

A Study on Kac-Moody Superalgebras

Kishor Chandra Pati

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The importance of being Lie

Discrete groups describe discrete symmetries.

Continues symmetries are described by so called **Lie group**.

eg. a regular hexagon: 6-rotations and 6 reflections

→ *dihedral group* D_{12} (discrete)

eg. a circle: infinitely many rotations and reflections

→ *orthogonal group* O_2 and $U(1)$ (Lie group)

Lie Algebra

Being continuous (smooth manifold), Lie groups allow calculus. Differentiating smooth curve through 1 at 1 gives the *Lie algebra*: A vector space equipped with a bracket $[\cdot, \cdot]$ satisfying

- $[x, y] = -[y, x]$,
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$

Canonical Example

Let $L(V)$ be the set of all linear operators on a vector space V . With this bracket operation $[A, B] = AB - BA$, turns $L(V)$ into a Lie algebra. This Lie algebra is called the general linear Lie algebra $gl(V)$

Lie algebras come into 2 flavours, *solvable* and *semisimple*.
A Lie algebra L is called solvable if $L_n = 0$ for some $n \in \mathbb{N}$.
Solvable Lie algebra are unclassifiable.
Semisimple Lie algebras are classifiable.

A subalgebra of a Lie algebra L consisting of semisimple elements is called *toral* subalgebra.

H as maximal toral subalgebra of L .

$ad_L H = \{ad\ h|h \in H\}$, is a commuting set of diagonalizable linear operators.

$$L_\alpha = \{x \in L \mid \forall h \in H : ad\ h(x) = \alpha(h)x\}$$

$$L = \bigoplus_\alpha L_\alpha$$

Theorem

Let L be a finite-dimensional complex semisimple Lie algebra and H a maximal toral subalgebra. Then L has with respect to $ad_L H$ the structure

$$L = C_L(H) \oplus_\alpha L_\alpha$$

As usual $H^* = \{\alpha | \alpha : H \rightarrow \mathbb{C}\}$ is the dual space of H .

A non-zero element $\alpha \in H^*$ for which $\dim L_\alpha \geq 1$ is called a *root* and the subspace $L_\alpha \neq 0$ is called *root space*.

Δ is the root system of L .

$\Pi = \{\alpha_1, \alpha_2 \dots \alpha_k\}$ forms a basis for Δ .

$\Pi^\nu = \{\alpha_1^\nu, \dots, \alpha_k^\nu\} \subset H$.

Cartan matrix

The Cartan matrix $A = (a_{ij})_{i,j=1}^k$ where

$$a_{ij} = \frac{2(\alpha_j|\alpha_i)}{(\alpha_i|\alpha_i)}$$

This A has the following properties:

- $\det A \neq 0$
- $a_{ii} = 2$
- $a_{ij} \in \{0, -1, -2, -3\}$ ($i \neq j$)
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
- $a_{ij} = -2 \Rightarrow a_{ji} = -1$
- $a_{ij} = -3 \Rightarrow a_{ji} = -1$

Note

$\langle \alpha_j, \alpha_i^\vee \rangle =: a_{ij}$ is the dual contraction between Π and Π^\vee .

For a FSLA L the following relations hold:

- $[\alpha_i^\vee, \alpha_j^\vee] = 0$
- $[\alpha_i^\vee, e_j] = \langle \alpha_j, \alpha_i^\vee \rangle e_j = a_{ij} e_j$
- $[\alpha_i^\vee, f_j] = \langle -\alpha_j, \alpha_i^\vee \rangle f_j = -a_{ij} f_j$
- $[e_i, f_j] = \delta_{ij} \alpha_j^\vee$
- $[e_j, e_\theta] = 0$

Also the operators $ad e_i$ and $ad f_i$ are nilpotent satisfying

- $(ade_j)^{1-a_{ji}} e_i = 0 \quad (i \neq j)$
- $(adf_j)^{1-a_{ji}} f_i = 0 \quad (i \neq j)$

FSLA can be represented by diagrams known as Dynkin diagrams. Dynkin diagram consists of vertices representing simple roots of the Lie algebra and lines connecting them. The no of lines connecting a pair of vertices and their possible orientation is determined by matrix elements a_{ij} of the Cartan matrix. Let $A(L)$ be a $k \times k$ Cartan matrix of L . Dynkin diagram $D(A)$ of L is constructed as follows:

- Draw k vertices, one for each simple root, vertex i ($i = 1, \dots, k$) corresponding to each simple root α_i .
- Connect the vertices i and j with η_{ij} lines where $\eta_{ij} = a_{ij}a_{ji}$
- If $|a_{ij}| > 1$ draw an arrow from j to i .

Any complex FSLA is one of the following:

- $A(n)$ or $sl(n+1, \mathbb{C})$
- $B(n)$ or $so(2n, \mathbb{C})$
- $C(n)$ or $sp(n, \mathbb{C})$
- $D(n)$ or $so(2n+1, \mathbb{C})$

or one of the exceptional algebra E_6, E_7, E_8, F_4, G_2

Lie Superalgebra

A Lie super algebra is a \mathbb{Z}_2 -graded generalisation:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \quad i, j \in \mathbb{Z}_2.$$

- $[x, y] = -(-1)^{\deg x \deg y} [y, x],$
- $[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \deg y} [y, [z, x]]$

Comments

In Physics, lie algebras describe *bosonic* degrees of freedom.
Lie super algebra allow *ferminionic* degrees of freedom.

Canonical Example

Declare as $(m+n) \times (m+n)$ block matrices $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$ to be even if $A_{01} = A_{10} = 0$ and odd if $A_{10} = A_{11} = 0$

The space of $(m+n) \times (m+n)$ block matrices is the Lie superalgebra $gl(m|n)$ with:

- $gl(m|n)_0 = (\text{even matrices})$ and $gl(m|n)_1 = (\text{odd matrices})$



$$[A, B] = \begin{cases} AB - BA & \text{if } A \text{ or } B \text{ is even} \\ AB + BA & \text{if } A \text{ and } B \text{ is odd} \end{cases}$$

List of all finite-dimensional simple lie-superalgebras

- $A(m, n)$ or $sl(m + 1, n + 1)$
- $B(m, n)$ or $osp(2m + 1, 2n)$ with $m \neq 0$
- $B(0, n)$ or $osp(1, 2n)$
- $D(m, n)$ or $osp(2m, 2n)$
- $D(2, 1; \alpha)$
- F_4
- G_3

A matrix $A = (a_{ij})$ is called a generalized Cartan matrix if it satisfies the following conditions:

- $a_{ii} = 2$
- $a_{ij} \in \{0, -1, -2, \dots\}$ for $(i \neq j)$
- $a_{ij} = 0 \Rightarrow a_{ji} = 0$

Given a $n \times n$ generalized Cartan matrix A . Let $(\mathfrak{h}, \Pi, \Pi^\nu)$ be a realisation of A and $\tilde{\mathfrak{g}}(A)$ be the Lie algebra on generators e_i, f_i ($i = 1, \dots, n$) and (\mathfrak{h}) and the defining relations

$$[e_i, f_j] = \delta_{ij} \alpha_i^\nu \quad (i, j = 1, \dots, n) \quad (1)$$

$$[h, h'] = 0 \quad (h, h' \in \mathfrak{h}) \quad (2)$$

$$[h_i, e_j] = a_{ij} e_j \quad (3)$$

$$[h_i, f_j] = -a_{ij} f_j \quad (4)$$

Let τ be the maximal ideal in $\mathfrak{g}(A)$ which intersects (\mathfrak{h}) trivially. Set $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$. The Lie algebra $\mathfrak{g}(A)$ corresponding to generalized Cartan matrix A is called Kac-Moody algebra.

Kac-Moody Superalgebra

Let τ be a subset of $I = \{1, 2, 3, \dots, r\}$.

To a given generalized Cartan matrix A , and subset τ we can associate Lie superalgebra called Kac-Moody algebra with $3r$ generators e_i, f_i, h_i and a Z_2 gradation defined by $\deg e_i = \deg f_i = 0$ if $i \notin \tau$, $\deg e_i = \deg f_i = 1$ if $i \in \tau$ and $\deg h_i = 0 \quad \forall i$. The generators satisfy these set of relations

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ij}e_j \\ [h_i, f_j] &= -a_{ij}f_j \\ [e_i, f_j] &= \delta_{ij}h_i \\ [e_i, e_j] &= 0, [f_i, f_j] = 0, [e_i, f_j] = 0 \quad \text{if } a_{ij} = 0 \end{aligned} \quad (5)$$

and the Serre relations

$$(ad e_i)^{1-\tilde{a}_{ij}}e_j, \quad (ad f_i)^{1-\tilde{a}_{ij}}f_j = 0; \quad \forall i \neq j, \quad (6)$$

Cont...

$$G = G_{\bar{0}} \oplus G_{\bar{1}}$$

Let $\mathcal{H} \subset G_{\bar{0}}$ be the subalgebra of G generated by h_i is known as Cartan subalgebra.

$$G = \bigoplus_{\alpha} G_{\alpha}$$

where

$$G_{\alpha} = \{x \in G \mid [h, x] = \alpha(h)x, h \in \mathcal{H}\}$$

is the root space w.r.t the root α .

$$\Delta = \{\alpha \in \mathcal{H}^* \mid G_{\alpha} \neq 0\}$$

be the set of roots of G .

If $G_{\alpha} \subset G_{\bar{0}} \rightarrow \alpha$ is even and $G_{\alpha} \subset G_{\bar{1}} \rightarrow \alpha$ is odd.

Definitions

- If α is a even or bosonic root then $(\alpha, \alpha) \neq 0$ and 2α is not a root.
- The odd fermionic roots are isotropic if $(\alpha, \alpha) = 0$.
- and the odd root is Non-isotropic if $(\alpha, \alpha) \neq 0$ and 2α is a root.

Cont...

G admits a Borel decomposition i.e.

$$G = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$$

where \mathcal{N}^+ and \mathcal{N}^- are subalgebra of G such that

$$[\mathcal{H}, \mathcal{N}^+] \subseteq \mathcal{N}^+$$

$$[\mathcal{H}, \mathcal{N}^-] \subseteq \mathcal{N}^-$$

$$\dim \mathcal{N}^+ = \dim \mathcal{N}^-$$

α is *positive root* if $G_\alpha \cap \mathcal{N}^+ \neq \phi$ and it is called *negative* if $G_\alpha \cap \mathcal{N}^- \neq \phi$.

A root is called *simple* if it can't be written as sum of two positive root.

Cont...

The Weyl group of G is $W = \langle r_i \rangle$ defined by

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i, \quad \forall j$$

The elements of the root system that are W -conjugate to simple root are called *real root*, remaining are called imaginary roots.

$$\Delta = \Delta_{re} + \Delta_{im}.$$

Kac-Moody algebras fall into three disjoint classes, which are characterized by generalized Cartan matrix. These are

- finite-dimensional simple Lie algebra
- affine Kac-Moody algebra
- indefinite Kac-Moody algebra

Let A be a Cartan matrix. Then A belongs to one and only one of the following three classes:

- **A is finite type:** A satisfies $\det A \neq 0$, $\exists u > 0$ such that $Au > 0, Au \geq 0 \Rightarrow (u > 0 \text{ or } u = 0)$
- **A is Affine type:** A satisfies $\det A = 0, \text{corank } A = 1$ $\exists u > 0$ such that $Au = 0, Au \geq 0 \Rightarrow Au = 0$
- **A is indefinite type:** A satisfies $\exists u > 0$ such that $Au < 0, Au \geq 0, u \geq 0 \Rightarrow u = 0$

Hyperbolic Kac-Moody super Algebra

Let $G(A, \tau)$ be an indefinite Kac-Moody superalgebra with generalized Cartan matrix A and non trivial \mathbb{Z}_2 -gradation τ corresponding to a connected Dynkin diagram. $G(A, \tau)$ is called a hyperbolic Kac-Moody (HKM) superalgebra if every leading principal submatrix of A decomposes into constituents of finite, or affine type or equivalently, if deleting a vertex of the Dynkin diagram, one gets the Dynkin diagrams of finite or affine type. If the Cartan matrix A is symmetric, the corresponding superalgebra is called simply laced Kac-Moody superalgebra.

Our Main Result

Given any simply laced hyperbolic Kac-Moody Superalgebra (with a degenerate odd root in the Dynkin Diagram) G there is a subalgebra(super) of of rank 6 hyperbolic Kac-Moody superalgebra $HD(4, 1)$ that is isomorphic to G .

Let $G = G(A)$ be a Kac-Moody superalgebra associated with a $n \times n$ symmetric generalized Kac-matrix A . Let ϕ denotes the root system of G with Cartan subalgebra \mathcal{H} and ϕ_{re}^+ denotes the set of real positive roots of G . Consider $\beta_1, \beta_2, \dots, \beta_k \in \phi_{re}^+$, $k \leq n$ chosen such a way that $\beta_i - \beta_j \notin \phi$, $\forall i \neq j$. Let G_{β_i} and $G_{-\beta_i}$ denote the one-dimensional root spaces corresponding to the positive real roots β_i and $-\beta_i$ respectively. $0 \neq e_{\beta_i} \in G_{\beta_i}$ and $0 \neq f_{\beta_i} \in G_{-\beta_i}$ are root vectors. Let $h_{\beta_i} = [e_{\beta_i}, f_{\beta_i}] \in \mathcal{H}$ where $[,]$ denotes the Lie super bracket. Then $\{e_{\beta_i}, f_{\beta_i}, h_{\beta_i} \mid i = 1, 2, \dots, k\}$ will generate a Kac-Moody subalgebra of rank k with Kac matrix with $C = [C_{ij}] = [(\beta_i | \beta_j)]$. The subalgebra is denoted as $G(\beta_1, \beta_2, \dots, \beta_k)$.







Definition

Suppose D denotes the Dynkin diagram of G and D' denotes the Dynkin diagram corresponding to the generalized Kac matrix C then D' is called as root subdiagram of D . It is denoted as $D' \preceq D$. For proving **OUR MAIN RESULT** it is sufficient to prove the following proposition:







Proposition

Every simply laced connected hyperbolic Dynkin diagram (with degenerate odd root only) occurs as root subdiagram of rank 6 Dynkin diagram $HD(4, 1)$

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THANK YOU