Indian J. pure appl. Math., 30(9) : 911-923, September 1999.

ON NONLINEAR VARIATIONAL-TYPE INEQUALITY PROBLEM

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(*Received* 15 *September* 1997; *after Revision* 5 *February* 1999; *Accepted* 22 *February* 1999)

In this paper, using a function introduced by Hanson we study some results on variational type inequality problem. Our results develop the results of Siddiqui, Ansari and Kazmi (*Indian J. pure appl. Math.* **25** (1994), 969-73).

Key Words : Variational Inequality Problem; Banach Space; KKM Map; Jensen's Inequality

1. INTRODUCTION

In the recent decades, the study of variational inequality and complementarity problem has become a very effective and powerful technique for studying a wide range of problems arising in both mathematical and engineering sciences. Precisely speaking there has been significant development in the theory of optimization techniques; the study of variational inequality and complementarity problem is also a part of this development because optimization problems can often be reduced to the solutions of variational inequality and complementarity problems. Also in the recent years there have been several generalizations of these problems[5,6,8,10,11,12].

Siddiqui, Ansari and Kazmil[12] have developed the existence theory of a nonlinear variational inequality problem (see Problem 1[12], p. 969) in the setting of reflexive real Banach spaces and Hausdorff topological vector spaces separately. This problem (Problem 1[12], p.969) is an interesting as well as explicit in nature and generalizes many results in the literature. Carbone[2] has extended the result of Siddiqui, Ansari and Kazmi[12] by using a geometric lemma of Ky Fan[4]. In this paper, our aim is to study some results of Siddiqui, Ansari and Kazmil[12] using a function $\theta(-,-)$ introduced by Hanson[6]. The function $\theta(-,-)$ is quite general in nature and is applicable to many cases of general interests. For the introduction, existence and significance of the function $\theta(-,-)$, we refer to the papers of Hanson[6] and Ben-Israel and Mond[1]. However, in our discussion, for simplicity, we drop the functional j and the nonlinear mapping A of the Problem 1 developed by Siddiqui, Ansari and Kasmil[12] (p. 969). We need the following definition and results which will be frequently used in the sequel.

Let X be a reflexive real Banach space and let X^* be its dual endowed with weak* topology. Let the value of $f \in X^*$ at $x \in X$ be denoted by (f, x).

Definition 1.1[7] — Let K be a convex subset of X. A map $f : K \to \mathbf{R}$ is said to be convex if for any $y_1, y_2 \in K$, and $0 \le t \le 1$, we have

$$f((1-t)y_1 + ty_2) \le (1-t)f(y_1) + tf(y_2).$$

This inequality is called Jensen's inequality after the Danish mathematician who first introduced it.

Definition 1.2[4] — Let K be any subset of X. A point-to-set map $F : K \to 2^X$ is called a KKM-map if for each finite subset

$$\{x_1, x_2, \ldots, x_n\} \subset K,$$

$$\text{Conv}\ (\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i)$$

where Conv(A) denotes the convex hull of A.

Theorem 1.3[4] — *Let K be an arbitrary nonempty set in a Hausdorff topological vector space X. Let the point-to-set map $F : K \to 2^X$ be a KKM-map such that $F(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$. Then*

$$\bigcap_{x \in K} F(x) \ne \phi.$$

Theorem 1.4[4] — *Let K be a nonempty compact convex set in a Hausdorff topological vector space X. Let L be a subset of $K \times K$ having the following properties :*

(a) For each $x \in K$, $(x, x) \in L$.

(b) For each fixed $y \in K$, the set

$$L(y) = \{x \in K : (x, y) \in L\}$$

is closed in K.

(c) For each $x \in K$, the set

$$M(x) = \{y \in K : (x, y) \notin L\}$$

is convex.

Then there exists $x_0 \in K$ such that

$$\{x_0\} \times K \subset L.$$

2. The Results

We begin by considering K to be a compact and convex subset of X and prove the following results on the nonlinear variational-type inequality problem :

Theorem 2.1 — *Let K be a nonempty compact convex subset of a reflexive real Banach space X and let X^* be the dueal of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that* *(a) for each* $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$ *, the map*

$$(Tx, \theta(-, x)) : K \to \mathbf{R},$$

is convex (it means that, in Definition 1.1, if we define $f : K \to \mathbf{R}$ *by*

$$f(y) = (Tx, \theta(y, x))$$

and use the convexity of f *then we have*

$$(Tx, \theta((1 - t)y_1 + ty_2, x))$$

$$= f((1 - t)y_1 + ty_2)$$

$$\leq (1 - t) f(y_1) + tf(y_2)$$

$$= (1 - t)\ (Tx, \theta(y_1, x)) + t(Tx, \theta(y_2, x))$$

for all $y_1, y_2 \in K$, *and* $0 \leq t \leq 1$*).*

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in \mathbf{K}$.

PROOF : Let

$$L = \{(x, y) \in K \times K : (Tx, \theta(y, x)) \geq 0\}.$$

By (a) $L \neq \phi$ since $(x, x) \in L$ for each $x \in K$. For each $y \in K$, (following the notation of Theorem 1.4) consider the set

$$L(y) = \{x \in K : (x, y) \in L\}$$

$$= \{x \in K : (Tx, \theta(y, x)) \geq 0\}.$$

We assert that $L(y)$ is closed for each $y \in K$. Let $x_n \in L(y)$ and $\lim_{n \to \infty} x_n = x$. Since $x_n \in L(y)$ we have

$$(Tx_n, \theta(y, x_n)) \geq 0$$

for all $y \in K$; and since T and θ are continuous taking limit $n \to \infty$ in the above inequality we obtain that

$$(Tx, \theta(y, x)) \geq 0$$

for all $y \in K$, showing $L(y)$ to be closed for each $y \in K$.

We assert that for each $x \in K$, the set

$$M(x) = \{y \in K : (x, y) \notin L\}$$

$$= \{y \in K : (Tx, \theta(y, x)) < 0\}.$$

is convex. Let $y_1, y_2 \in M(x), 0 < t < 1$ and

$$w = (1-t)y_1 + ty_2.$$

By (b),

$$\begin{aligned}
&(Tx, \theta(w, x)) \\
&= (Tx, \theta((1-t)y_1 + ty_2, x)) \\
&\leq (1-t)\,(Tx, \theta(y_1, x)) + t(Tx, \theta(y_2, x)) \\
&< 0;
\end{aligned}$$

showing that $w \in M(x)$.

Thus all the conditions of Theorem 1.4 are satisfied and hence there exists $x_0 \in K$ such that $\{x_0\} \times K \subset L$, i.e.,

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. This completes the proof of Theorem 2.1.

We can generalize Theorem 2.1 by considering K to be a locally compact and convex subset of X.

***Theorem* 2.2** — *Let K be a nonempty locally compact and convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$, the map

$$(Tx, \theta(-x)) : K \to \boldsymbol{R},$$

is convex and

(c) for at least one $r > 0$, there exists $u \in D_r^0$ such that

$$(Ty, \theta(u, y)) \leq 0$$

for all $y \in S_r$, where

$$D_r = \{x \in K : \|x\| \leq r\},$$

$$D_r^0 = \{x \in K : \|x\| < r\}$$

and

$$S_r = \{x \in K : \|x\| = r\}.$$

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

PROOF : Since D_r is a compact and convex subset of K, by Theorem 2.1, there exists $x_0 \in D_r$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0 \qquad \text{... (1)}$$

for all $y \in D_r$. We have two cases :

Case I — $\| x_0 \| < r$: For each $y \in K$, choose $t \in (0, 1)$ such that

$$y_t = ty + (1-t)x_0 \in D_r.$$

We guarantee that such a choice of t is possible. If $y \in D_r$, then it is clear that we can always choose $t \in (0, 1)$ such that

$$y_t = ty + (1-t)x_0 \in D_r$$

as D_r is convex. If $y \notin D_r$, then we have $\| y \| > r$. Also we have $r - \| x_0 \| > 0$ and $\| y - x_0 \| \neq 0$. Let

$$t = \frac{r - \| x_0 \|}{\| y - x_0 \|} .$$

Clearly $t > 0$. We assert that $t < 1$. Suppose to the contrary that $t \geq 1$. Then we have

$$r - \| x_0 \| \geq \| y - x_0 \| \geq \| y \| - \| x_0 \|$$

and hence $r \geq \| y \|$, which is a contradiction. Hence, $t \in (0, 1)$. With the above choice of t it is clear that $\| y_t \| \leq r$, that is,

$$y_t = ty + (1-t)x_0 \in D_r.$$

Putting $y = y_t$ in (1) we have

$$0 \leq (Tx_0, \theta(y_t, x_0))$$

$$= (Tx_0, \theta(ty + (1-t)x_0, x_0))$$

$$\leq t(Tx_0, \theta(t, x_0)) + (1-t)\,(Tx_0, \theta(x_0, x_0))$$

$$= t(Tx_0, \theta(y, x_0)),$$

showing

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

Case II — $\| x_0 \| = r$: Putting $y = u$ in (1) and using the hypothesis we obtain

$$(Tx_0, \theta(u, x_0)) = 0.$$

For any $z \in K$, choose $t > 0$ sufficiently small such that

$$u_t = tz + (1-t)u \in D_r.$$

Putting $y = u_t$ in (1), we obtain

$$0 \leq (Tx_0, \theta(u_t, x_0))$$

$$= (Tx_0, \theta(tz + (1-t)u, x_0))$$

$$\leq t(Tx_0, \theta(z, x_0)) + (1-t)(Tx_0, \theta(u, x_0))$$

$$= t(Tx_0, \theta(z, x_0)),$$

showing

$$(Tx_0, \theta(z, x_0)) \geq 0$$

for all $z \in K$. This completes the proof of Theorem 2.2.

Theorem 2.3 — *Let K be a locally compact and convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$, the map

$$(Tx, \theta(-, x)) : K \to \mathbf{R},$$

is convex and

(c) there exists a nonempty, compact and convex subset C of K and $u \in C$ such that

$$(Ty, \theta(u, y)) < 0$$

for all $y \in K - C$.

Then there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

PROOF : Since C is compact, there exists $r > 0$ such that $y \in D_r^0$ for all $y \in C$. By Theorem 2.1, there exists $x_0 \in D_r$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0 \quad \text{... (2)}$$

for all $y \in C \subset D_r$. In fact $x_0 \in C$. If $x_0 \notin C$. (i.e., $x_0 \in K - C$) then by hypothesis there exists $u \in C$ such that

$$(Tx_0, \theta(u, x_0)) < 0,$$

which contradicts (2) when $y = u$. Thus $\| x_0 \| < r$. Now for any given $y \in K$, choose $0 < t < 1$ sufficiently small such that

$$z = ty + (1 - t)\, x_0 \in D_r \subset K.$$

By (a), (b) and (2) we have

$$0 \leq (Tx_0, \theta(z, x_0))$$

$$= (Tx_0, \theta(ty + (1 - t)x_0, x_0))$$

$$\leq t(Tx_0, \theta(y, x_0)) + (1 - t)\,(Tx_0, \theta(x_0, x_0))$$

$$= t(Tx_0, \theta(y, x_0)),$$

for all $y \in K$. Since $t > 0$ we see that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. This completes the proof of Theorem 2.3.

As have been done in Lemma 1 by Siddiqui, Ansari and Kazmi[12] we prove the following lemma, a generalization due to Minty[10] (also see Chipot[3]) which will be needed in the proof of Theorem 2.5. Once more we point out that for simplicity we have dropped the nonlinear mapping T and the functional j of Lemma 1[12].

Lemma 2.4 — Let K be a nonempty closed, convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$;

(b) for each $x \in K$, the map

$$(Tx, \theta(-, x)) : K \to \mathbf{R},$$

is convex; and

(c) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ for all $x, y \in K$.

Then the following are equivalent :

(A) $x_0 \in K, (Tx_0, \theta(y, x_0)) \geq 0$ for all $y \in K$.

(B) $x_0 \in K, (Ty, \theta(x_0, y)) \leq 0$ for all $y \in K$.

PROOF : Suppose that $x_0 \in K$ and

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. By (c)

$$(Ty, \theta(x_0, y)) \leq -(Tx_0, \theta(y, x_0)) \leq 0$$

for all $y \in K$.

Conversely, suppose that $x_0 \in K$ and

$$(Ty, \theta(x_0, y)) \leq 0$$

for all $y \in K$. For an arbitrary $y \in K$, let

$$y_t = ty + (1-t)x_0,$$

$0 < t < 1$. Since K is convex $y_t \in K$. Using (a), (b) and (B) we obtain

$$0 = (Ty_t, \theta(y_t, y_t))$$

$$= (Ty_t, \theta(ty + (1-t)x_0, y_t))$$

$$\leq t(Ty_t, \theta(y, y_t)) + (1-t)(Ty_t, \theta(x_0, y_t))$$

$$\leq t(Ty_t, \theta(y, y_t))$$

$$< (Ty_t, \theta(y, y_t));$$

since T and θ are continuous, taking limit $t \to 0$ in the above inequality we get

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. This completes the proof of Lemma 2.4.

Using the above Lemma we can generalize Theorem 2.2 when K is closed, convex and bounded.

Theorem 2.5 — *Let K be a nonempty closed, convex and bounded subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$, the map

$$(Tx, \theta(-, x)) : K \to \mathbf{R},$$

is convex, and

(c) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ for all $x, y \in K$.

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

PROOF : Define a set valued map

$$E : K \to 2^X$$

by the rule

$$E(y) = \{x \in K : (Ty, \theta(x, y)) \leq 0\}.$$

By (a), $E(y)$ is nonempty for each $y \in K$. It is easy to see that for each $y \in K$, $E(y)$ is a closed and convex subset of K. Since K is a closed, convex and bounded subset of a reflexive real Banach space X, it is also weakly compact; hence $E(y)$ is also weakly compact for each $y \in K$.

Define another set valued map

$$F : K \to 2^X$$

by the rule

$$F(y) = \{x \in K : (Tx, \theta(y, x)) \geq 0\}.$$

By (c), it is clear that for each $y \in K$, $F(y) \subset E(y)$. We assert that F is a KKM-map. If F is not a KKM-map then there exists

$$\{x_1, x_2, \ldots, x_n\} \subset K$$

and

$$a_i \geq 0, \; 1 \leq i \leq n$$

with

$$\sum_{i=1}^{n} a_i = 1$$

such that

$$\sum_{i=1}^{n} a_i x_i \notin \bigcup_{j=1}^{n} F(x_j)$$

i.e.,

$$\sum_{i=1}^{n} a_i x_i \notin F(x_j)$$

for any $j = 1, 2, ..., n$. Thus

$$\left(T \sum_{i=1}^{n} a_i x_i, \theta\left(x_j, \sum_{i=1}^{n} a_i x_i\right)\right) < 0$$

for each $j = 1, 2, ..., n$. By the convexity of the map

$$y \mapsto (Tx, \theta(y, x))$$

of K into R, it follows that

$$\left(T \sum_{i=1}^{n} a_i x_i, \theta\left(\sum_{j=1}^{n} a_j x_j, \sum_{i=1}^{n} a_i x_i\right)\right)$$

$$\leq \sum_{j=1}^{n} a_j \left(T \sum_{i=1}^{n} a_i x_i, \theta\left(x_j, \sum_{i=1}^{n} a_i x_i\right)\right)$$

$$< 0,$$

which contradicts (a). Thus F is a KKM-map and hence E is also a KKM-map. By Theorem 1.3

$$\bigcap_{y \in K} E(y) \neq \phi$$

i.e., there exists $x_0 \in K$ such that

$$(Ty, \theta(x_0, y)) \leq 0$$

for all $y \in K$. By Lemma 2.4, this is equivalent to saying that there exists $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$. This completes the proof of Theorem 2.5.

The following result generalizes Theorem 2.5 when K is an unbounded, closed and convex subset of X.

Theorem 2.6 — *Let K be a nonempty closed, convex and unbounded subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$, the map

$$(Tx, \theta(-, x)) : K \to \mathbf{R},$$

is convex.

(c) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ *for all* $x, y \in K$,

(d) for at least one $r > 0$, *there exists* $u \in D_r^0$ *such that*

$$(Ty, \theta(u, y)) \leq 0$$

for all $y \in S_r$.

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

PROOF : D_r is a closed, convex and bounded subset of K. By Theorem 2.5, there exists $x_0 \in D_r$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in D_r$. The remaining part of the proof is similar to the corresponding part of the proof of Theorem 2.2. This completes the proof of Theorem 2.6.

The following result generalizes Theorem 2.3 when K is a closed, convex and unbounded subset of X.

Theorem 2.7 — *Let* K *be a nonempty closed, convex and bounded subset of a reflexive real Banch space* X *and let* X^* *be the dual of* X. *Let* $T : K \to X^*$ *and* $\theta : K \times K \to X$ *be two continuous maps such that* ·

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$, *the map*

$$(Tx, \theta(-x)) : K \to \boldsymbol{R},$$

is convex,

(c) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ *for all* $x, y \in K$, *and*

(d) there exists a nonempty, compact and convex subset C *of* K *and* $u \in C$ *such that*

$$(Ty, \theta(u, y)) < 0$$

for all $y \in K - C$.

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

PROOF : Since C is compact, there exists $r > 0$ such that $y \in D_r^0$ for all $y \in C$. Since D_r is closed, convex and bounded subset of K, by Theorem 2.5, there exists $x_0 \in D_r$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in C \subset D_r$. The remaining part of the proof is similar to the corresponding part of the proof of Theorem 2.3. This completes the proof of Theorem 2.7.

3. Uniqueness of Solution

In this section, under reasonable conditions we show that the solution $x_0 \in K$ as obtained in Theorem 2.1 is unique.

Theorem 3.1 — *Let K be a nonempty compact and convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two continuous maps such that*

(a) for each $x \in K$, $(Tx, \theta(x, x)) = 0$,

(b) for each $x \in K$, the map

$$(Tx, \theta(-, x)) : K \to \mathbf{R},$$

is convex,

(c) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) \leq 0$ for all $x, y \in K$, and

(d) $(Tx, \theta(y, x)) + (Ty, \theta(x, y)) = 0$ implies $x = y$.

Then there exists a unique $x_0 \in K$ such that

$$(Tx_0, \theta(y, x_0)) \geq 0$$

for all $y \in K$.

Proof : Existence of the solution $x_0 \in K$ is already proved in Theorem 2.1. Let $x_1, x_2 \in K$ be such that

$$(Tx_1, \theta(y, x_1)) \geq 0 \text{ and } (Tx_2, \theta(y, x_2)) \geq 0$$

for all $y \in K$; putting $y = x_2$ and $y = x_1$ in the former and later inequality respectively and on adding we get

$$(Tx_1, \theta(x_2, x_1)) + (Tx_2, \theta(x_1, x_2) \geq 0$$

and this combined with inequalty given in (c) gives

$$(Tx_1, \theta(x_2, x_1)) + (Tx_2, \theta(x_1, x_2) = 0.$$

Now by (d) we have $x_1 = x_2$. This completes the proof of Theorem 3.1.

It is a pleasure to thank the referee for his comments which resulted in an improved presentation of the paper.

ACKNOWLEDGEMENT

The research was supported in part by the Ministry of Human Resource Development, Department of Education, Government of India.

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