
ARITHMETIC PROGRESSION OF SQUARES AND SOLVABILITY OF THE DIOPHANTINE EQUATION $8x^4 + 1 = y^2$

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Abstract

There is no arithmetic progression consisting of square terms and with a square common difference. Alternatively, the diophantine equation $1 + x^4 = 2y^2$ has no solution in positive integers. Consequently, the diophantine equation $8x^4 + 1 = y^2$ has no positive integral solution other than $x = 1, y = 3$, a clear indication that no balancing number other than 1 is a perfect square.

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1 INTRODUCTION

Balancing numbers and balancers, originally introduced by Behera and Panda [1], are respectively natural numbers n and r satisfying the equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r).$$

This equation when simplified yields

$$n^2 = \frac{(n + r)(n + r + 1)}{2},$$

leading to the conclusion that a natural number n is a balancing number if and only if n^2 is a triangular number, or, equivalently, $8n^2 + 1$ is a perfect square. Since $8 \cdot 1^2 + 1 = 9$, Panda [4] and Panda and Ray [5] accepted as 1 as the first balancing number, though the definition for the balancing numbers suggests that every balancing number must be greater than 1. The n^{th} balancing number is denoted by B_n and is well known that the balancing numbers satisfy the recurrence relation (also see [1])

$$B_{n+1} = 6B_n - B_{n-1}, \quad B_1 = 1, \quad B_2 = 6.$$

The first fifteen balancing numbers are 1, 6, 35, 204, 1189, 6930, 40391, 1372105, 7997214, 46611179, 271669860, 1583407981, 9228778026, 53789260175 and except the first one, none is a square. This leads to a natural question "Is it possible for any other balancing number to be a perfect square?" The answer to this question lies in the solvability of the diophantine equation $8x^4 + 1 = y^2$ in natural number with $x > 1$, which, in turn, depends on the solvability of a problem on arithmetic progressions, consist of finding three distinct natural numbers P, Q and R in arithmetic progression such that each of P, Q and R and the common difference $d = Q - P = R - Q$ are perfect squares.

2 THREE PERFECT SQUARES IN ARITHMETIC PROGRESSION

Let x, y and z be distinct natural numbers such that x^2, y^2 and z^2 are in arithmetic progression. Then x, y and z satisfy the Pythagorean-type equation

$$x^2 + z^2 = 2y^2.$$

Observe that both x and z must be of the same parity. If both x and z are even, then y must be even and hence, we can cancel the highest power of 2 that is common to each of x^2, y^2 and z^2 . We can assume, both x and z are odd and hence y is also odd; else,

$$x^2 + z^2 \equiv 2(\text{mod } 4)$$

and

$$2y^2 \equiv 0(\text{mod } 4)$$

contradict

$$x^2 + z^2 = 2y^2.$$

Since x, y and z are distinct, we may assume without loss of generality that $x < z$. If d is the greatest common divisor of x and z then certainly, d divides y and we can cancel d^2 from both sides of the equation. Thus, we also assume without loss of generality that x, y and z are pairwise coprimes. Since both x and z are odd, $\frac{z+x}{2}$ and $\frac{z-x}{2}$ are natural numbers and the diophantine equation $x^2 + z^2 = 2y^2$ reduces to the Pythagorean equation

$$\left(\frac{z+x}{2}\right)^2 + \left(\frac{z-x}{2}\right)^2 = y^2.$$

Since x and z are coprimes, $z+x$ and $z-x$ are either coprime or their greatest common divisor is 2. Thus, in the present case, $\frac{z+x}{2}$ and $\frac{z-x}{2}$ are coprimes. Let (A, B, C) be any arbitrary primitive Pythagorean triple, that is A, B and C are pairwise coprimes and $A^2 + B^2 = C^2$. Assume that $A > B$. Then we have $\frac{z+x}{2} = A$, $\frac{z-x}{2} = B$ and $y = C$, implying that $z = A + B$, $x = A - B$ and $y = C$ leading to

$$(A - B)^2 + (A + B)^2 = 2C^2.$$

We also note here that if x^2, y^2 and z^2 are in arithmetic progression then, $(kx)^2, (ky)^2$ and $(kz)^2$ are also in arithmetic progression for each natural number k .

From the above discussion it is clear that using primitive Pythagorean triples we can always generate three pairwise coprime perfect squares to form arithmetic progressions. Conversely, if three perfect squares are in arithmetic progression, we can always find a Pythagorean triple from these squares. Thus if x^2, y^2 and z^2 are in arithmetic progression, then x, y and z must be of same parity and taking

$$A = \frac{z+x}{2}, B = \frac{z-x}{2}, C = y$$

we have

$$A^2 + B^2 = C^2.$$

3 ARITHMETIC PROGRESSIONS WITH SQUARE TERMS AND SQUARE COMMON DIFFERENCE

Let x, y and z be distinct natural numbers such that x^2, y^2 and z^2 are in arithmetic progression with a perfect square common difference. We can assume without loss of generality that x, y and z are pairwise coprimes. This leads us to consider the diophantine equation $x^2 + z^2 = 2y^2$ such that the common difference $y^2 - x^2 = z^2 - y^2$ is a perfect square. Let y be the smallest number to satisfy this property. In view of the results of last section, we have

$$x = A - B, y = C, z = A + B$$

where A, B and C are natural numbers forming a primitive Pythagorean triple. It is well known that each primitive solution of the Pythagorean equation $A^2 + B^2 = C^2$ is of the form

$$A = a^2 - b^2, B = 2ab, C = a^2 + b^2$$

where a and b are two coprime natural number of opposite parity, see [2] and [3, p.584]. We may assume without loss of generality that $a > b$. The common difference of the arithmetic progression takes the form

$$y^2 - x^2 = z^2 - y^2 = 2AB = 4ab(a^2 - b^2) = 4ab(a + b)(a - b).$$

Since a and b are coprimes and of opposite parity, the numbers $a, b, a + b$ and $a - b$ are also pairwise coprimes and hence, all must be perfect squares. Thus there exist natural numbers u, v, p and q such that

$$a = u^2, b = v^2, a + b = p^2, a - b = q^2,$$

leading to the simultaneous diophantine equation

$$u^2 + v^2 = p^2, u^2 - v^2 = q^2,$$

a clear indication that $u^2 - v^2, u^2, u^2 + v^2$ are in arithmetic progression with common difference v^2 . But in view of

$$u^2 = a \leq a^2 < a^2 + b^2 = C < C^2 = y^2,$$

it follows that $u < y$, contradicting the assumption that y is the smallest number such that x^2, y^2 and z^2 are in arithmetic progression with a perfect square common difference. Hence three squares cannot be in arithmetic progression if their common difference is a square.

4 THE DIOPHANTINE EQUATION $x^4 + 1 = 2y^2$

Our next objective is to show that no balancing number other than 1 is a perfect square, or, equivalently, the diophantine equation $8x^4 + 1 = y^2$ has no positive integral solution except $x = 1, y = 3$. But before we do so, we need to study the diophantine equation $x^4 + 1 = 2y^2$.

The diophantine equation $x^4 + 1 = 2y^2$ has the trivial solution in positive integers $x = y = 1$. To take care of other solutions, we assume that $x \neq y$. Further, we observe that if $x \geq 2$, then we have

$$4x^2 \leq x^4 < x^4 + 1 = 2y^2$$

leading to $x < y$. We next convert $x^4 + 1 = 2y^2$ to a Pythagorean equation. Adding $2x^2$ to both sides we get

$$(x^2 + 1)^2 = 2(x^2 + y^2) = (x + y)^2 + (y - x)^2.$$

Observe that if $x^4 + 1 = 2y^2$, then x is odd and hence y is also odd. Further if d is the greatest common divisor of x and y then d^2 divides $2y^2 - x^4 = 1$, implying that $d = 1$ and hence that x and y are coprimes. Thus, we have the equation

$$\left(\frac{x + y}{2}\right)^2 + \left(\frac{x - y}{2}\right)^2 = \left(\frac{x^2 + 1}{2}\right)^2.$$

It is also well known that if a and b are coprimes then $a + b$ and $a - b$ are either coprime or their greatest common divisor is 2. Since for the case at hand, x and y are odd $\frac{x+y}{2}$ and $\frac{y-x}{2}$ are coprimes. We claim that $\frac{x+y}{2}, \frac{y-x}{2}$ and $\frac{x^2+1}{2}$ are pairwise coprimes. Observe that if d divides $\frac{x+y}{2}$ and $\frac{x^2+1}{2}$ then d^2 must divide

$$\left(\frac{x^2 + 1}{2}\right)^2 - \left(\frac{x + y}{2}\right)^2 = \left(\frac{y - x}{2}\right)^2$$

and then $\frac{x+y}{2}$ and $\frac{y-x}{2}$ cannot be coprimes. Thus there exist coprime natural numbers a and b of opposite parity with $a > b$ such that the solution of the equation

$$\left(\frac{x + y}{2}\right)^2 + \left(\frac{y - x}{2}\right)^2 = \left(\frac{x^2 + 1}{2}\right)^2$$

is given by

$$\frac{x+y}{2} = 2ab, \quad \frac{y-x}{2} = a^2 - b^2, \quad \frac{x^2+1}{2} = a^2 + b^2$$

or

$$\frac{x+y}{2} = a^2 - b^2, \quad \frac{y-x}{2} = 2ab, \quad \frac{x^2+1}{2} = a^2 + b^2.$$

In either case, we have

$$y = a^2 - b^2 + 2ab, \quad x = |a^2 - b^2 - 2ab|, \quad x^2 = 2(a^2 + b^2) - 1.$$

This gives

$$(a^2 - b^2 - 2ab)^2 = 2(a^2 + b^2) - 1,$$

which, on simplification yields

$$(a^2 + b^2 - 1)^2 = 4ab(a+b)(a-b).$$

Since a and b are coprimes and of opposite parity $a+b$ and $a-b$ are also coprimes, so that each of $a, b, a+b$ and $a-b$ are perfect squares. Thus there exist natural numbers u, v, p and q such that

$$a = u^2, \quad b = v^2, \quad a+b = p^2, \quad a-b = q^2$$

leading to

$$u^2 + v^2 = p^2, \quad u^2 - v^2 = q^2,$$

implying that q^2, u^2 and p^2 are in arithmetic progression with common difference v^2 . By virtue of results of last section, this is impossible. Hence the diophantine equation $x^4 + 1 = 2y^2$ has no solution in positive integers other than $x = y = 1$.

5 THE DIOPHANTINE EQUATION $8x^2 + y^2 = z^2$

To prove the impossibility of solution of the diophantine equation $8x^4 + 1 = y^2$ other than $x = 1$ and $y = 3$, we further need to study a similar type of diophantine equation $8x^2 + y^2 = z^2$ when both y and z are odd natural numbers. The equation $8x^2 + y^2 = z^2$ generates balancing and Lucas-balancing numbers [4, 5, 6] when $y = 1$.

The diophantine equation $8x^2 + y^2 = z^2$ is also a variant of the Pythagorean equation $x^2 + y^2 = z^2$; however the method of solution we present here does not use Pythagorean triples. We assume without loss of generality that y and z are coprimes; else if d is the greatest common divisor of y and z then d^2 divides $z^2 - y^2$ and hence d^2 divides x^2 ; so d^2 can be canceled from each of x^2, y^2 and z^2 and the equation $8x^2 + y^2 = z^2$ is reduced to another equation of the same type.

We rewrite the equation $8x^2 + y^2 = z^2$ as $\frac{z+y}{2} \cdot \frac{z-y}{2} = 2x^2$. Since y and z are both odd and coprimes $\frac{z+y}{2}$ and $\frac{z-y}{2}$ are also coprimes. Hence there exist natural numbers u and v such that

$$\frac{z+y}{2} = u^2, \quad \frac{z-y}{2} = 2v^2$$

or

$$\frac{z+y}{2} = 2v^2, \quad \frac{z-y}{2} = u^2$$

leading to

$$z = u^2 + 2v^2, \quad y = |u^2 - 2v^2|, \quad x = uv$$

and we finally have

$$8u^2v^2 + (u^2 - 2v^2)^2 = (u^2 + 2v^2)^2.$$

Note that if (x, y, z) is a solution of $8x^2 + y^2 = z^2$ and k is any natural number, then (kx, ky, kz) is a solution.

6 PERFECT SQUARE BALANCING NUMBERS AND THE DIOPHANTINE EQUATION $8x^4 + 1 = y^2$

We know that the first balancing number 1 is a square. As remarked in Section 1, there is no square balancing number in a visible distance. It is also well known that a natural number x is a balancing number if and only if $8x^2 + 1$ is a perfect square and hence, for any natural number x , x^2 is a balancing number if and only if $8x^4 + 1$ is a perfect square. This leads to consider the diophantine equation $8x^4 + 1 = y^2$, which has the trivial solution $x = 1, y = 3$. We try for solutions with $x > 1$. In view of the results of last section, the solutions of $8x^4 + 1 = y^2$ can be written as

$$x^2 = uv, y = u^2 + 2v^2$$

such that $|u^2 - 2v^2| = 1$, where the numbers u and v are coprimes. Thus there exist natural numbers a and b such that

$$u = a^2, v = b^2, |a^4 - 2b^4| = 1.$$

The equation $a^4 - 2b^4 = 1$ is equivalent to

$$(b^4 + 1)^2 = a^4 + b^8$$

which, clearly has no solutions in natural numbers [3, p.591] and if $2b^4 - a^4 = 1$ then we have $1 + a^4 = 2b^4$ and by virtue of results of Section 4, this equation has also no solution in natural numbers other than $a = b = 1$. But $a = b = 1$ corresponds to $u = v = 1$ and finally $x = 1, y = 3$. Thus the diophantine equation $8x^4 + 1 = y^2$ has the only solution $x = 1$ and $y = 3$. This suggests that other than 1, no balancing number is a perfect square.

References

- [1] A. Behera and G.K. Panda, *On the square roots of triangular numbers*, The Fib Quart, 37(2) (1999), 98-105.
- [2] P.W. Haggard, *Pythagorean triples and sums of triangular numbers*, Int. J. Math. Educ. Sci. Technol., Vol28(1),(1997),109-116.
- [3] Thomas Koshy, *Elementary number theory with applications*, Academic press, (2007).
- [4] G.K. Panda, *Some fascinating properties of balancing numbers*, Applications of Fibonacci Numbers, Congressus Numerantium, Vol 194(2006), 185-190.
- [5] G.K. Panda, and P.K. Ray, *Cobalancing numbers and cobalancers*, Int. J. Math. Math. Sci. 8 (2005), 11891200.
- [6] G.K. Panda, and P.K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bulletin of the Institute of Mathematics, Academia Sinica (New Series), 6(1) (2011), 41-72.