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VARIATIONAL INEQUALITY PROBLEM IN HAUSDORFF TOPOLOGICAL VECTOR SPACES

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Browder[4] has proved the variational inequality problem in reflexive real Banach spaces. Isac[10] has proved the same variational inequality problem in Hausdorff topological vector spaces. The authors[2] have already obtained a generalization of Browder's result. In this paper a certain generalization of Isac's result is presented.

1. INTRODUCTION

There have been significant development in the theory of optimization techniques in the recent decades. The study of variational inequalities and complementarity problems is also a part of this development because optimization problems can often be reduced to the solution of variational inequalities and complementarity problems. Also in recent years there have been several generalizations of these problems[2, 3, 8, 9, 11, 12, 15, 17]. In this paper our aim is to use the techniques of Hanson[9] and Ben-Israel and Mond[3] for generalizing a certain variational inequality problem.

Browder has proved the variational inequality problem in reflexive real Banach spaces[4]. Isac has proved the same variational inequality problem in Hausdorff topological vector spaces[10]. The authors have already obtained a generalization of Browder's result[2]. In this paper our aim is to obtain a generalization of Isac's result in the line of the generalization of Browder's result. For obtaining the said generalization, we use a function $\theta(-, -)$ introduced by Hanson[9]. The function $\theta(-, -)$ is quite general in nature and applicable to many cases of general interests. For the introduction, existence and significance of the function $\theta(-, -)$ we refer to the papers of Hanson[9] and Ben-Israel and Mond[3].

Let X be a reflexive real Banach space and let X^* be its dual endowed with weak* topology. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let K be a closed convex set in X, with $0 \in K$. For each $r \geq 0$ we write

$$D_r = \{x \in K : \|x\| \leq r\}$$

$$D_r^0 = \{x \in K : \|x\| < r\}$$

$$S_r = \{x \in K : \|x\| = r\}.$$

The following result is proved by Browder[4] (see also Chipot[5], Mosco[14]).

Theorem 1.1 — Let T be a monotone and hemicontinuous map of a closed convex set K in X, with $0 \in K$, into X^*, and if K is not bounded, let T be coercive on K. Then there exists an $x_0 \in K$ such that

$$(Tx_0, y - x_0) \geq 0 \qquad \text{... (1)}$$

for all $y \in K$.

The inequalities of the form (1) are called variational inequalities and applicable to many cases of general interests.

The above theorem is generalized in the following way[2].

Theorem 1.2 — Let K be a closed convex set in a reflexive real Banach space X, with $0 \in K$, and X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that

(i) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(ii) for each fixed $y \in K$, the function

$$(Ty, \theta(-, y)) : K \to \mathbb{R}$$

is convex.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0 \qquad \text{... (2)}$$

for all $y \in K$, under each of the following conditions :

(a) For at least one $r > 0$, there exists $u \in D_r^0$ such that

$$(Ty, \theta(u, y)) \leq 0$$

for all $y \in S_r$.

(b) There exist a nonempty, compact and convex subset C of K and $u \in C$ such that

$$(Ty, \theta(u, y)) < 0$$

for every $y \in K - C$.

Remark 1.3 : If $\theta(y, x_0) = y - x_0$, then (2) reduces to (1).

In this paper, our aim is to show the existence of the solution x_0 belonging to a convex compact subset of K (K not being necessarily closed), in the presence of the condition (b) of Theorem 1.2. For doing this we consider a Haudorff topological vector space X.

In fact strictly speaking we obtain a generalization of the following result proved by Isac[10] (Theorem 4.3.2., p. 116) which is parallel to Theorem 1.1, proved by Browder.

Theorem 1.4 — Let K be a nonempty convex subset in a Hausdorff topological vector space X and let X^* be the dual of X. Let $T : K \to X^*$ be a mapping such that

(i) $x \mapsto (Tx, y - x)$ is upper semicontinuous on K for every $y \in K$,

(ii) there exist a nonempty, compact and convex subset $L \subset K$ and $u \in L$ such that

$$(Ty, u - y) < 0$$

for all $y \in K - L$.

Then there exists $x_0 \in L$ such that

$$(Tx_0, y - x_0) \geq 0 \qquad \text{... (3)}$$

for all $y \in K$.

Furthermore, the following result proved by Isac[10] (Proposition 6.2.2, p. 170) can also be obtained as a particular case of our result.

Theorem 1.5 — Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let X^* be the dual of X. Let $T : K \to X^*$ and $g : K \to K$ be two continuous mappings such that

$$(Tx, x - g(x)) \geq 0$$

for all $x \in X$. Then there exists $x_0 \in K$ such that

$$(Tx_0, y - g(x_0)) \geq 0 \qquad \text{... (4)}$$

for all $y \in K$.

2. Main Result

We prove the following result.

Theorem 2.1 — Let K be a nonempty convex subset in a Hausdorff topological vector space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be any two maps such that

(A) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(B) for each fixed $y \in K$, the function

$$(Ty, \theta(-, y)) : K \to \mathbb{R}$$

is convex,

(C) $x \mapsto (Tx, \theta(y, x))$ is upper semicontinuous for each $y \in K$,

(D) there exist a nonempty, compact and convex subset L of K and $u \in L$ such that

$$(Ty, \theta(u, y)) < 0$$

for every $y \in K - L$.

Then there exists $x_0 \in L$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0 \qquad \dots (5)$$

for all $y \in K$.

Remark 2.2 : If $\theta(y, x_0) = y - x_0$, then (5) reduces to (3).

We need the following definition and result which are fundamental to prove Theorem 2.1.

Definition 2.3[7] — A point-to-set map $F : K \to 2^X$ is called a KKM-map if for each finite subset

$$\{x_1, x_2, \dots, x_n\} \subset K,$$

$$\text{conv}\ (\{x_1, x_2, \dots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i)$$

where conv(A) denotes the convex hull of A.

Theorem 2.4[7] — Let K be an arbitrary nonempty set in a Hausdorff topological vector space X. Let the point-to-set map $F : K \to 2^X$ be a KKM-map such that $F(x)$ is closed for all $x \in K$ a..d is compact for at least one $x \in K$. Then

$$\bigcap_{x \in K} F(x) \neq \phi.$$

3. Proof of Theorem 2.1

Proof of Theorem 2.1 — For each $y \in K$ define a set valued map

$$E : K \to 2^X$$

by the rule

$$E(y) = \{x \in L : (Tx, \theta(y, x)) \geq 0\}.$$

By (A) we note that for each $y \in K$, $E(y)$ is nonempty (since $y \in E(y)$). By hypothesis (C) since $x \mapsto (Tx, \theta(y, x))$ is upper semicontinuous $E(y)$ is closed and consequently $E(y)$ is compact.

We claim that the family $\{E(y) : y \in K\}$ has the finite intersection property. Let $y_1, y_2, \dots, y_m$ be arbitrary elements of K and denote

$$H = \text{conv}(L \cup \{y_1, y_2, \dots, y_m\}).$$

Obviously H is a compact convex subset of K. For every $y \in K$ define another set valued map

$$F : K \to 2^X$$

by the rule

$$F(y) = \{x \in H : (Tx, \theta(y, x)) \geq 0\}.$$

Again by (A) we note that for each $y \in K$, $F(y)$ is nonempty (since $y \in F(y)$). By hypothesis (C) since $x \mapsto (Tx, \theta(y, x))$ is upper semicontinuous $F(y)$ is closed and consequently $F(y)$ is compact.

We assert that F is a KKM-map. If F is not a KKM-map, then there exist

$$\{x_1, x_2, \ldots, x_n\} \subset H$$

and

$$a_i \geq 0, \quad 1 \leq i \leq n$$

with

$$\sum_{i=1}^{n} a_i = 1$$

such that

$$\sum_{i=1}^{n} a_i x_i \notin \bigcup_{j=1}^{n} F(x_j)$$

i.e.,

$$\sum_{i=1}^{n} a_i x_i \notin F(x_j)$$

for any $j = 1, 2, \ldots, n$. Thus

$$\left(T\left(\sum_{i=1}^{n} a_i x_i \right), \theta\left(x_j, \sum_{i=1}^{n} a_i x_i \right) \right) < 0$$

for any $j = 1, 2, \ldots, n$. By the convexity of $(Ty, \theta(-, y))$ (hypothesis (B)) we get

$$\left(T\left(\sum_{i=1}^{n} a_i x_i \right), \theta\left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} a_i x_i \right) \right) < 0$$

which is a contradiction to (A). So F is a KKM-map.

Hence by Theorem 2.4

$$\bigcap_{y \in H} F(y) \neq \phi.$$

Thus there exists

$$\tilde{x}_0 \in \bigcap_{y \in H} F(y)$$

i.e., $\tilde{x}_0 \in F(y)$ for all $y \in H$. Hence $\tilde{x}_0 \in H$ and

$$(T\tilde{x}_0, \theta(y, \tilde{x}_0)) \geq 0 \quad \text{... (6)}$$

for all $y \in H$. In fact $\tilde{x}_0 \in L$. Suppose to the contrary that $\tilde{x}_0 \notin L$ (i.e., $\tilde{x}_0 \in H - L \subset K - L$). By hypothesis (D) there exists $u \in L$ such that

$$(Ty, \theta(u, y)) < 0$$

for all $y \in K - L$; putting $y = \tilde{x}_0$ in the above inequality we obtain

$$(T\tilde{x}_0, \theta(u, \tilde{x}_0)) < 0,$$

which is a contradiction to (6).

Thus $\tilde{x}_0 \in L$ and in particular $\tilde{x}_0 \in E(y_i)$ for $i = 1, 2, .., m$, i.e.,

$$\tilde{x}_0 \in \bigcap_{i=1}^{m} E(y_i).$$

Hence the family $\{E(y) : y \in K\}$ has the finite intersection property, i.e., there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. Since $x_0 \in E(y)$ for each $y \in K$ and $E(y) \subset L$, we see that $x_0 \in L$. This completes the proof of Theorem 2.1.

Remark 3.1 : If the maps $T : K \to X^*$ and $\theta : K \times K \to X$ are continuous then the hypothesis (C) in Theorem 2.1 may be dropped and in that case we have the following theorem.

Theorem 3.2 — Let K be a nonempty convex subset in a Hausdorff topological vector space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be any two continuous maps such that

(α) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(β) for each fixed $y \in K$, the function

$$(Ty, \theta(-, y)) : K \to \mathbb{R}$$

is convex,

(γ) there exist a nonempty, compact and convex subset $L \subset K$ and $u \in L$ such that

$$(Ty, \theta(u, y)) < 0$$

for every $y \in K - L$.

Then there exists $x_0 \in L$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0 \qquad \ldots (7)$$

for all $y \in K$.

PROOF : The proof is exactly similar to the proof of Theorem 2.1 except the arguments for proving the compactness of $E(y)$ and $F(y)$ as defined in the proof of Theorem 2.1; by the continuity of the maps $T : K \to X^*$ and $\theta : K \times K \to X$ it is easy to check that $E(y)$ and $F(y)$ are closed and consequently compact and hence we omit the proof.

Remark 3.3 : If $\theta(y, x_0) = y - g(x_0)$, then (7) reduces to (4).

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