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A PROOF OF CAUCHY'S THEOREM

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In this paper, we present a straightforward proof of the homology version of Cauchy's Theorem without using winding numbers.

1. INTRODUCTION

Ever since Cauchy's Theorem first appeared in 1825 (Cauchy[2]) different versions of the theorem have appeared in the literature from time to time (see Conway[3] and the references contained in there). As far as we know the homology version — the most general version of Cauchy's theorem, is usually proved by using winding numbers (see Ahlfors[1] and Conway[3]). In this paper we present a straightforward proof of the homology version of Cauchy's theorem without the use of winding numbers.

2. PRELIMINARIES

Definitions 2.1 — Let X be a topological space and $I = [0, 1]$. A continuous map

$$\alpha : I^n \to X \quad (n \geq 0)$$

is called a 'singular n-simplex' (or simply a n-simplex).

A formal linear combination

$$c = n_1 \alpha_1 + n_2 \alpha_2 + \ldots + n_k \alpha_k$$

of n-simplices in X with integer coefficients is called a n-chain in X. Let

$$i_1, i_2, i_3, i_4 : I \to I^2$$

be defined by

$$i_1(t) = (t, 0), \; i_2(t) = (t, 1),$$

$$i_3(t) = (1, t), \; i_4(t) = (0, t).$$

For a 2-simplex α in X, $\partial\alpha$, the 'boundary' of α, is defined by

$$\partial\alpha = \alpha \circ i_1 - \alpha \circ i_2 + \alpha \circ i_3 - \alpha \circ i_4$$

(which is a 1-chain in X). The boundary ∂c of a 2-chain

$$c = n_1\alpha_1 + n_2\alpha_2 + \ldots + n_k\alpha_k$$

is defined by

$$c = \sum_{i=1}^{k} n_i \, \partial\alpha_i$$

A 1-chain β in X is said to be 'homologous' to zero if $\beta = \partial c$ for some 2-chain c in X (see Spanier[4]).

3. Cauchy's Theorem

Let G be an open subset of the complex plane and let $f : G \to \mathbb{C}$ be analytic over G. For a 1-chain

$$\beta = c_1\,\beta_1 + c_2\,\beta_2 + \ldots + c_k\,\beta_k$$

in G with each β_i rectifiable, define

$$\int_\beta f = \sum_{i=1}^{k} c_i \int_{\beta_i} f.$$

The following result is well-known.

Cauchy's Theorem 3.1 (first version) — If a complex valued function f is analytic over

$$B_r(z_0) = \{z \in \mathbb{C} : d(z, z_0) = |z - z_0| < r\}$$

then

$$\int_\beta f = 0$$

for every closed rectifiable curve β in $B_r(z_0)$.

The following lemmas will be used in the sequel.

Lemma 3.2 — For an open subset G of the complex plane and a compact subset K of G, $d(K, G^c) > 0$.

Proof : Define $g : G \to \mathbb{R}$ by $g(z) = d(z, G^c)$. Since $K \cap G^c = \phi$ and G^c is closed, $g(z) > 0$ for each $z \in G$. Since K is compact, there exists a point $z \in K$, such that

$$0 < g(z) = \inf\{g(z) : z \in K\} = d(K, G^c).$$

Lemma 3.3 — Let G be an open subset of the complex plane and α a 2-simplex in G such that $\partial\alpha$ is rectifiable. If f is analytic over G then

$$\int_{\partial\alpha} f = 0.$$

PROOF : Let $2\delta = d(\alpha(I^2), G^c)$; by Lemma 3.2, $\delta > 0$. This implies that for each $z_0 \in \alpha(I^2)$, $B_\delta(z_0) \subset G$. For each $\alpha(x) \in \alpha(I^2)$ let

$$U_x = \alpha^{-1}(B_\delta(\alpha(x)));$$

clearly, $\{U_x : x \in I^2\}$ is an open cover of I^2. Since I^2 is compact, this open cover has a Lebesgue number $\varepsilon > 0$. This means that every subset of I^2 of diameter $< \varepsilon$ is contained in some U_x; hence its image under α is contained in some $B_\delta(\alpha(x))$.

Now choose a positive integer m such that $\sqrt{2}/m < \varepsilon$. Divide I^2 into m^2 squares (each having side $1/m$) so that each square is of diameter $< \varepsilon$. Let the vertices of this grating have coordinates (i, j), $0 \leq i, j \leq m$. Let $z_{i,j}$ be the image of the point (i, j) under the map α. Now define $L_{(i,j),(i',j')}$ as follows :

For $j = 0, 1, ..., m-1$

$$L_{(i,j),(i,j+1)} = \begin{cases} [z_{i,j}, z_{i,j+1}] & 0 < i < m \\ \alpha[(i,j),(i,j+1)] & i = 0, m. \end{cases}$$

For $i = 0, 1, ..., m-1$

$$L_{(i,j),(i+1,j)} = \begin{cases} [z_{i,j}, z_{i+1,j}] & 0 < j < m \\ \alpha[(i,j),(i+1,j)] & j = 0, m. \end{cases}$$

For $0 \leq i, j \leq m-1$, let

$$C_{ij} = L_{(i,j),(i+1,j)} + L_{(i+1,j),(i+1,j+1)} - L_{(i,j+1),(i+1,j+1)} - L_{(i,j),(i,j+1)}.$$

It is now easy to see that

(i) each C_{ij} is a closed rectifiable curve;

(ii) each C_{ij} is of diameter $< \varepsilon$; hence $\alpha(C_{ij})$ is contained in some $B_\delta(z_0) \subset G$ and finally

(iii) $\sum_{i,j} C_{ij} = \partial\alpha.$

By Theorem 3.1,

$$\int_{\alpha(C_{ij})} f = 0;$$

hence $$\int_{\partial\alpha} f = 0.$$

4. Conclusion

The homology version of the theorem now follows easily from Lemma 3.3.

Theorem 4.1 — Let G be an open subset of the complex plane and f an analytic function over G. If β is a 1-chain in G which is homologous to zero, that is,

$$\beta = \partial(n_1\,\alpha_1 + n_2\,\alpha_2 + \ldots + n_k\,\alpha_k),$$

with each $\partial\alpha_i$ rectifiable then

$$\int_\beta f = 0.$$

The open subsets of the complex plane for which the integral of an analytic function around a 1-chain is always zero can be characterized.

Definition 4.2 — An open subset G of the complex plane is called 'simply connected' if G is connected and every 1-chain in G is homologous to zero.

Corollary 4.3 — If G is simply connected then for every 1-chain β in G and every analytic function f over G,

$$\int_\beta f = 0.$$

References

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