Indian J. pure appl. Math., **26**(8) : 773-776, August 1995

ON GENERALIZED NONLINEAR COMPLEMENTARITY PROBLEM

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(*Received* 11 *October* 1994; *accepted* 23 *March* 1995)

A generalized version of the existence theorem for a generalized nonlinear complementarity problem is established.

1. INTRODUCTION

Let R denote the set of real numbers and R^n the n-dimensional Euclidean space with the usual norm and inner-product. Let K be a closed convex cone in R^n with $0 \in K$ and K^* be the polar cone of K. The following existence theorem on a generalized nonlinear complementarity problem is proved by Mishra and Nanda[2].

Theorem 1.1 — Let $T : K \to R^n$ be a continuous map and $g : K \to R^n$ be a continuous map such that

$$x \in K, \ (g(x), x) \le 0 \Rightarrow x = 0.$$

If $G(x) = T(x) - T(0)$ satisfies

$$(G(tx), x) \ge c(t)\,(g(x), x)$$

for some mapping $c : R^+ \to R$ with $c(t) \to +\infty$ as $t \to +\infty$, then there exists $x_0 \in R^n$ such that

$$x_0 \in K, \ T(x_0) \in K^*, \ (Tx_0, x_0) = 0. \qquad \ldots (1)$$

The purpose of this note is to obtain a generalization of Theorem 1.1.

2. MAIN RESULT

We generalize Theorem 1.1 as follows.

Theorem 2.1 — Let K be a closed convex cone in R^n with $0 \in K$. Let $T : K \to R^n$, $g : K \to R^n$ and $\theta : K \times K \to R^n$ be continuous maps such that

(i) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(ii) for each fixed $y \in K$, the function

$(Ty, \theta(-, y)) : K \to R$ is convex,

(iii) $x \in K$, $(g(x), \theta(y, x)) \leq 0 \Rightarrow x = 0$ for all $y \in K$. ... (2)

If $G(x) = T(x) - T(0)$ satisfies

$$(G(tx), x) \geq c(t)\,(g(x), x) \qquad \text{... (3)}$$

for some mapping $c : R^+ \to R$ with $c(t) \to +\infty$ as $t \to +\infty$, then there exists $x_0 \in R^n$ such that

$$x_0 \in K, \quad Tx_0 \in K^*, \quad (Tx_0, \theta(y, x_0)) = 0. \qquad \text{... (4)}$$

Remark 2.2 : If $\theta(y, x_0) = y - x_0$ and $y = 2x_0$, then (4) reduces to (1).

3. Auxiliary Results

We need the following result to prove Theorem 2.1.

Theorem 3.1 — Let $T : K \to R^n$ and $\theta : K \times K \to R^n$ be continuous maps and for each fixed $y \in K$ the function $(Ty, \theta(-, y)) : K \to R$ be convex. If (4) has no solution, then there exists a sequence $\{a_i\}$ of positive real numbers and a convergent sequence $\{u_i\} \subset K$ such that

(i) $\lim u_i = u$, $u \neq 0$, $u \in K$

(ii) $(T(a_i u_i), \theta(y, u_i)) < 0$ for all i.

The following result will be used in proving Theorem 3.1.

Theorem 3.2 — (Behera and Panda[1], Theorem 2.2) — Let K be a compact convex set in a reflexive real Banach space X, with $0 \in K$, and let X^* denote the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that

(i) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(ii) for each fixed $y \in K$, the function

$(Ty, \theta(-, y)) : K \to R$ is convex.

Then there exists $x_0 \in K$ such that

$$(Ty, \theta(y, x_0) \geq 0$$

for all $y \in K$.

4. Proofs of Theorem 3.1 and 2.1

Proof of Theorem 3.1 — The sets

$$K_a = \{y \in K : (y, z) \leq a \text{ for all } z \in K\}$$

are nonempty, compact and convex for $0 < a < \infty$. Hence it follows from Theorem 3.2 that there exists $x_a \in K_a$ with

$$(Tx_a, \theta(y, x_a)) \geq 0$$

for all $y \in K_a$. Since $0 \in K$, the Slater condition for the feasible domain K_a (cf. Sposito and David[4]) is satisfied and applying the duality theory of linear programming over convex cone domain[4] we get a scalar $s_a \in R$ such that

$$Tx_a + s_a z \in K^*, \quad x_a \in K, \qquad \ldots (5)$$

$$(Tx_a + s_a z, \ \theta(y, x_a)) = 0, \qquad \ldots (6)$$

$$\left.\begin{array}{l}(a - (z, \theta(y, x_a)))\, s_a = 0, \\ (z, \theta(y, x_a)) \leq a, \ s_a \geq 0.\end{array}\right\} \qquad \ldots (7)$$

Now if $s_a = 0$ for some a, then it is clear from (5) and (6) that x_a is a solution to (4). Therefore, we conclude that if (4) has no solution then $s_a > 0$ for all a, $0 < a < \infty$. Now by (7), $(z, \theta(y, x_a)) = a$ for all a. Let

$$u_a = \frac{\theta(y, x_a)}{a}$$

and thus $u_a \in K$ and $(z, u_a) = 1$. Since the points u_a, $0 < a < \infty$, lie in the compact set

$$C = \{x \in K : (z, x) = 1\}$$

there is a sequence $\{a_i\}$ such that $\{u_i\} = \{u_{a_i}\}$ converges to a vector $u \in K$ satisfying $(z, u) = 1$. From (5) and (6), we get

$$0 < s_a = -\frac{1}{a}(Tx_a, \ \theta(y, x_a))$$

$$Tx_a + s_a z \in K^* \text{ for all } a \in \{a_i\}.$$

Now substituting $x_a = au_a$ in the above relations, we obtain (ii). This completes the proof of Theorem 3.1.

Proof of Theorem 2.1 — Suppose that (4) has no solution. Then by Theorem 3.1, we get a sequence $\{a_i\}$ of positive real numbers and a convergent sequence $\{u_i\} \subset K$ such that

$$\lim u_i = u, u \neq 0, \quad u \in K$$

$$(T(a_i, u_i), \ \theta(y, u_i)) < 0 \text{ for all } i.$$

Now by (3) we obtain

$$
\begin{aligned}
0 &> (T(a_i u_i),\ \theta(y, u_i)) \\
&= (G(a_i u_i) + T(0),\ \theta(y, u_i)) \\
&= (G(a_i u_i),\ \theta(y, u_i)) + (T(0),\ \theta(y, u_i)) \\
&\geq c(a_i)\,(g(u_i),\ \theta(y, u_i)) + (T(0),\ \theta(y, u_i))
\end{aligned}
$$

for all i. It now follows that

$$(g(u),\ \theta(y, u)) \leq 0 \text{ for } 0 \neq u \in K.$$

Thus we get a contradiction to assumption (2). Hence we conclude that (4) has a solution and this completes the proof of Theorem 2.1.

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