# KKT Points and Non-convexity

Suvendu Ranjan Pattanaik

July 24, 2014

# Outline

# Convex Programming

$$\min_{x} f(x) \quad \text{subject to} \quad x \in K$$

where

$$K = \{x \in \mathbb{R}^n | g_i(x) \leq 0, \quad i = 1, \ldots, m\}$$

and $f$ and $g_i$ both are convex and differentiable function.

# Note

1. Every local minimum point is global minimum point.

# Note

1. Every local minimum point is global minimum point.
2. Ben-Tal and Nemirovsky [1] show that convex optimization requires only convex feasible set without precising its representation, i.e., convex inequalities.
3. Every local minimum is global minimum and derivation of this fact uses only the geometry of the feasible set not its representation.

# Note

1. Every local minimum point is global minimum point.
2. Ben-Tal and Nemirovsky [1] show that convex optimization requires only convex feasible set without precising its representation, i.e., convex inequalities.
3. Every local minimum is global minimum and derivation of this fact uses only the geometry of the feasible set not its representation.
4. With Slater's condition, the KKT optimality condition is both necessary and sufficient.

# KKT Optimality Condition & Slater's Constraint Qualification

The point $\bar{x} \in K$ is said to be KKT point of the problem (CP) if there exists scalars $\lambda_i \geq 0, i = 1, \ldots, m$ such that

(i) $0 = \nabla(f)(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla(g_i)(\bar{x})$

(ii) $\lambda_i g_i(\bar{x}) = 0, \quad \forall i = 1, 2, \ldots, m.$

Problem (CP) satisfies Slater's constraint qualification condition, i.e., there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$ for all $i = 1, \ldots, m$.

# From convexity to Non-convexity

Lasserre [4, 5] provides with Slater's condition and a mild non-degeneracy condition, KKT optimality condition is both necessary and sufficient for a convex feasible set described by inequalities which are differentiable but not necessarily convex.

# From convexity to Non-convexity

Lasserre [4, 5] provides with Slater's condition and a mild non-degeneracy condition, KKT optimality condition is both necessary and sufficient for a convex feasible set described by inequalities which are differentiable but not necessarily convex.

For every $j = 1, \ldots, m$

$$\nabla g_j(x) \neq 0$$

whenever $x \in K$ and $g_j(x) = 0$.

For instance, the set

$$K := \{x \in R^2 : 1 - x_1 x_2 = 0; x \geq 0\}$$

is convex but the function $x \to 1 - x_1 x_2$ is not convex on $\mathbb{R}^2_+$.

# Differentiable to Lipschitz Continuous

Dutta and Lalitha [3] extend to the feasible set described by inequalities which are locally Lipschitz and regular in the sense of Clarke [2].

# Differentiable to Lipschitz Continuous

Dutta and Lalitha [3] extend to the feasible set described by inequalities which are locally Lipschitz and regular in the sense of Clarke [2].
By Dutta and Lalitha[3]For every $j = 1, \ldots, m$

$$0 \notin \partial^C g_j(x)$$

whenever $x \in K$ and $g_j(x) = 0$.

# Lipschitz Continuous to contineous

We extend to the feasible set described by inequalities which are continuous only.

# Sharper Results

**Example**

function $g_1 : \mathbb{R}^2 \to \mathbb{R}$ is defined as $g_1(x_1, x_2) = -|x_1| + x_2$ and function $g_2 : \mathbb{R}^2 \to \mathbb{R}$ is defined as $g_2(x_1, x_2) = x_1$. Clearly, K is convex but Clarke generalized subdifferential of $g_1$ at origin is $\{(\xi, 1) | \xi \in \mathbb{R})\}$.

# Regular and Fréchet upper subdifferential

Regular Subdifferential

$$\hat{\partial} f(\bar{x}) = \left\{ v | f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\|x - \bar{x}\|) \quad \text{for all} \quad x \in \mathbb{R}^n \right\}, \quad (1)$$

Fréchet upper subdifferential

$$\hat{\partial}^+ f(\bar{x}) = \left\{ v \in \mathbb{R}^n | f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle \leq o(\| x - \bar{x} \|) \right\}.$$

$$\hat{\partial}^+ f(\bar{x}) = -\hat{\partial}(-f)(\bar{x})$$

# Framing Problems

$$\max f(x), \quad \text{subject to} \quad x \in K;$$

$$K = \left\{ x \in \mathbb{R}^n : g_i(x) \geq 0, \quad i = 1, \ldots, m \right\}, \tag{2}$$

where each $g_i : \mathbb{R}^n \to \mathbb{R}$ is continuous function. Further, we assume problem (CP) satisfies Slater's constraint qualification condition, i.e., there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) > 0$ for all $i = 1, \ldots, m$.

# KKT point

The point $\bar{x} \in K$ is said to be KKT point of the problem (CP) if there exists scalars $\lambda_i \geq 0, i = 1, \ldots, m$ such that

(i) $0 \in \hat{\partial}^+(-f)(\bar{x}) + \sum_{i=1}^m \lambda_i \hat{\partial}^+(-g_i)(\bar{x})$

(ii) $\lambda_i g_i(\bar{x}) = 0, \quad \forall i = 1, 2, \ldots, m.$

We say that the assumption (A) holds if

$$0 \notin -\hat{\partial}^+ g_i(x), \quad \text{whenever} \quad x \in K \quad \text{and} \quad g_i(x) = 0.$$

**Theorem**

*Let us consider the problem (CP). Assume that the Slater constraint qualification holds and the assumption (A) is satisfied. Then $\bar{x} \in K$ is a global minimizer of $f$ over $K$ if and only if it is a KKT point.*

**Lemma**

*Let the set $K$ be given as in the problem (CP). Assume that Slater constraint qualification and the assumption (A) hold. Then $K$ is convex if and only if for every $i = 1, 2 \ldots m$,*

$$-\hat{\partial}^{+} g_i(\bar{x}) \subset N_K(\bar{x}) \quad \text{for all} \quad \bar{x} \quad \text{with} \quad g_i(\bar{x}) = 0. \tag{3}$$

1. We have extended Locally Lipschitz function to Continuous function.
2. Also We do not require any regularization condition
3. By using Fréchet upper subdifferential we get sharper results.

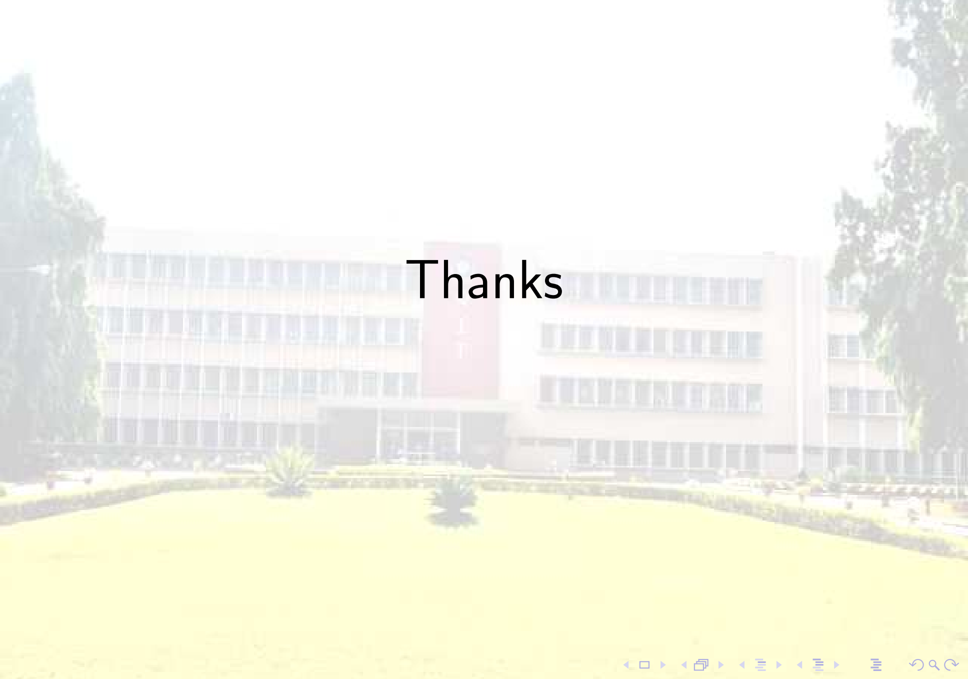
# Extra condition

1. By Lasserre[4] For every $j = 1, \ldots, m$
$$\nabla g_j(x) \neq 0$$
whenever $x \in K$ and $g_j(x) = 0$.
2. By Dutta and Lalitha[3]For every $j = 1, \ldots, m$
$$0 \notin \partial^C g_j(x)$$
whenever $x \in K$ and $g_j(x) = 0$.

$$0 \notin -\hat{\partial}^+ g_i(x), \quad \text{whenever} \quad x \in K \quad \text{and} \quad g_i(x) = 0.$$

# Bibliography


- Ben-Tal, A., Nemirovsky, A.: Lecture on Morden Convex Optimization: Analysis, Algorithims and Engineering Applications. SIAM, Philadelphia (2001).
- Clarke, F. H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983).
- Dutta, J.,Lalitha, C. S.: Optimality conditions in convex optimization revisited, Optim. Lett.,7, 221-229 (2013).
- Lasserre, J.B.: On representation of the feasible set in convex optimization, Optim. Lett. 4, 1-5 (2010).
- Lasserre, J. B.: On convex optimization without convex representation, Optim. Lett., 5, 549-556, (2011).
- Mordukhovich, B. S.: variational Analysis and Generalized Differention I: Basic Theory, Springer, New York (2006).



# Thanks