FREE VIBRATION ANALYSIS OF A PRETWISTED FUNCTIONALLY GRADED MATERIAL CANTILEVER TIMOSHENKO BEAM

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Abstract: In the present work the free vibration of a functionally graded ordinary (FGO) pre-twisted cantilever Timoshenko beam has been investigated. Finite element shape functions are established from differential equations of static equilibrium. Expressions for element stiffness and mass matrices are obtained from energy considerations. The material properties along the thickness of the beam are assumed to vary according to the power law. Increase in the value of power law index decreases the first two mode frequencies of the beam. Increase in pretwist angle increases first mode frequency and decreases the second mode frequency. The effect of pretwist angle and power law index on the first mode frequency is marginal.

Key words: FGO; Power law; Pretwist angle.

1. Introduction:

Some machine components are twisted for functional point of view. The blades of helicopter rotor, turbine and aircraft propeller are given pre-twist and usually are subjected to vibration. The study of free vibration of pre-twisted components therefore remains an important area of research.

Carnegie [1, 2] studied the dynamic behaviour of a pre-twisted cantilever blade taking into account torsion, bending rotary inertia and shear deformation. Dawson, Ghosh and Carnegie [3] investigated the effect of shear deformation and rotary inertia on the natural frequencies of pre-twisted cantilever beams using numerical technique. They also verified their results conducting experiments. Carnegie and Thomas[4] used finite difference method to study the vibration of uniform and tapered pre-twisted cantilevers. Subrahmanyam and Rao[5] applied Reissner method to study the vibration of tapered pre-twisted cantilever beam. Onipede and Dong[6] studied vibration of pre-twisted inhomogeneous beam of arbitrary cross-section by using variational method. Vibration and stability of a spinning pre-twisted thin walled composite beam were studied by Song, Jeong and Librescu[7] considering a number of non-classical features such as transverse shear, anisotropy and pre-twist. Vielsack[8] has shown that the influence of small pre-twist on lateral vibration of beams depends on the ratio of bending stiffness about principal axes. Lin , Wu and Lee[9] have studied the coupled bending–bending vibration of a rotating pre-twisted beam with an elastically restrained root and a tip mass, subjected to the external transverse forces and rotating at a constant angular velocity. Yardimoglu and Yildirim[10] have developed a finite element model with reduced number of nodal degree of freedom to investigate the vibration of a pre-twisted cantilever beam. The effect of pre-twist angle of an aerofoil blade simplified as a rotating Euler as well as Timoshenko beam has been investigated by Subuncu and Ervan[11,12] using finite element method. Jhung and Jo[13] have studied the vibration characteristics of a rectangular twisted beam with pins surrounded with liquid . Mohanty[14] has studied parametric instability of pre twisted cantilever with localized damage. Hsu[15] has investigated dynamic behavior of pre-twisted beams using spline collocation method. Liu, Friend and Yeo[16] have carried out an investigation on the coupled axial-torsional vibration of pre-twisted beams. Leung and Fan[17] have studied the influence of multiple kinds of initial stresses due to compression, shears, moments and torque on the natural vibration of pre-twisted straight beam based on the Timoshenko theory. Chen[18,19] has found the influence of thickness- to-width ratio, twist angle, spinning speed and axial load on the natural frequency, buckling load and instability zone of a pre-twisted Timoshenko beams by using finite element method. Mohanty, Dash and Rout[20,21] studied static and dynamic behaviour of Timoshenko beam on Winkler's foundation using finite element method.

FGM made of the mixture of turbine material and ceramics can be applied as a coating on the surface of the blade such that the surface of the coated blade is rich in ceramics to improve the performance of the blade in eroding environment. This article presents a study of free vibration of a functionally graded pre-twisted Timoshenko beam of cantilever type.

2. Formulation:

A functionally graded pre-twisted beam is considered for analysis as shown in Fig.1(a). The mid-longitudinal(x-y, x-z) planes are chosen as the reference planes for expressing the displacements as shown in fig. 1(b). The two transverse displacement of a point on the reference plane are, v and w and the corresponding rotation of cross-sections are ϕ and θ. Fig. 1(c) shows a two nodded beam finite element having four degrees of freedom per node.

2.1 Shape functions

The differential equation of motion [2, 3, 4] for the FGSW beam element is given as

$$\frac{\partial M_y}{\partial x} + V_z = -\left(\rho_1 \cos^2 \alpha(x) + \rho_2 \sin^2 \alpha(x)\right)\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{2}(\rho_2 - \rho_1)\sin 2\alpha(x)\frac{\partial^2 \theta}{\partial t^2} \quad (1)$$

$$\frac{\partial M_z}{\partial x} + V_y = \left(\rho_2 \cos^2 \alpha(x) + \rho_1 \sin^2 \alpha(x)\right)\frac{\partial^2 \theta}{\partial t^2} - \frac{1}{2}(\rho_2 - \rho_1)\sin 2\alpha(x)\frac{\partial^2 \phi}{\partial t^2} \quad (2)$$

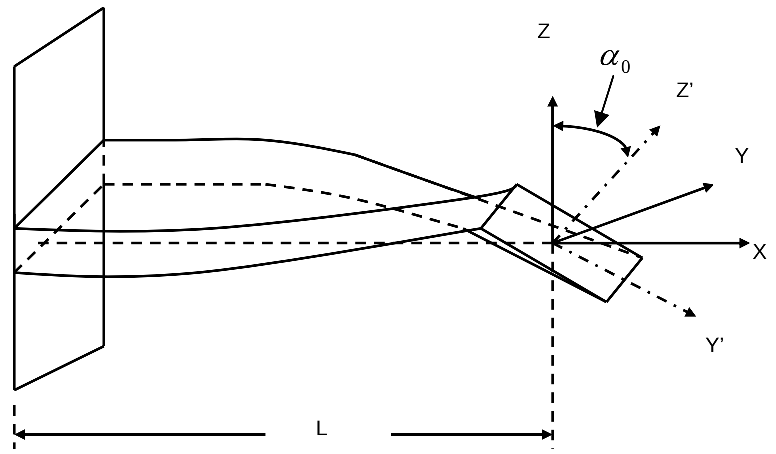

Figure 1(a) Pre-twisted cantilever beam subjected to dynamic axial force.

$$\frac{\partial V_y}{\partial x} = \rho_0 \frac{\partial^2 v}{\partial t^2} \tag{3}$$

$$\frac{\partial V_z}{\partial x} = \rho_0 \frac{\partial^2 w}{\partial t^2} \tag{4}$$

where

$$M_y = -\left(I_1 \cos^2 \alpha(x) + I_2 \sin^2 \alpha(x)\right)\frac{\partial \phi}{\partial x} - \frac{1}{2}(I_2 - I_1)\sin 2\alpha(x)\frac{\partial \theta}{\partial x}$$

$$M_z = \left(I_2 \cos^2 \alpha(x) + I_1 \sin^2 \alpha(x)\right)\frac{\partial \theta}{\partial x} + \frac{1}{2}(I_2 - I_1)\sin 2\alpha(x)\frac{\partial \phi}{\partial x}$$

$$V_z = -G_0\left(\frac{\partial w}{\partial x} - \phi\right)$$

$$V_y = G_0\left(\frac{\partial v}{\partial x} - \theta\right)$$

$$I_1 = \int E(z) z^2 dA,\ I_2 = \frac{I_0 b^2}{12},\ I_0 = \int E(z) dA,$$

$$\rho_2 = \frac{\rho_0 b^2}{12},\ \rho_1 = \int \rho(z) z^2 dA$$

$$\rho_0 = \int \rho(z) dA \quad ,\ G_0 = k\int G(z) dA.$$

$$\alpha(x) = \alpha_0 + \varphi x$$

α_0 is the twist angle at the starting end and φ is the change in twist angle per unit length of the beam.

The material properties along thickness (z-direction) is assumed to vary according to power law as given below.

$$R(z) = (R_t - R_b)\left(\frac{z}{h} + \frac{1}{2}\right)^n + R_b \tag{5}$$

where, $R(z)$ denotes a material property such as, E, G, ρ etc., R_t and R_b denote the values of the properties at topmost and bottommost layer of the beam respectively, and *n* is an index.

Eq.(1,2,3,4) in homogeneous form is given as

$$\frac{\partial M_y}{\partial x} + V_z = 0 \tag{6}$$

$$\frac{\partial M_z}{\partial x} + V_y = 0 \tag{7}$$

$$\frac{\partial V_y}{\partial x} = 0 \tag{8}$$

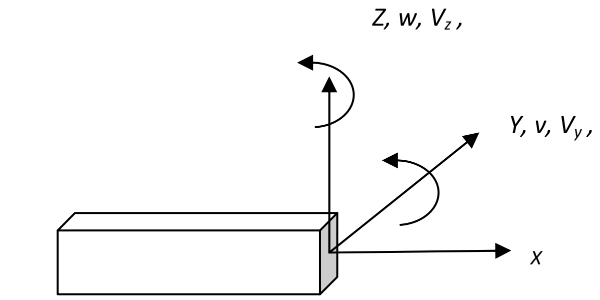

Figure 1(b) The coordinate system with generalized forces and displacements for the FGSW beam element

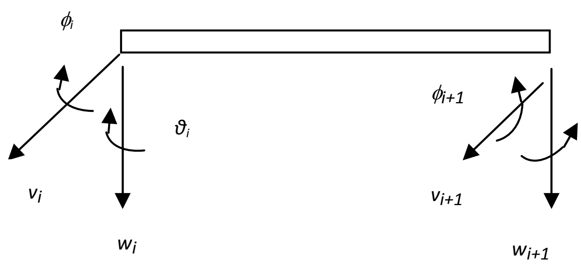

Figure 1(c) Beam element showing generalized degrees of freedom for i^{th} element.

$$\frac{\partial V_z}{\partial x} = 0 \tag{9}$$

The following polynomials may be assumed for the displacement field.

$$\left.\begin{aligned} v &= a_1 + a_2 x + a_3 x^2 + a_4 x^3, \\ w &= a_5 + a_6 x + a_7 x^2 + a_8 x^3, \\ \phi &= b_1 + b_2 x + b_3 x^2, \\ \theta &= b_4 + b_5 x + b_6 x^2. \end{aligned}\right\} \tag{10}$$

$$\{\bar{u}_d\} = \begin{bmatrix} [N_v] \\ [N_w] \end{bmatrix}\{a\} \tag{11}$$

$$\{\bar{u}_r\} = \begin{bmatrix} [N_\phi] \\ [N_\theta] \end{bmatrix}\{b\} \tag{12}$$

where, $\overline{u}_d = \left[v \;\; w \right]^T$ and $\overline{u}_r = \left[\phi \;\; \theta \right]^T$ are deflection and rotation vector respectively.

N_v, N_w, N_ϕ and N_θ are the corresponding shape functions.

The coefficient vector b can be expressed in terms of the coefficient vector a by substituting eq. (16) in eq. (12) and eq. (13) and by equating the coefficients of same powers of x. The mentioned procedure yields the following.

$$\left.\begin{array}{l} b_1 = \beta_1 a_3 + \beta_2 a_4 + a_6 + \beta_3 a_7 + \beta_4 a_8 \,, \\ b_2 = \beta_5 a_4 + 2a_7 + \beta_6 a_8 \,,\; b_3 = 3a_8 \,, \\ b_4 = a_2 + \beta_7 a_3 + \beta_8 a_4 + \beta_1 a_7 + \beta_2 a_8 \,, \\ b_5 = 2a_3 + \beta_9 a_4 + \beta_5 a_8 \,,\; b_6 = 3a_4 \,. \end{array}\right\} \tag{13}$$

Where, $\beta_1 = 2\varphi I_c \cos 2\alpha(x)$,

$\beta_2 = 6\varphi I_c \cos 2\alpha(x)\left(\varphi I_c\left(\sin 2\alpha(x) - \sin 2\alpha(x)\right)\right) + 3I_c \sin 2\alpha(x)$,

$\beta_3 = 2\varphi I_c \sin 2\alpha(x)$, $\beta_4 = 6\varphi^2 I_c^2\left(\sin^2 2\alpha(x) + \cos^2 2\alpha(x)\right)$,

$\beta_5 = 6\varphi I_c \cos 2\alpha(x)$, $\beta_6 = 6\varphi I_c \sin 2\alpha(x)$,

$\beta_7 = -\beta_3$, $\beta_8 = \beta_4$, $\beta_9 = -\beta_6$

$I_c = \left(I_2 - I_1\right)/G_0$

The coefficient vector b can be expressed in terms of the coefficient vector a in matrix form as

$$\{b\} = [B]\{a\} \tag{14}$$

$$[B] = \begin{bmatrix} 0 & 0 & \beta_1 & \beta_2 & 0 & 1 & \beta_3 & \beta_4 \\ 0 & 0 & 0 & \beta_5 & 0 & 0 & 2 & \beta_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & \beta_7 & \beta_8 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & 2 & \beta_9 & 0 & 0 & 0 & \beta_5 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A two noded Timoshenko beam finite element with 4-degree of freedom per node as shown in fig. 1(c) is considered for analysis. The nodal degree of freedom vector of the i[th] element is given as

$$\{\hat{u}\} = \left[v_i \;\; w_i \;\; \phi_i \;\; \theta_i \;\; v_{i+1} \;\; w_{i+1} \;\; \phi_{i+1} \;\; \theta_{i+1} \right]^T \tag{15}$$

Using eq. (16) and eq. (19) the nodal degree of freedom vector can be expressed as

$$\{\hat{u}\} = [C]\{a\}$$

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 & 0 & 1 & \beta_3 & \beta_4 \\ 0 & 1 & \beta_7 & \beta_8 & 0 & 0 & \beta_1 & \beta_2 \\ 1 & L & L^2 & L^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L & L^2 & L^3 \\ 0 & 0 & \beta_1 & \beta_2 + \beta_5 L & 0 & 1 & \beta_3 + 2L & \beta_4 + \beta_6 L + 3L^2 \\ 0 & 1 & \beta_7 + 2L & \beta_8 + \beta_9 L + 3L^2 & 0 & 0 & \beta_1 & \beta_2 + \beta_5 L \end{bmatrix}$$

Substituting eq. (22) in eq. (18) we get

$$\phi = \lfloor N_\phi \rfloor [B]\{a\} \tag{17}$$

$$\theta = [N_\theta][B]\{a\} \tag{18}$$

2.2 Element elastic stiffness matrix

The strain energy of the element [4] is given as

$$S = \frac{1}{2}\int_0^L \left[\begin{array}{l} \left(I_1 \cos^2\alpha(x) + I_2 \sin^2\alpha(x)\right)\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(I_2 - I_1\right)\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \theta}{\partial x}\right)\sin 2\alpha(x) \\ + \left(I_2 \cos^2\alpha(x) + I_1 \sin^2\alpha(x)\right)\left(\frac{\partial \theta}{\partial x}\right)^2 + G_0\left\{\left(\frac{\partial v}{\partial x} - \theta\right)^2 + \left(\frac{\partial w}{\partial x} - \phi\right)^2\right\} \end{array} \right] dx \tag{19}$$

Substituting eq. (9, 10, 17 and 27) into eq. (28) the expression of strain energy is given as

$$S = 0.5\{\hat{u}\}^T [k_e]\{\hat{u}\} \tag{20}$$

Where $[k_e]$ is the element elastic stiffness matrix and is given as follows.

$$[k_e] = \int_0^L [C]^{-T}[k][C]^{-1} dx$$

$$[k] = [B]^T \left[\begin{array}{l} \left(I_1 \cos^2\alpha(x) + I_2 \sin^2\alpha(x)\right)[N'_\phi]^T[N'_\phi] \\ + \left(I_2 \cos^2\alpha(x) + I_1 \sin^2\alpha(x)\right)[N'_\theta]^T[N'_\theta] \\ + \left(I_2 - I_1\right)\sin 2\alpha(x)\left([N'_\phi]^T[N'_\theta] + [N'_\theta]^T[N'_\phi]\right) \end{array} \right][B]$$

$$+ G_0\left[[N'_v]^T[N'_v] + [N'_w]^T[N'_w] + [B]^T\left[[N_\phi]^T[N_\phi] + [N_\theta]^T[N_\theta]\right][B]\right]$$

$$- G_0\left[[B]^T\left[[N_\phi]^T[N'_w] + [N_\theta]^T[N'_v]\right] + \left[[N'_w]^T[N_\phi] + [N'_v]^T[N_\phi]\right][B]\right] \tag{21}$$

2.3 Element mass matrix

The kinetic energy of the beam element [4] is expressed as follows

$$T = \frac{1}{2}\int_0^L \left[\begin{array}{l} \rho_0\left(\left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right) + \left(\rho_1 \cos^2\alpha(x) + \rho_2 \sin^2\alpha(x)\right)\left(\frac{\partial \phi}{\partial t}\right)^2 \\ + 2\left(\rho_2 - \rho_1\right)\left(\frac{\partial \phi}{\partial t}\right)\left(\frac{\partial \theta}{\partial t}\right) + \left(\rho_2 \cos^2\alpha(x) + \rho_1 \sin^2\alpha(x)\right)\left(\frac{\partial \theta}{\partial t}\right)^2 \end{array} \right] dx \tag{22}$$

Substituting eq. (9, 10, 17 and 27) in eq. (32) the kinetic energy is expressed as follows.

$$T = 0.5\{\hat{u}\}^T [m_e]\{\hat{u}\} \tag{23}$$

where $[m]$ is the element mass matrix which is presented below

$$[m_e] = \int_0^L [C]^{-T}[D][C]^{-1} dx \tag{24}$$

$$[D] = \rho_0\left[[N_v]^T[N_v] + [N_w]^T[N_w]\right]$$

$$+ [B]^T \left[\begin{array}{l} \left(\rho_1 \cos^2\alpha(x) + \rho_2 \sin^2\alpha(x)\right)[N_\phi]^T[N_\phi] + \\ \left(\rho_2 - \rho_1\right)\left[[N_\phi]^T[N_\theta] + [N_\theta]^T[N_\phi]\right] \\ + \left(\rho_2 \cos^2\alpha(x) + \rho_1 \sin^2\alpha(x)\right)[N_\theta]^T[N_\theta] \end{array} \right][B] \tag{25}$$

2.4 Governing equation of motion

The equation of motion for the element can be expressed in terms of nodal degrees of freedom as

$$[m_e]\{\ddot{\hat{u}}\} + [k_e]\{\hat{u}\} = 0 \tag{26}$$

Assembling the element matrices as used in eq. (40), the equation in global matrix form which is the equation of motion for the beam, can be expressed as

$$[M]\{\ddot{\hat{U}}\} + [[K_e]]\{\hat{U}\} = 0 \tag{27}$$

$[M]$, $[K_e]$, are global mass, elastic stiffness matrices respectively and $[\hat{U}]$ is global displacement vector.

3. Results and discussion

A steel-alumina FGO beam with steel-rich bottom, fixed at one and free at other is considered for vibration and dynamic analysis. The length, breadth and thickness of the beam are 50.8 cm, 2.54 cm and 0.03175 cm respectively. The material properties of the constituent phases are given below. Properties of steel: E =2.1x10^{11} Pa, G =3/8kE ρ =7.8x10^{3}kg/m^{3}. Properties of alumina: E =3.9x10^{11} Pa, G =3/8kE , ρ =2.707x10^{3}kg/m^{3}. The shear correction factor is assumed as k=0.833.

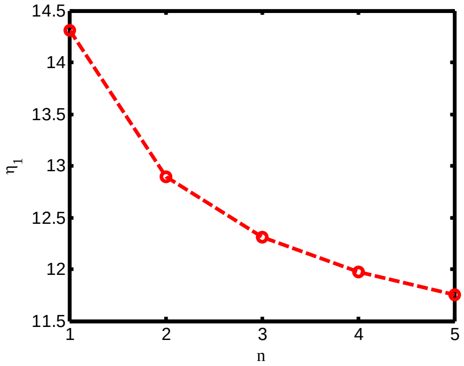

Figure 2(a) Effect of power law index on first mode frequency of steel-alumina pre-twisted FGO beam with steel-rich bottom. Twist angle α_0 =45^{0}.

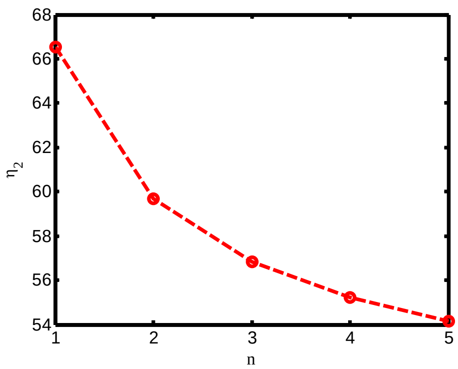

Figure 2(b) Effect of power law index on second mode frequency of steel-alumina pre-twisted FGO beam with steel-rich bottom. Twist angle α_0 =45^{0}.

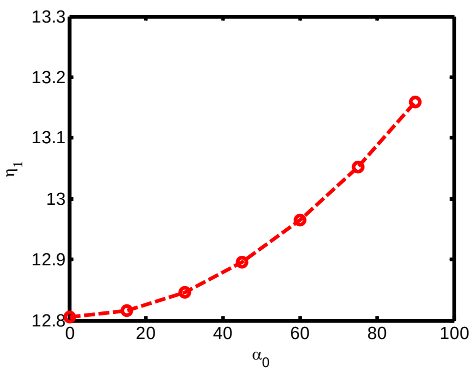

Figure 3(a) Effect of pre-twist angle on first mode frequency of steel-alumina pre-twisted FGO beam with steel-rich bottom. n=2.

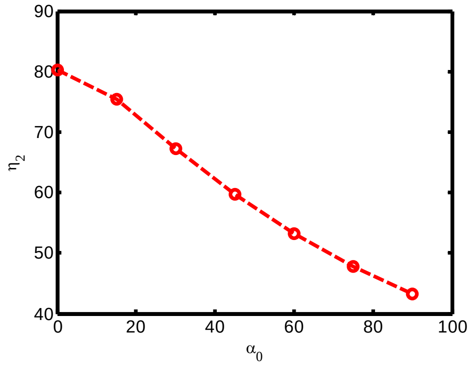

Figure 3(b) Effect of pre-twist angle on second mode frequency of steel-alumina pre-twisted FGO beam with steel-rich bottom. n=2.

The effect of power law index on the natural frequencies of the FGO beam is investigated and is presented in fig. 2(a) and 2(b) for first and second mode respectively. It is observed from the figure that the increase in power law index decreases the first and second mode frequencies. This may be due to the fact that increase in index causes increase in the relative amount of steel which in turn decreases the elastic stiffness of the beam.

The Fig. 3(a) and 3(b) represent the variation of first and second mode frequencies respectively with the pre-twist angle. The increase in pre-twist angle increases the first mode frequency of the beam where as it decreases the second mode frequency of the beam.

4. Conclusion

Finite element method in conjunction with static equilibrium approach is used to investigate the free vibration behaviour of a steel-alumina functionally graded cantilever beam with steel-rich bottom. The material properties along the thickness of the beam are assumed to vary according to power law. Increase in the value of power law index decreases the first two modal frequencies of the beam. Increase in pretwist angle increases first mode frequency and decreases the second mode frequency.

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Nomenclature:

$\{a\}$	Independent coefficient vector
$\{b\}$	Dependent coefficient vector
a_1, a_2, a_3, a_4	Polynomial coefficients of the linear displacement on xy plane
b_1, b_2, b_3, b_4	Polynomial coefficients of the linear displacement on xz plane
I_0, I_1, I_2	Stiffness coefficients
ρ_0, ρ_1, ρ_2	Mass moments
M_y, M_z	Bending moments about Y and Z axis
V_y, V_z	Shear forces along Y and Z axis
R_b	Material property at the bottommost layer
R_t	Material property at topmost layer
$R(z)$	A material property at location z
η	Natural frequency
ϕ	Rotation of cross-section plane about Y-axis
θ	Rotation of cross-section plane about Z-axis
α_0	Twist angle at the free end
φ	Increase in twist angle per unit length
$\{\bar{u}_d\}, \{\bar{u}_r\}$	Deflection and Rotation vectors
$\rho(z)$	Density of material at location z
$[K_e]\ [M]$	Global elastic stiffness and mass matrices
$N_v, N_w, N_\phi, N_\theta$	Shape function matrix
$\{\hat{u}\}$	Nodal displacement vector
$\{\hat{U}\}$	Global nodal displacement vector
$[k_e]\ [m]$	Element elastic stiffness and mass matrices
$[\breve{G}]$	Material constant matrix