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A GENERALIZATION OF MINTY'S LEMMA

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A certain generalization of Minty's Lemma is established.

Key Words : Monotone and Hemicontinuous Operator; Reflexive Real Banach Space; Minty's Lemma; Convex Set.

1. INTRODUCTION

In the recent decades, there have been a great deal of development in the theory of optimization techniques. The study of variational inequalities and complementarity problems is also a part of this development because optimization problems can often be reduced to the solution of variational inequalities and complementarity problems.

One of the important results of variational inequality theory is Minty's Lemma, which has interesting applications in the study of obstacles problems, confined plasmas, filtration phenomena, free-boundary problems, plasticity and viscoplasticity phenomena, elasticity problems and stochastic optimal control problems. In this paper, following the traditional proof the Minty's lemma, we obtain a certain generalization of Minty's lemma. Furthermore we apply this result to obtain the solution of a certain variational-like inequality.

2. PRELIMINARIES

Let X be a reflexive real Banach space and let X^* be its dual endowed with weak* topology. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x). Let K be a nonempty closed convex set in X and $T : K \to X^*$ an operator.

An operator $T : K \to X^*$ is said to be monotone[1] if

$$(Tx - Ty, x - y) \geq 0$$

for all $x, y \in K$.

A map $S : K \to Y$, where Y is a Banach space, is said to be hemicontinuous[1] at $x \in X$ if any sequence $\{x_n\}$ converging to x along a line in K, the sequence $\{Sx_n\}$ converges weakly to Sx in Y.

A map $S : K \to Y$ is said to be continuous on finite dimensional subspaces[2] if for every finite dimensional subspace M of X the mapping $S : K \cap M \to Y$ is weakly continuous i.e., $\{x_n\}$ converges to x in $K \cap M$ implies that $\{Sx_n\}$ converges weakly to Sx in Y.

When $Y = X^*$, the convergence of $\{Sx_n\}$ in the definitions of hemicontinuity and continuity on finite dimensional subspaces refer to convergence in the weak* topology of X^*.

A set valued map $F : K \to 2^X$ is called a KKM-map[3] if for each finite subset

$$\{x_1, x_2, \ldots, x_n\} \subset K$$

$$\text{conv}\ (\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i)$$

where conv(A) denotes the convex hull of A.

Theorem 2.1 (Fan[4]) — *Let K be an arbitrary nonempty set in a Hausdorff topological vector space X. Let the set valued map $F : K \to 2^X$ be a KKM-map such that $F(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$. Then*

$$\bigcap_{x \in K} F(x) \neq \phi.$$

3. Minty's Lemma

Minty's lemma is stated as the following.

Theorem 3.1 (*Chipot*[2], *p.* 6; *also see Browder*[5]) — *Let K be a nonempty closed convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ be a monotone operator which is continuous on finite dimensional subspaces (or at least hemicontinuous). Then the following are equivalent :*

(*a*) $x_0 \in K, (Tx_0, y - x_0) \geq 0$ for all $y \in K$.

(*b*) $x_0 \in K, (Ty, y - x_0) \geq 0$ for all $y \in K$.

Our aim is to obtain a generalization of Theorem 3.1.

4. The Result

We generalize Minty's lemma as follows :

Theorem 4.1 — *Let X be a nonempty closed convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two maps such that*

(i) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(ii) the map

$$x \mapsto (Tx, \theta(y, x))$$

of K into $\mathbb{R}$ is continuous on finite dimensional subspaces (or at least hemicontinuous), for each $y \in K$,

(iii) the map

$$y \mapsto (Tx, \theta(y, x))$$

of K into $\mathbb{R}$ is convex for each $x \in K$,

(iv) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ for all $x, y \in K$.

Then the following are equivalent :

(A) $x_0 \in K, (Tx_0, \theta(y, x_0)) \geq 0$ for al! $y \in K$.

(B) $x_0 \in K, (Ty, \theta(x_0, y)) \leq 0$ for all $y \in K$.

PROOF : Suppose that $x_0 \in K$ and

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. By (iv)

$$(Ty, \theta(x_0, y)) \leq -(Tx_0, \theta(y, x_0)) \leq 0 \text{ for all } y \in K.$$

Conversely suppose that $x_0 \in K$ and

$$(Ty, \theta(x_0, y)) \leq 0$$

for all $y \in K$. For any arbitrary $x \in K$, let

$$y_t = tx + (1 - t)\, x_0,\ 0 < t < 1.$$

Since K is convex, $y_t \in K$. Putting $y = y_t$ in (B) we get

$$(Ty_t, \theta(x_0, y_t)) \leq 0. \qquad \ldots (1)$$

Also by (i) we have

$$(Ty_t, \theta(y_t, y_t)) \geq 0$$

and by the convexity of the map

$$y \mapsto (Tx, \theta(y, x))$$

we have

$$\begin{aligned} 0 &\leq (Ty_t, \theta(y_t, y_t)) \\ &= (Ty_t, \theta(tx + (1 - t)\, x_0, y_t)) \leq t(Ty_t, \theta(x, y_t)) \\ &\qquad + (1 - t)\,(Ty_t, \theta(x_0, y_t)). \end{aligned}$$

Thus

$$(Ty_t, \theta(x, y_t))$$
$$\geq -\frac{1-t}{t}(Ty_t, \theta(x_0, y_t))$$
$$\geq 0, \text{ by } (1).$$

So

$$(Ty_t, \theta(x, y_t)) \geq 0.$$

Since the map

$$x \mapsto (Tx, \theta(y, x))$$

of K into $\mathbb{R}$ is continuous on finite dimensional subspaces (or at least hemicontinuous), taking limit as $t \mapsto 0$ in the above inequality, we get

$$(Tx_0, \theta(x, x_0)) \geq 0.$$

Since $x \in K$ is arbitrary, the required inequality follows. This completes the proof of Theorem 4.1.

Note 4.2 : If we define the map $\theta : K \times K \to X$ by the rule

$$\theta(x, y) = x - y$$

then Theorem 3.1 follows as a direct consequence of Theorem 4.1.

5. An Application

In this section we apply Theorem 4.1 to prove the existence of a solution of variational-like inequality.

Theorem 5.1. — *Let K be a nonempty closed, convex and bounded subset of a reflexive real Banach space X and X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two maps such that*

(i) $(Ty, \theta(y, y)) = 0$ for all $y \in K$,

(ii) the map

$$x \mapsto (Tx, \theta(y, x))$$

of K into $\mathbb{R}$ is continuous on finite dimensional subspaces (or at least hemicontinuous), for each $y \in K$,

(iii) the map

$y \mapsto (Tx, \theta(y, x))$ of K into $\mathbb{R}$ is convex for each $x \in K$,

(iv) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ for all $x, y \in K$.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

PROOF : Define a set valued map

$$E : K \to 2^X$$

by the rule

$$E(y) \;=\; \{x \in K : (Ty, \theta(x, y)) \leq 0\}.$$

By (i) $E(y)$ is nonempty for each $y \in K$. It is easy to see that for each $y \in K$, $E(y)$ is a closed and convex subset of K. Hence $E(y)$ is weakly compact for each $y \in K$.

Define another set valued map

$$F : K \to 2^X$$

by the rule

$$F(y) \;=\; \{x \in K : (Tx\,, \theta(y, x)) \geq 0\}.$$

By (iv) it is clear that for each $y \in K$, $F(y) \subset E(y)$. We assert that F is a KKM-map. If F is not a KKM-map, then there exist

$$\{x_1, x_2, \ldots, x_n\} \;\subset K$$

and

$$a_i \geq 0,\; 1 \leq i \leq n$$

with

$$\sum_{i=1}^{n} \; a_i = 1$$

such that

$$\sum_{i=1}^{n} \; a_i\, x_i \notin \; \bigcup_{j=1}^{n} \; F(x_j)$$

i.e.,

$$\sum_{i=1}^{n} \; a_i\, x_i \notin \; F(x_j)$$

for any j = 1, 2, ..., n. Thus

$$\left(T \sum_{i=1}^{n} a_i x_i, \theta\left(x_j, \sum_{i=1}^{n} a_i x_i\right)\right) < 0$$

for each $j = 1, 2, ..., n$. Now by the convexity of the map

$$y \mapsto (Tx, \theta(y, x))$$

of K into $\mathbb{R}$, it follows that

$$\left(T \sum_{i=1}^{n} a_i x_i, \theta\left(\sum_{j=1}^{n} a_j x_j, \sum_{i=1}^{n} a_i x_i\right)\right)$$

$$\leq \sum_{i=1}^{n} a_i \left(T \sum_{i=1}^{n} a_i x_i, \theta\left(x_j, \sum_{i=1}^{n} a_i x_i\right)\right) < 0,$$

which contradicts (i). Thus F is a KKM-map and hence E is also a KKM-map. Now by Theorem 2.1

$$\bigcap_{y \in K} E(y) \neq \phi$$

i.e., there exists $x_0 \in K$ such that

$$(Ty, \theta(x_0, y)) \leq 0$$

for all $y \in K$. By Theorem 4.1, this is equivalent to saying that there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. This completes the proof of Theorem 5.1.

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