STEADY REVOLVING FLOW OF A REINER-RIVLIN FLUID

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ABSTRACT

In this paper the steady revolving flow, otherwise known as the Bödewadt flow of a non-Newtonian Reiner-Rivlin fluid is considered. A second order finite difference method (FDM) has been adopted to solve the resulting fully coupled and highly nonlinear system of differential equations. The effects of non-Newtonian cross-viscous parameter (K) on the velocity field has been studied in detail and shown graphically. It is interesting to find that an increase in K decreases the torque required to maintain the disk at rest. One of the important findings of the present investigation is that when the non-Newtonian parameter K is increased, solutions to the boundary value problem tend to approach their far-field asymptotic boundary values more rapidly.

Introduction

Both Newtonian and non-Newtonian flows past rotating disks have drawn the attention of many researchers due to their fundamental and immense engineering and industrial applications. Von Kármán (11) considered the steady flow of a viscous incompressible fluid due to a rotating disk. The inverse problem arising when a viscous fluid rotates with a uniform angular velocity at a larger distance from a stationary disk (revolving flow) is one of the few problems in fluid dynamics for which the Navier-Stokes equation admits an exact solution. This problem was initially studied by Bödewadt (1) by making boundary layer approximations. That is why the flow is well known as Bödewadt flow. In this case it is observed that the fluid particles near a disk flow radially inwards and for reasons of continuity this flow is compensated by an axial flow upwards, away from the disk (Fig. 1). The general problem of an infinite rotating disk in fluid of which the above two problems are particular cases has been later investigated by Hannah (3), Rogers and Lance (7; 8). A comprehensive review of earlier works on flow and heat transfer due to a single and two parallel rotating disks up to 1989 has been included in a monograph by Owen and Rogers (4).

In all of the above studies the fluid is assumed to be Newtonian. Many materials such as polymer solutions or melts, drilling mud, clastomers, certain oils and greases and many other emulsions are classified as non-Newtonian fluids. For these kind of fluids, the commonly accepted assumption of a linear relationship between the stress and the rate of strain does not hold. Most of the fluids used in industries are non-Newtonian fluids. The non-Newtonian fluids have been modeled by constitutive equations which vary greatly in complexity. The non-Newtonian fluid considered in the present paper is that for which the stress tensor τ^i_j is related to the rate of strain tensor e^i_j as (6; 2)

$$\tau^i_j = -p\delta^i_j + 2\mu e^i_j + 2\mu_c e^i_k e^k_j, \quad e^j_j = 0 \tag{1}$$

where p denotes the pressure, μ is the coefficient of viscosity and μ_c is the coefficient of cross viscosity. This model was introduced by Reiner (6) to describe the behavior of wet sand, but was at one time considered as a possible model for non-Newtonian fluid behavior. A detailed discussion up to 1991 regarding the flow of non-Newtonian fluids due to rotating disks can be found in the review paper by Rajagopal (5). Further one can refer the work of Sahoo (9) and the references therein regarding the Kármán flow and heat transfer of Reiner-Rivlin and other non-Newtonian fluids. A thorough survey shows that though literature on Kármán flow of different non-Newtonian fluids is abundant, very little information exists on Bödewadt flow of non-Newtonian fluids.

In this paper the steady flow of a non-Newtonian Reiner-Rivlin fluid (presented by (1)), which rotates with a uniform angular velocity at a larger distance from a stationary disk (known as the Bödewadt flow) is studied. The resulting system of highly nonlinear differential equations for the velocity field is solved by a second order finite difference method.

Governing equations

We consider a non-Newtonian Reiner-Rivlin fluid occupying the space $z > 0$ over an infinite stationary disk, coinciding with $z = 0$ (see Figure 1). The motion is due to the rotation of the fluid like rigid body with constant angular velocity Ω at large distance from the disk. It is natural to describe the flow in the cylindrical polar coordinates (r, ϕ, z).

In view of the rotational symmetry, $\frac{\partial}{\partial \phi} \equiv 0$. Taking $\mathbf{V} = (u, v, w)$ for the steady flow and using the following well known Kármán transformations (11)

$$u = r\Omega F(\zeta), \quad v = r\Omega G(\zeta), \quad w = \sqrt{\Omega\nu}H(\zeta), \quad z = \sqrt{\frac{\nu}{\Omega}}\zeta$$

the equations of continuity and motion are reduced (4; 9) to the following system of fully coupled and highly nonlinear

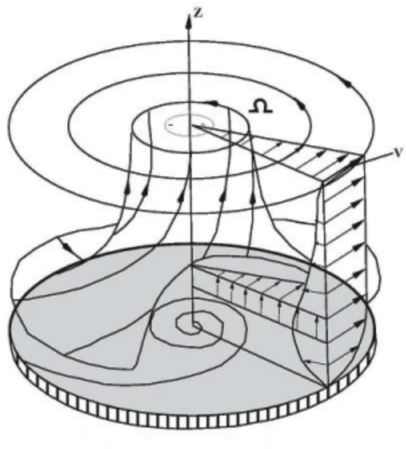

Figure 1. Geometric representation of the flow domain.

differential equations:

$$\frac{dH}{d\zeta}+2F=0, \tag{2}$$

$$\frac{d^2F}{d\zeta^2}-H\frac{dF}{d\zeta}-F^2+G^2-\frac{1}{2}K\Big[\Big(\frac{dF}{d\zeta}\Big)^2-3\Big(\frac{dG}{d\zeta}\Big)^2 \tag{3}$$

$$-2F\frac{d^2F}{d\zeta^2}\Big]=1 \tag{4}$$

$$\frac{d^2G}{d\zeta^2}-H\frac{dG}{d\zeta}-2FG+K\Big(\frac{dF}{d\zeta}\frac{dG}{d\zeta}+F\frac{d^2G}{d\zeta^2}\Big)=0, \tag{5}$$

$$\frac{d^2H}{d\zeta^2}-H\frac{dH}{d\zeta}-\frac{7}{2}K\frac{dH}{d\zeta}\frac{d^2H}{d\zeta^2}+\frac{dP}{d\zeta}=0. \tag{6}$$

which has to be solved subject to the following no-slip boundary conditions:

$$\zeta=0: \quad F=0, \quad G=0, \quad H=0, \tag{7a}$$

$$\zeta\to\infty: \quad F\to 0, \quad G\to 1 \tag{7b}$$

where F, G, H and P are non-dimensional functions of ζ, ν is the kinematic viscosity ($\nu=\frac{\mu}{\rho}$) of the fluid and $K=\frac{\mu_c\Omega}{\mu}$ is the parameter that describes the non-Newtonian characteristic of the fluid. The above system (2)-(5) with the prescribed boundary conditions (7a)-(7b) are sufficient to solve for the three components of the flow velocity. Equation (6) can be used to solve for the pressure distribution at any point.

Numerical Solution

A finite value, large enough, has been substituted for ζ_∞, the numerical infinity to ensure that the solutions are not affected by imposing the asymptotic conditions at a finite distance. The value of ζ_∞ has been kept invariant during the run of the program.

Now we introduce a mesh defined by

$$\zeta_i=ih(i=0,1,\ldots n), \tag{8}$$

h being the mesh size, n is a sufficiently large finite value. The equations (2)-(5) are discretized using the central difference approximations for the derivatives, then the following equations are obtained.

$$\frac{F_{i+1}-2F_i+F_{i-1}}{h^2}-H_i\Big(\frac{F_{i+1}-F_{i-1}}{2h}\Big)-F_i^2+G_i^2-\frac{1}{2}K\Big[\Big(\frac{F_{i+1}-F_{i-1}}{2h}\Big)^2$$
$$-3\Big(\frac{G_{i+1}-G_{i-1}}{2h}\Big)^2-2F_i\Big(\frac{F_{i+1}-2F_i+F_{i-1}}{h^2}\Big)\Big]-1=0 \tag{9}$$

$$\frac{G_{i+1}-2G_i+G_{i-1}}{h^2}-H_i\Big(\frac{G_{i+1}-G_{i-1}}{2h}\Big)-2F_iG_i$$
$$+K\Big[\Big(\frac{F_{i+1}-F_{i-1}}{2h}\Big)\Big(\frac{G_{i+1}-G_{i-1}}{2h}\Big)+F_i\Big(\frac{G_{i+1}-2G_i+G_{i-1}}{h^2}\Big)\Big]=0 \tag{10}$$

$$H_{i+1}=H_i-h(F_i+F_{i+1}) \tag{11}$$

Rest part of the solution scheme can be seen in our published work on Kármán flow (9).

Results and Discussions

The value of ζ_∞, the numerical infinity has been taken larger as compared to the Kármán flow problem (9; 10) and kept invariant through out the run of the program. Although, the results are shown only from the disk surface $\zeta=0$ to $\zeta=14.0$, the numerical integrations were performed over a substantially larger domain in order to assure that the outer asymptotic boundary conditions are satisfied. To see if the program runs correctly, the values of F, G and H are compared with (see Table 1) those reported by Owen and Rogers (4) for a viscous fluid ($K\to 0$), and have been plotted graphically in Figure 2.

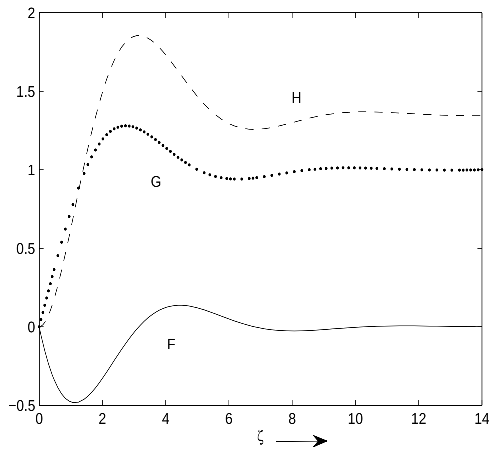

Figure 2. Velocity profile for the Newtonian flow.

Figures 3-5 present, respectively, the steady state profiles of F, G, and H, plotted against ζ for various values of K. Figure 3 depicts that the radial component of velocity is negative near the disk and reverses direction away from the disk. It is clear that the increase in the values of K decreases the radial component of velocity F in magnitude up to a significant distance from the disk and then increases and eventually reaches the asymptotic value 0. Figure 4 presents, the steady state profile of the transverse component of velocity

G with K. It is interesting to find that the magnitude of G decreases with increasing K near the disk and increases away from the disk. This accounts for a crossover of the profiles of G. The steady state profile of the axial component of velocity H with various values of K is shown in figure 5. It is observed that increasing K, decreases H for all values of ζ. It is also clear that the profiles of F, G and H becomes flatter as K is increased. In other words, when the non-Newtonian parameter K is increased, solutions to the momentum equations tend to approach their far-field boundary values more rapidly.

Another interesting quantity is the turning moment for the disk. The expression of the dimensionless moment coefficient C_M is given by:

$$C_M = \frac{-\pi G'(0)}{\sqrt{Re}} \qquad (12)$$

with $Re = \Omega R^2/\nu$ the rotational Reynolds number based on the disk radius R and the maximum velocity (ΩR). This definition of C_M is the extension of the finite disk problem, which supposes that the disk radius is large enough. Figure 6 shows the variation of C_M with K for $Re = 1$. It's clear that whatever the flow parameters, C_M exhibits negative values. The value of C_M decreases in magnitude with an increase in K and approaches the asymptotic value zero for sufficiently high value of the non-Newtonian parameter K. The Von Kármán flow considered by Sahoo (9) is precisely the inverse problem, which explains the different sign.

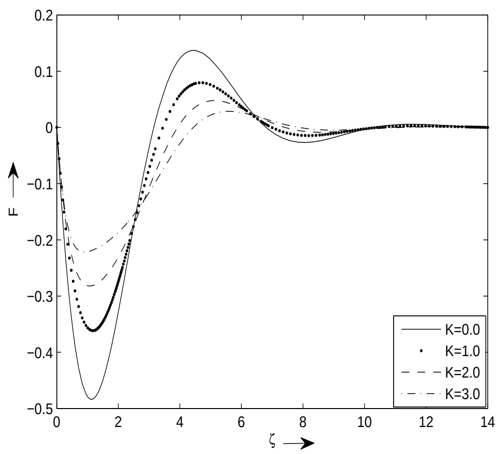

Figure 3. Variation of F with K.

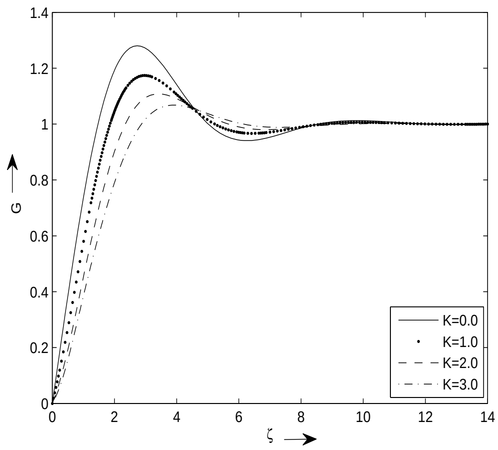

Figure 4. Variation of G with K.

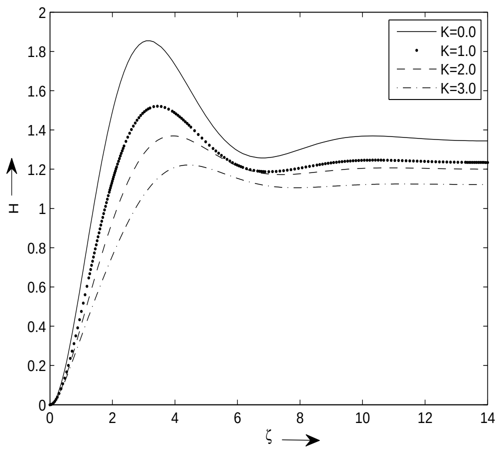

Figure 5. Variation of H with K.

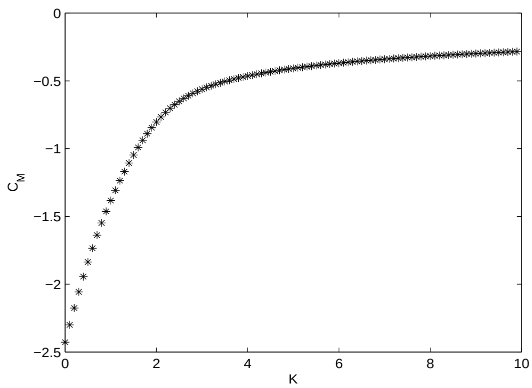

Figure 6. Variation of C_M with K.

Table 1. Variations of F, G and H.

	F		G		H	
ζ	Current result	Owen & Rogers (4)	Current result	Owen & Rogers (4)	Current result	Owen & Rogers (4)
0.0	0.000000	0.0000	0.000000	0.0000	0.000000	0.0000
0.5	-0.348650	-0.3487	0.383430	0.3834	0.194373	0.1944
1.0	-0.478766	-0.4788	0.735429	0.7354	0.624103	0.6241
1.5	-0.449633	-0.4496	1.013401	1.0134	1.098743	1.0987
2.0	-0.328745	-0.3287	1.192367	1.1924	1.492875	1.4929
2.5	-0.176206	-0.1762	1.272136	1.2721	1.745869	1.7459
3.0	-0.036086	-0.0361	1.271405	1.2714	1.849641	1.8496
3.5	0.066310	0.0663	1.218219	1.2182	1.830807	1.8308
9.5	-0.010216	-0.0102	1.011849	1.0118	1.361698	1.3617
10.0	-0.003282	-0.0033	1.012120	1.0121	1.368328	1.3683
10.5	0.001819	0.0018	1.009906	1.0099	1.368882	1.3689
11.0	0.004738	0.0047	1.006537	1.0065	1.365423	1.3654
11.5	0.005681	0.0057	1.003090	1.0031	1.360067	1.3601
12.0	0.005170	0.0052	1.000271	1.0003	1.354546	1.3545
12.5	0.003827	0.0038	0.998411	0.9984	1.350003	1.3500
20.0	0.000102	-	0.999893	-	1.349325	-
25.0	0.000009	-	1.000014	-	1.349457	-
25.5	0.000011	-	1.000007	-	1.349447	-
26.0	0.000010	-	1.000001	-	1.349437	-
26.5	0.000008	-	0.999997	-	1.349428	-
28.0	0.000000	0.0000	1.000000	1.0000	1.349421	1.3494

Conclusions

In the present paper the steady revolving flow of a non-Newtonian Reiner-Rivlin fluid has been considered. The constitutive equation of the fluid gives rise to momentum equations, which, when transformed using the similarity variables, reduce to highly non-linear system of boundary value problem. A second order finite difference technique has been used to solve the system of resulting equations. The effects of non-Newtonian fluid parameter K on the velocity field has been studied in detail. It is interesting to find that the parameter K results in a crossover of the transverse velocity profile. One of the important findings of the present investigation is that when the non-Newtonian parameter K is increased, solutions to the boundary value problem tend to approach their far-field asymptotic boundary values more rapidly. The profiles of the moment coefficient C_M for the revolving (Bödewadt) flow and the Kármán flow (9) are just opposite to each other as was expected.

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