BALANCING-LIKE SEQUENCES ASSOCIATED WITH INTEGRAL STANDARD DEVIATIONS OF CONSECUTIVE NATURAL NUMBERS

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ABSTRACT

The variance of first n natural numbers is $(n^2-1)/12$ and is a natural number if n is odd, $n>1$ and is not a multiple of 3. The values of n corresponding to integral standard deviations constitute a sequence behaving like the sequence of Lucas-balancing numbers and the corresponding standard deviations constitute a sequence having some properties identical with balancing numbers. The factorization of the standard deviation sequence results in two other interesting sequences sharing important properties with the two original sequences.

Key words: Binary recurrence, Diophantine equations, balancing sequence

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1. INTRODUCTION

The concept of balancing numbers was first given by Behera and Panda [1] in connection with the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$, wherein, they call n a balancing number and r the balancer corresponding to n. The n^{th} balancing number is denoted by B_n and the balancing numbers satisfy the binary recurrence $B_{n+1}=6B_n-B_{n-1}$ with $B_0=0$ and $B_1=1$ [1]. In [3], Panda explored many fascinating properties of balancing numbers, some of them are similar to the corresponding results on Fibonacci numbers, while some others are more exciting.

A detailed study of balancing and related number sequences is available in [5]. In a latter paper [4], as a generalization of the sequence of balancing numbers, Panda and Rout studied a class of binary recurrences defined by $x_{n+1}=Ax_n-Bx_{n-1}$ with $x_0=0$ and $x_1=1$ where A and B are any natural numbers. They proved that when $B=1$ and $A\neq 1,2$, sequences arising out of these recurrences have many important and interesting properties identical to those of balancing numbers. We therefore prefer to call this class of sequences as balancing-like sequences.

For each natural number n, $8B_n^2+1$ is a perfect square and $C_n=\sqrt{8B_n^2+1}$ is called a Lucas-balancing number [5]. We can, therefore, call $\{C_n\}$, the Lucas-balancing sequence. In a similar manner, if $\{x_n\}$ is a balancing-like sequence, kx_n^2+1 is a perfect square for some natural number k and for all n and $y_n=\sqrt{kx_n^2+1}$, we call $\{y_n\}$ a Lucas-balancing-like sequence.

Khan and Kong [2] called sequences arising out of the above class of recurrences corresponding to $B=1$ as generalized natural numbers sequences because of its similarity with natural numbers with respect to certain properties. Observe that, the sequence of balancing numbers is a member of this class corresponding to $A = 6, B = 1$. In this paper, we establish the close association of another sequence of this class to an interesting Diophantine problem of basic statistics.

The variance of the real numbers $x_1, x_2, \cdots, x_n$ is given by $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the mean of $x_1, x_2, \cdots, x_n$. Using the above formula, it can be checked that the variance of first n natural numbers (and hence the variance of any n consecutive natural numbers) is ${s_n}^2 = (n^2 - 1)/12$. It is easy to see that this variance is a natural number if and only if n is odd but not a multiple of 3. Our focus is on those values of n that correspond to integral values of the standard deviation s_n. Observe that for some N, s_N is a natural number say, $s_N = \sigma$ if $N^2 - 1 = 12\sigma^2$ which is equivalent to the Pell's equation $N^2 - 12\sigma^2 = 1$. The fundamental solution corresponds to $N_1 = 7$ and $\sigma_1 = 2$. Hence, the totality of solutions is given by

$$N_k + 2\sqrt{3}\sigma_k = \left(7 + 4\sqrt{3}\right)^k; k = 1,2, \cdots.$$

This gives

$$N_k = \frac{\left(7 + 4\sqrt{3}\right)^k + \left(7 - 4\sqrt{3}\right)^k}{2}$$

and

$$\sigma_k = \frac{\left(7 + 4\sqrt{3}\right)^k - \left(7 - 4\sqrt{3}\right)^k}{4\sqrt{3}}.$$

Because (N_k, σ_k) is a solution of the Pell's equation $N^2 - 12\sigma^2 = 1$, both N_k and σ_k are natural numbers for each k.

2. RECURRENCE RELATIONS FOR N_k AND σ_k

In the last section, we obtained the Binet forms for N_k and σ_k where σ_k is the standard deviation of N_k consecutive natural numbers. Notice that the standard deviation of a single number is zero and hence we may assume that $N_0 = 1$ and $\sigma_0 = 0$, and indeed, from the last section, we already have $x_1 = 7$ and $y_1 = 2$. Observe that $u_n = \left(7 + 4\sqrt{3}\right)^n$ and $v_n = \left(7 - 4\sqrt{3}\right)^n$ both satisfy the binary recurrences

$$u_{n+1} = 14u_n - u_{n-1},\ v_{n+1} = 14v_n - v_{n-1};$$

hence, the linear binary recurrences for both N_k and σ_k sequences are given by

$$N_{k+1} = 14N_k - N_{k-1};\ N_0 = 1, N_1 = 7$$

and

$$\sigma_{k+1} = 14\sigma_k - \sigma_{k-1};\ \sigma_0 = 0, \sigma_1 = 2.$$

The first five terms of both sequences are thus $N_1 = 7, N_2 = 97, N_3 = 1351, N_4 = 18817, N_5 = 262087$ and $\sigma_1 = 2, \sigma_2 = 28,\ \sigma_3 = 390, \sigma_4 = 5432, \sigma_5 = 75658$. Using the above binary recurrences for N_k and σ_k, some useful results can be obtained. The following theorem deals with two identities in which N_k and σ_k behave like hyperbolic functions.

2.1 Theorem. *For natural numbers k and l, $\sigma_{k+l} = \sigma_k N_l + N_k \sigma_l$ and $N_{k+l} = N_k N_l + 12\sigma_k \sigma_l$.*

Proof. Since identity

$$N_k + 2\sqrt{3}\sigma_k = \left(7 + 4\sqrt{3}\right)^k$$

holds for each natural number k, it follows that

$$\begin{aligned} N_{k+l} + 2\sqrt{3}\sigma_{k+l} &= \left(7 + 4\sqrt{3}\right)^{k+l} = \left(7 + 4\sqrt{3}\right)^k \left(7 + 4\sqrt{3}\right)^l \\ &= \left(N_k + 2\sqrt{3}\sigma_k\right)\left(N_l + 2\sqrt{3}\sigma_l\right) \\ &= (N_k N_l + 12\sigma_k\sigma_l) + 2\sqrt{3}(\sigma_k N_l + N_k\sigma_l). \end{aligned}$$

Comparing the rational and irrational parts, the desired results follow. ■

The following corollary is a direct consequence of Theorem 2.1.

2.2 Corollary. *If* $k \in \mathbb{N}$, $\sigma_{k+1} = 7\sigma_k + 2N_k$, $N_{k+1} = 7N_k + 24\sigma_k$, $\sigma_{2k} = 2\sigma_k N_k$ *and* $N_{2k} = {N_k}^2 + 12{\sigma_k}^2$.

Theorem 2.1 can be used for the derivation of another similar result. The following theorem provides formulas for σ_{k-l} and N_{k-l} in terms of N_k, N_l, σ_k and σ_l.

2.3 Theorem. *If k and l are natural numbers with $k > l$, then* $\sigma_{k-l} = \sigma_k N_l - N_k\sigma_l$ *and* $N_{k-l} = N_k N_l - 12\sigma_k\sigma_l$.

Proof. By virtue of Theorem 2.1,

$$\sigma_k = \sigma_{(k-l)+l} = \sigma_{k-l}N_l + N_{k-l}\sigma_l$$

and

$$N_k = N_{(k-l)+l} = 12\sigma_{k-l}\sigma_l + N_{k-l}N_l.$$

Solving these two equations for σ_{k-l} and N_{k-l}, we obtain

$$\sigma_{k-l} = \frac{\begin{vmatrix} \sigma_k & \sigma_l \\ N_k & N_l \end{vmatrix}}{\begin{vmatrix} N_l & \sigma_l \\ 12\sigma_l & N_l \end{vmatrix}} = \frac{\sigma_k N_l - N_k\sigma_l}{{N_l}^2 - 12{\sigma_l}^2}$$

and

$$N_{k-l} = \frac{\begin{vmatrix} N_l & \sigma_k \\ 12\sigma_l & N_k \end{vmatrix}}{\begin{vmatrix} N_l & \sigma_l \\ 12\sigma_l & N_l \end{vmatrix}} = \frac{N_k N_l - 12\sigma_k\sigma_l}{{N_l}^2 - 12{\sigma_l}^2}.$$

Since for each natural number l, (N_l, σ_l) is a solution of the Pell's equation ${N_l}^2 - 12{\sigma_l}^2 = 1$, the proof is complete. ■

The following corollary follows from Theorem 2.3 in the exactly same way Corollary 2.2 follows from Theorem 2.1.

2.4 Corollary. *For any natural number* $k > 1$, $\sigma_{k-1} = 7\sigma_k - 2N_k$ *and* $N_{k-1} = 7N_k - 24\sigma_k$.

Theorems 2.1 and 2.3 can be utilized to form interesting higher order non-linear recurrences for both $\{N_k\}$ and $\{\sigma_k\}$ sequences. The following theorem is crucial in this regard.

2.5 Theorem. *If k and l are natural numbers with $k > l$, $\sigma_{k-l} \cdot \sigma_{k+l} = {\sigma_k}^2 - {\sigma_l}^2$ and $N_{k-l} \cdot N_{k+l} + 1 = {N_k}^2 + {N_l}^2$.*

Proof. By virtue of Theorems 2.1 and 2.3,

$$\sigma_{k-l} \cdot \sigma_{k+l} = {\sigma_k}^2 {N_l}^2 - {N_k}^2 {\sigma_l}^2$$

and since for each natural number r, ${N_r}^2 = 12{\sigma_r}^2 + 1$,

$$\sigma_{k-l} \cdot \sigma_{k+l} = {\sigma_k}^2(12{\sigma_l}^2 + 1) - {\sigma_l}^2(12{\sigma_k}^2 + 1) = {\sigma_k}^2 - {\sigma_l}^2.$$

Further,

$$N_{k-l} \cdot N_{k+l} = {N_k}^2 {N_l}^2 - 144{\sigma_k}^2{\sigma_l}^2 = {N_k}^2{N_l}^2 - 144 \cdot \frac{{N_k}^2 - 1}{12} \cdot \frac{{N_l}^2 - 1}{12}$$

implies

$$N_{k-l} \cdot N_{k+l} + 1 = {N_k}^2 + {N_l}^2. \quad \blacksquare$$

The following corollary is a direct consequence of Theorem 2.5.

2.6 Corollary. *For any natural number $k > 1$, $\sigma_{k-1} \cdot \sigma_{k+1} = {\sigma_k}^2 - 4$ and $N_{k-1} \cdot N_{k+1} = {N_k}^2 + 48$.*

In view of Theorem 2.5, we also have ${\sigma_{k+1}}^2 - {\sigma_k}^2 = 2\sigma_{2k+1}$. Adding this identity for $k = 0, 1, \cdots, l-1$, we get the identity

$$2(\sigma_1 + \sigma_3 + \cdots + \sigma_{2l-1}) = {\sigma_l}^2.$$

This proves

2.7 Corollary. *Twice the sum first l odd ordered terms of the standard deviation sequence is equal to the variance of first N_l natural numbers.*

The following corollary is also a direct consequence of Theorem 2.5.

2.8 Corollary. *For each natural number k, $7(N_1 + N_3 + \cdots + N_{2k-1}) + k = 2\left(N_1^2 + N_2^2 + \cdots + N_{k-1}^2\right) + N_k^2$.*

3. BALANCING-LIKE SEQUENCES DERIVED FROM $\{N_k\}$ AND $\{\sigma_k\}$

The linear binary recurrences for the sequences $\{N_k\}$ and $\{\sigma_k\}$ along with their properties suggest that $\{\sigma_k/2\}$ it is a balancing-like sequence whereas $\{N_k\}$ is the corresponding Lucas-balancing-like sequence [see 3]. In addition, these sequences are closely related to two other sequences that can also be described by similar binary recurrences.

The following theorem deals with a sequence derived from $\{N_k\}$, the terms of which are factors of corresponding terms of the sequence $\{\sigma_k\}$.

3.1 Theorem. *For each natural number k, $(N_k + 1)/2$ is a perfect square. Further, $M_k = \sqrt{(N_k + 1)/2}$ divides σ_k.*

Proof. By virtue of Theorem 2.1 and the Pell's equation $N^2 - 12\sigma^2 = 1$

$$\frac{N_{2k}+1}{2} = \frac{{N_k}^2 + 12{\sigma_k}^2 + 1}{2} = {N_k}^2$$

implying that $M_{2k} = N_k$. Since $\sigma_{2k} = 2\sigma_k N_k$, M_{2k} divides σ_{2k} for each natural number k. Further,

$$\frac{N_{2k+1}+1}{2} = \frac{7N_{2k} + 24\sigma_{2k} + 1}{2} = \frac{7\left({N_k}^2 + 12{\sigma_k}^2\right) + 48\sigma_k N_k + 1}{2}$$

$$= 84{\sigma_k}^2 + 24\sigma_k N_k + 4 = 36{\sigma_k}^2 + 24\sigma_k N_k + 4{N_k}^2 = (6\sigma_k + 2N_k)^2$$

$$= (7\sigma_k + 2N_k - \sigma_k)^2 = (\sigma_{k+1} - \sigma_k)^2$$

from which we obtain $M_{2k+1} = \sigma_{k+1} - \sigma_k$. By virtue of Theorem 2.5, ${\sigma_{k+1}}^2 - {\sigma_k}^2 = 2\sigma_{2k+1}$ and thus

$$\sigma_{2k+1} = \frac{\sigma_{k+1} + \sigma_k}{2} \cdot (\sigma_{k+1} - \sigma_k) = \delta_k(\sigma_{k+1} - \sigma_k)$$

where $\delta_k = \frac{\sigma_{k+1}+\sigma_k}{2}$ is a natural number since σ_k is even for each k and hence M_{2k+1} divides σ_{2k+1}. ■

We have shown while proving Theorem 3.1 that $M_{2k+1} = \sigma_{k+1} - \sigma_k$. This proves

3.2 Corollary. *The sum of first l odd terms of the sequence $\{M_k\}$ is equal to the standard deviation of the first N_l natural numbers.*

By virtue of Theorem 3.1, M_k divides σ_k for each natural number k. Therefore, it is natural to study the sequence $L_k = \sigma_k / M_k$. From the proof of Theorem 3.1, it follows that $L_{2k} = 2\sigma_k$ and $L_{2k+1} = (\sigma_{k+1} + \sigma_k)/2$.

Our next objective is to show that the sequence $\{L_k\}_{k=1}^{\infty}$ is a balancing-like sequence and $\{M_k\}_{k=1}^{\infty}$ is the corresponding Lucas-balancing-like sequence. This claim is validated by the following theorem.

3.3 Theorem. *For each natural number k, ${M_k}^2 = 3{L_k}^2 + 1$. Further, the sequences $\{L_k\}_{k=1}^{\infty}$ and $\{M_k\}_{k=1}^{\infty}$ satisfy the binary recurrences $L_{k+1} = 4L_k - L_{k-1}, k \geq 1$ with $L_0 = 0$ and $L_1 = 1$ and $M_{k+1} = 4M_k - M_{k-1}, k \geq 1$ with $M_0 = 1$ and $M_1 = 2$.*

Proof. In view of the Pell's equation $N^2 - 12\sigma^2 = 1$, Corollary 2.4 and the discussion following Corollary 3.2,

$$3{L_{2k}}^2 + 1 = 3(2\sigma_k)^2 + 1 = {N_k}^2 = {M_{2k}}^2$$

and

$$3{L_{2k-1}}^2 + 1 = 3\left(\frac{\sigma_k + \sigma_{k-1}}{2}\right)^2 + 1 = 3(4\sigma_k - N_k)^2 + 1 = (6\sigma_k - 2N_k)^2 = (\sigma_k - \sigma_{k-1})^2 = {M_{2k-1}}^2.$$

To this end, using Corollary 2.2, we get

$$4M_{2k+1} - M_{2k} = 4(\sigma_{k+1} - \sigma_k) - N_k = 4(6\sigma_k + 2N_k) - N_k = N_{k+1} = M_{2k+2}$$

and

$$4M_{2k} - M_{2k-1} = 4N_k - (\sigma_{k+1} - \sigma_k) = 4N_k - (-6\sigma_k + 2N_k) = 6\sigma_k + 2N_k = \sigma_{k+1} - \sigma_k = M_{2k+1}.$$

Thus, the sequence M_k satisfies the binary recurrence $M_{k+1} = 4M_k - M_{k-1}$. Similarly, the identities

$$4L_{2k+1} - L_{2k} = 2(\sigma_{k+1} + \sigma_k) - 2\sigma_k = 2\sigma_{k+1} = L_{2k+2}$$

and

$$4L_{2k} - L_{2k-1} = 8\sigma_k - \frac{\sigma_k + \sigma_{k-1}}{2} = 8\sigma_k - (4\sigma_k - N_k) = 4\sigma_k + N_k = \frac{\sigma_{k+1} + \sigma_k}{2} = L_{2k+1}$$

confirm that the sequence L_k satisfies the binary recurrence $L_{k+1} = 4L_k - L_{k-1}$. ■

It is easy to check that the Binet forms of the sequences $\{L_k\}$ and $\{M_k\}$ are respectively

$$L_k = \frac{\left(2+\sqrt{3}\right)^k - \left(2-\sqrt{3}\right)^k}{2\sqrt{3}}$$

and

$$M_k = \frac{\left(2+\sqrt{3}\right)^k + \left(2-\sqrt{3}\right)^k}{2}$$

$k = 1,2,\cdots$. Using the Binet forms or otherwise, the interested reader is invited to verify the following identities.

(a) $L_1 + L_3 + \cdots + L_{2n-1} = {L_n}^2$,
(b) $M_1 + M_3 + \cdots + M_{2n-1} = L_{2n}/2$,
(c) $L_2 + L_4 + \cdots + L_{2k} = L_k L_{k+1}$,
(d) $M_2 + M_4 + \cdots + M_{2k} = (L_{2k+1} - 1)/2$,
(e) $L_{x+y} = L_x M_y + M_x L_y$,
(f) $M_{x+y} = M_x M_y + 3L_x L_y$.

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