The detailed proof is given in [9].

V. NUMERICAL EXAPERIMENT AND DISCUSSION

To show the applicability and efficiency of the present method, it has been implemented to the following test problem.

Example 5.1 *Consider the following nonlinear singularly perturbed problem*

$$\begin{cases}\varepsilon u'(x)+2u-\exp(-u)+x^2+\lambda+\tanh(\lambda+x)=0, \\ x\epsilon\Omega=(0,1), \\ u(0)=1, \ u(1)=0.\end{cases} \qquad (5.1)$$

The exact solution is not available for the BVP (5.1). In order to calculate the maximum pointwise error $E_{\varepsilon,u}^N$ and the rate of convergence $r_{\varepsilon,u}^N$, we use interpolation. Define $\overline{U}_j^{2N}$ piecewise linear interpolation to U_j^N in Ω_N. We use the double mesh principle to estimate the errors and to compute the experiment rates of convergence. For any value of N, define $E_{\varepsilon,u}^N=\underbrace{max}_{j}\left|U_j^N-\overline{U}_j^{2N}\right|$ as the maximum pointwise error with respect to the variable u. Similarly for the parameter λ, the maximum pointwise error is definned as $E_{\varepsilon,\lambda}^N=\underbrace{max}_{j}\left|\lambda^N-\bar{\lambda}^{2N}\right|$. The corresponding rate of convergenceare calculated by

$$r_{\varepsilon,u}^N=log_2\left(\frac{E_{\varepsilon,u}^N}{E_{\varepsilon,u}^{2N}}\right), \quad r_{\varepsilon,\lambda}^N=log_2\left(\frac{E_{\varepsilon,\lambda}^N}{E_{\varepsilon,\lambda}^{2N}}\right).$$

Figure 1(a) displays the movement of the adaptive mesh through equidistribution principle which moves towards left and Figure 1(b) represents the corresponding computed mesh. Clearly, a nonuniform mesh is generated adaptively. Figure 2(a) represents the numerical solution and the exact solution of Example(5.1) $\varepsilon=1e-2$, and $N=20$ and Figure 2(b) represents the corresponding error. The maximumpointwise error and the rate of convergence related to solution u and related to the parameter λ ispresented in Table 1 and in Table 2 respectively. The numerical results are clear illustrations of the convergence estimate. In Table 3, we have compared our result with the numerical result obtained by [1], which indicates the sharpness and more accuracy of our proposed method.

In this paper, a novel approach for solving parameterized SPBVP has been discussed. The key to the success to this method is that without any prior knowledge an appropriate nonuniform mesh is generated. Further more, numerical experiment indicated that the present method enjoys higher accuracy than those developed by other authors.

Table 1: Maximum point-wise errors $E_{\varepsilon,u}^N$ and the corresponding rate of convergence $r_{\varepsilon,u}^N$

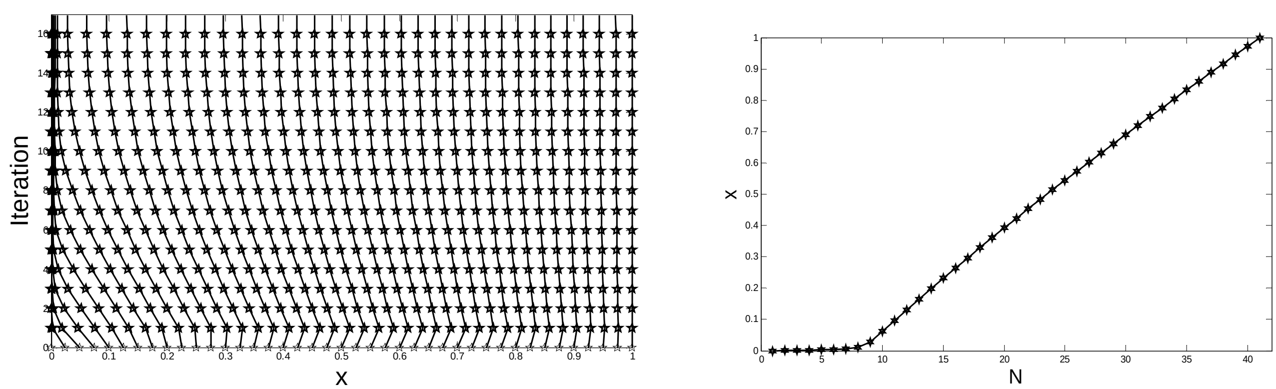
N	$\varepsilon=1e-4$	$\varepsilon=1e-8$
16	1.3692e-2 0.67	1.3705e-2 0.68
32	8.5672e-3 0.85	8.5717e-3 0.85
64	4.7630e-3 0.91	4.7648e-3 0.91
128	2.5373e-3 0.93	2.5386e-3 0.92
256	1.3363e-3 0.95	1.3374e-3 0.95
512	6.9033e-4 0.97	6.9101e-4 0.97
1024	3.5261e-4	3.5325e-4

Table 2: Maximum point-wise errors $E_{\varepsilon,\lambda}^{N}$ and the corresponding rate of convergence $r_{\varepsilon,\lambda}^{N}$

N	$\varepsilon = 1e-4$	$\varepsilon = 1e-8$
16	1.5487e-07 1.10	1.5492e-11 1.10
32	7.2069e-08 1.07	7.2205e-12 1.05
64	3.4274e-08 1.21	3.4854e-12 1.02
128	1.4782e-08 1.09	1.7130e-12 1.01
256	6.9363e-9 1.06	8.5010e-13 1.00
512	3.3358e-09 1.17	4.2316e-13 1.05
1024	1.4826e-09	2.0512e-13

Table 3: Comparison of numerical results with the result given in [1] for Example 5.1.

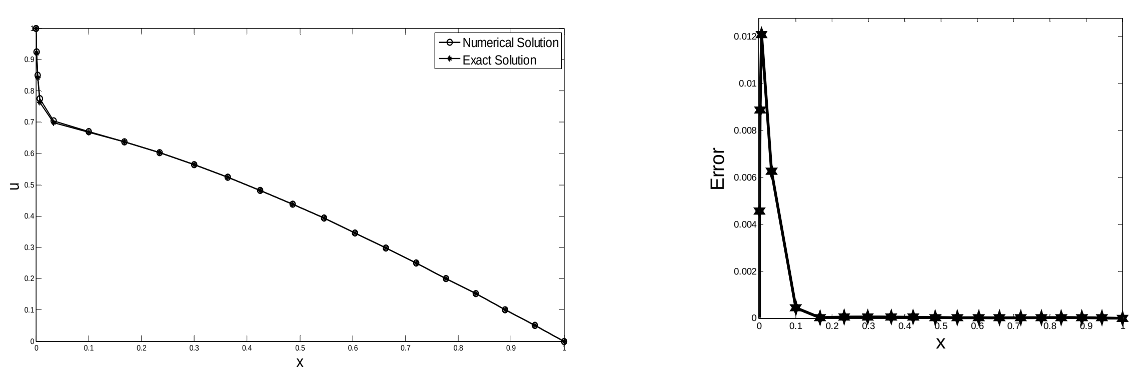
N		$\varepsilon = 1e-4$		$\varepsilon = 1e-6$	
		Result given in[1]	Our result	Result given in[1]	Our result
16	$E_{\varepsilon,\lambda}^{N}$	3.5500e-06	1.5487e-07	6.0000e-08	1.5492e-09
	$r_{\varepsilon,\lambda}^{N}$	1.01	1.10	1.00	1.10
32	$E_{\varepsilon,\lambda}^{N}$	1.7600e-06	7.2069e-08	3.0000e-08	7.2206e-10
	$r_{\varepsilon,\lambda}^{N}$	1.01	1.07	1.00	1.05



(a) Mesh movement

(b) Final computed mesh

Figure 1: Mesh movement of Example 5.1 for $\varepsilon = 1e-2$, and $N = 40$.



(a) Solutions

(b) Error

Figure 2: Numerical solution the error of Example 5.1 for $\varepsilon = 1e-2$, and $N = 20$.

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