Estimating Quantiles of Normal Populations Under Order Restrictions

by

Dr. Manas Ranjan Tripathy

Department of Mathematics
NIT Rourkela, India

Co-author: Dr. Somesh Kumar
Department of Mathematics
IIT Khargapur, India

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Introduction

1 Introduction

- The estimation of ordered parameters when the ordering is unknown arises in various ranking and selection problems. One may refer to Dudewicz and Koo (1982) for a detailed bibliography on these problems. The estimation of parameters when they are known to follow an *a priori* ordering has also been studied by many authors in the past. This problem has applications in various agricultural, industrial, economic and medical experiments.
- In testing for a new drug, doses in increasing order are administered to animals in successive stages. If $\theta_1, \theta_2, \ldots, \theta_k$ denote the mortality rates at k successive stages, then it is expected that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$ for some k. Let $\mu_1, \mu_2, \ldots, \mu_k$ denote the inflation rates of a developing economy when the corresponding petroleum prices are $\theta_1, \theta_2, \ldots, \theta_k$; where $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$. It is then natural to expect that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k$.

Introduction

- A detailed account of results on maximum likelihood estimators (MLEs) of ordered parameters can be found in Barlow et al. (1972) and Robertson et al. (1988).
- Brown and Cohen (1968) introduced the decision theoretic approach in the problem of estimating ordered parameters. They obtained sufficient conditions for the admissibility and minimaxity of the generalized Bayes estimator of the location parameter (μ_1, μ_2) with respect to the Lebesgue prior on the space $\{(\mu_1, \mu_2) : \mu_1 \leq \mu_2\}$.
- Sackrowitz (1970) and Cohen and Sackrowitz (1970) obtained some minimaxity and admissiblity results when estimating the last mean of a monotone sequence for both discrete and continuous cases.
- Sackrowitz and Strawederman (1974) and Sackrowitz (1982) established some admissibility results for the MLE of parameters of binomial distribution when an ordering among the parameters is known *a priori*.

Introduction

- Kumar and Sharma (1988) considered two normal populations $N(\mu_i, \sigma_i^2), i = 1, 2$ with known and common σ_i^2 and $\mu_1 \leq \mu_2$. They considered the simultaneous estimation of (μ_1, μ_2) with respect to the loss function as the sum of squared errors. They introduced a class of mixed estimators and some of these estimators were shown to be minimax. A class of admissible estimators has been obtained within this class. They have also shown that some of these estimators dominate the MLE when the variances are unequal.
- Kumar and Sharma (1989) considered k normal populations with means $\mu_1, \mu_2, \ldots, \mu_k$; $\mu_1 \leq \mu_2 \ldots \leq \mu_k$ and the common variance unity. They proved that the Pitman estimator for estimating $(\mu_1, \mu_2, \ldots, \mu_k)$ is minimax. Also they have discussed the admissibility of the Pitman estimator in certain subclass of estimators. For $k = 2$, the components of the Pitman estimator for estimating μ_1 and μ_2 are minimax but for $k = 3$, the components for estimating μ_1 and μ_3 are shown to be not minimax. The questions of the minimaxity of the second component was settled in affirmative by Kumar and Tripathi (2005) using a simulation study.

Introduction

- For simultaneous estimation of k ordered normal means with unequal but known variances, the Pitman estimator is shown to be minimax with respect to a scale invariant loss function (Kumar and Sharma (1993)), but non-minimax with respect to the loss function as the sum of squared errors (Kumar et al.(2005a)). Further improvements over the Pitman estimator and MLE of ordered normal means are obtained in Kumar et al.(2005b).
- Gupta and Singh (1992) considered two normal populations with means $\mu_1, \mu_2; \mu_1 \leq \mu_2$ and common unknown variance σ^2. They showed that the MLEs of μ_1, μ_2 and σ^2 when $\mu_1 \leq \mu_2$; dominate the standard MLEs of these parameters (that is, when there is no ordering among μ_1, μ_2) with respect to the Pitman nearness criterion.
- Elfessi and Pal (1992) considered estimation of the common mean of two normal populations, when the variances are known to be ordered. They proposed an estimator dominating the well known Graybill-Deal estimator (Graybill and Deal (1959)) in the terms of stochastic dominance. Misra and van der Meulen (1997) generalized the results of Elfessi and Pal (1992) to k normal populations. They also proved dominance of the new estimator with respect to the Pitman nearness criteria.

2 A Minimaxity Result and the Maximum Likelihood Estimator

- Suppose $(X_{11}, X_{12}, \ldots, X_{1n}), (X_{21}, X_{22}, \ldots, X_{2n}), \ldots, (X_{k1}, X_{k2}, \ldots, X_{kn})$ are independent random samples drawn from k normal populations with unknown means $\mu_1, \mu_2, \ldots, \mu_k$ respectively and common unknown variance σ^2. We consider simultaneous estimation of quantiles of the k populations; $\underline{\theta} = (\theta_1, \ldots, \theta_k)$, where $\theta_i = \mu_i + \eta\sigma$; $i = 1, 2, \ldots, k$. The means follow a known ordering $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k$.
- This is equivalent to saying that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_k$. The loss function is taken to be the sum of scale invariant quadratic losses;

$$L(\underline{\mu}, \sigma, \underline{a}) = \sum_{i=1}^{k} \left(\frac{a_i - \theta_i}{\sigma} \right)^2, \tag{2.1}$$

where $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_k), \underline{a} = (a_1, a_2, \ldots, a_k)$.

A Minimaxity Result

- Let $\bar{X}_i = \frac{1}{n}\sum_{j=1}^n X_{ij}$ and $S_i^2 = \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2$ denote the mean and the sum of squares of deviations from the mean respectively of the ith sample. It can be seen that $(\underline{T}, S)$ is a complete sufficient statistic, where $\underline{T} = (\bar{X}_1, \ldots, \bar{X}_k)$ and $S^2 = \sum_{i=1}^k S_i^2$.
- We introduce affine invariance in the estimation problem. Consider the transformation $X_{ij} \to aX_{ij} + b_i$, where $a > 0, b_i \in \mathcal{R}, j = 1, \ldots, n; i = 1, \ldots, k$. The decision problem is invariant under this transformation and this further induces transformations $\bar{X}_i \to a\bar{X}_i + b_i, S^2 \to a^2S^2, \mu_i \to a\mu_i + b_i, \sigma^2 \to a^2\sigma^2, \theta_i \to a\theta_i + b_i, i = 1, \ldots, k$.
- The form of an affine equivariant estimator of $\underline{\theta}$ is seen to be

$$\begin{aligned} \underline{d}(\underline{T}, S) &= \underline{T} + \underline{c}S \\ &= \underline{d}_{\underline{c}} \text{ say,} \end{aligned} \tag{2.2}$$

where $\underline{c} = (c_1, \ldots, c_k) \in \mathcal{R}^k$.

A Minimaxity Result

- The mnimizing choice for $\underline{c}$ has been obtained with respect to the loss (2.1) and is obtained as $\underline{c}^* = \underline{e}\eta b_{k(n-1)+1}$, and $b_\nu = \frac{\Gamma(\nu/2)}{\sqrt{2}\Gamma(\nu+1)/2}$ for $\nu = 2, 3, \ldots$ and $\underline{e} = (1, \ldots, 1)$.
- The estimator $\underline{d}_{\underline{c}^*}$ is minimax for estimating $\underline{\theta}$, when there are no restrictions on μ_i's and the loss function is (2.1); (see Zidek (1971)). The risk of $\underline{d}_{\underline{c}^*}$ is constant and is calculated as

$$
\begin{aligned}
R(\underline{\mu}, \sigma, \underline{d}_{\underline{c}^*}) &= \frac{k}{n} + k\eta^2[1 - k(n-1)b^2_{k(n-1)+1}] \\
&= R, \text{ say.} \qquad (2.3)
\end{aligned}
$$

- Next we prove a general result on the minimaxity of estimators for quantiles $\underline{\theta}$ when μ_i's are known to be ordered. The result is an extension of Theorem 2.1 of Kumar and Sharma (1988), which applies to the problem of simultaneous estimation of ordered location parameters.

A Minimaxity Result

- **Theorem 2.1** *Let* Ω *be a subset of* $\mathcal{R}^k$ *such that there exists a sequence* $\{\underline{a}_n = (a_{n1}, \ldots, a_{nk}) : n \geq 1\}$ *for which*

  (2.4) $\lim_{n\to\infty} \inf\{(\mu_1, \ldots, \mu_k) : (\mu_1 + a_{n1}, \ldots, \mu_k + a_{nk}) \in \Omega\} = \mathcal{R}^k.$

  Let $\underline{d}$ *be an estimator with* $R(\underline{\mu}, \sigma, \underline{d}) \leq R < \infty$, *whenever* $\underline{\mu} \in \Omega, \sigma > 0$, *where R is the constant risk of the estimator* $\underline{d}_{\underline{c}^*}$. *Then* $\underline{d}$ *is a minimax estimator of* $\underline{\theta}$ *for* $\underline{\mu} \in \Omega, \sigma > 0$.

- **Corrolary 2.1** *Let* $\underline{d}$ *be an estimator of* $\underline{\theta}$ *with*

  $$R(\underline{\mu}, \sigma, \underline{d}) \leq R \quad \textit{for} \quad \mu_1 \leq \ldots \leq \mu_k, \sigma > 0,$$

  where R *is the constant risk of* $\underline{d}_{\underline{c}^*}$. *Then* $\underline{d}$ *is a minimax estimator of* $\underline{\theta}$ *for* $\mu_1 \leq \ldots \leq \mu_k, \sigma > 0$.

- **Remark 2.1** *Taking* $\underline{d} = \underline{d}_{\underline{c}^*}$ *in Corollary* 2.1, *it is seen that* $\underline{d}_{\underline{c}^*}$ *is also a minimax estimator of* $\underline{\theta}$ *for* $\mu_1 \leq \ldots \leq \mu_k, \sigma > 0$.

Maximum Likelihood Estimator

- The maximum likelihood estimators of $\underline{\mu}$ and σ, when there are no restrictions on μ_is are $\hat{\mu}_i = \bar{X}_i, i = 1, \ldots, k$; $\hat{\sigma}^2 = \frac{S^2}{kn}$. Using these and the invariance property of MLE, one gets the MLE of $\underline{\theta}$ as $\underline{\hat{\theta}} = (\hat{\theta}_1, \ldots, \hat{\theta}_k)$, where $\hat{\theta}_i = \hat{\mu}_i + \eta\hat{\sigma}, i = 1, \ldots, k$.
- However, when μ_is are ordered, the MLEs of $\underline{\mu}$ and σ^2 get modified. Following representations in Lee (1981) and Gupta and Singh (1992), we can write these restricted MLEs as

$$\tilde{\mu}_i = \min_{i \le t} \max_{s \le i} Av(s,t) \quad \text{and} \quad \tilde{\sigma}^2 = \frac{1}{kn} \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - \tilde{\mu}_i)^2,$$

where $Av(s,t) = \frac{1}{t-s+1} \sum_{j=s}^{t} \bar{X}_j$. Therefore, the MLE of $\underline{\theta}$ under order restrictions on μ_is is $\underline{\tilde{\theta}} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_k)$, where $\tilde{\theta}_i = \tilde{\mu}_i + \eta\tilde{\sigma}, i = 1, \ldots, k$.

Maximum Likelihood Estimator

- For the case of $k = 2$ populations, the expressions for the MLEs of μ_1, μ_2 and σ^2, when $\mu_1 \le \mu_2$, reduce to

$$\begin{aligned} \tilde{\mu}_1 &= \min\{\bar{X}_1, \frac{1}{2}(\bar{X}_1 + \bar{X}_2)\}, \\ \tilde{\mu}_2 &= \max\{\bar{X}_2, \frac{1}{2}(\bar{X}_1 + \bar{X}_2)\}, \\ \text{and} \quad \tilde{\sigma}^2 &= \frac{S^2}{2n}, \quad \text{if } \bar{X}_1 \le \bar{X}_2; \\ &= \frac{S^2}{2n} + \frac{1}{4}(\bar{X}_1 - \bar{X}_2)^2, \quad \text{if } \bar{X}_1 > \bar{X}_2. \end{aligned}$$

- Lee (1981) proved that the restricted MLE $\tilde{\mu}_i$ is better than the usual MLE $\hat{\mu}_i$, $i = 1, \ldots, k$ under the mean squared error criteria. Gupta and Singh (1992) showed that the restricted MLE $\tilde{\sigma}^2$ is better than $\hat{\sigma}^2$ under the squared error as well as Pitman Nearness criterion. We have shown using a simulation study that $\underline{\tilde{\theta}}$ improves over $\underline{\hat{\theta}}$ under the loss function (2.1), when $\mu_1 \le \mu_2$.

3 The Mixed Estimators

- Let $\underline{\delta} = (\delta_1, \delta_2)$ be an estimator of $\underline{\theta}$. When $\theta_1 \le \theta_2$, the mixed estimator of $\underline{\delta}$ is defined by

$$(3.1) \qquad \underline{\delta}_{\alpha^+}(\underline{\delta}) = (\alpha\delta_1 + (1-\alpha)\delta_2, (1-\alpha)\delta_1 + \alpha\delta_2),$$

  where

$$\begin{aligned} \alpha &= 1 \quad \text{when} \quad \delta_1 \le \delta_2, \\ &= \alpha^+ \quad \text{when} \quad \delta_1 > \delta_2, \end{aligned}$$

  where α^+ is a constant. We state a lemma due to Katz (1963) and Kumar and Sharma (1988).

- **Lemma 3.1** *Let the loss function for estimating* $\underline{\theta} = (\theta_1, \theta_2)$ *be of the form* $W(\delta_1 - \theta_1) + W(\delta_2 - \theta_2)$*, where W is a convex, even and nonnegative function. If* $P_{\underline{\theta}}(\delta_1 > \delta_2) > 0$ *for some* $\underline{\theta} = (\theta_1, \theta_2), \theta_1 \le \theta_2$*, then the mixed estimator* $\underline{\delta}_{\alpha^+}(\underline{\delta})$ *given by* (3.1) *improves* $\underline{\delta}$ *when* $\theta_1 \le \theta_2$ *and* $0 \le \alpha^+ < 1$.

The Mixed Estimators

- **Remark 3.1** *It is easily seen that the above lemma is applicable to the quantile estimation problem considered in this paper. Henceforth we will use the term mixed estimator for the mixed estimator of the best affine equivariant estimator* $\underline{d}_{\underline{c}*}$. *Combining Remark* 2.1 *and Lemma* 3.1, *we get the following result.*
- **Theorem 3.1** *For estimating quantiles* $\underline{\theta} = (\theta_1, \theta_2)$ *with respect to the loss function* (2.1), *the class of mixed estimators of* $\underline{d}_{\underline{c}*}$, *that is,* $\{\underline{\delta}_{\alpha^+}(\underline{d}_{\underline{c}*}) : 0 \leq \alpha^+ \leq 1\}$ *is a class of minimax estimators when* $\mu_1 \leq \mu_2$.
- The risk of the mixed estimator is given by

$$\begin{aligned} R(\underline{\mu}, \sigma, \underline{\delta}_{\alpha^+}(\underline{d}_{\underline{c}*})) &= \frac{2}{n}[1 + 2(1-\alpha^+)^2\xi(\phi(\xi) + \xi\Phi(\xi)) - 2\alpha^+(1-\alpha^+)\Phi(\xi)] \\ \text{(3.2)} \qquad & \quad +2\eta^2[1 - 2(n-1)b_{2n-1}^2] \end{aligned}$$

- **Remark 3.2** *Comparing the risk expressions in* (2.4) *and* (3.2) *and using the fact that* $\phi(\xi) + \xi\Phi(\xi) > 0$ *for all* ξ, *it is seen that the risk of* $\underline{\delta}_{\alpha^+}(\underline{d}_{\underline{c}*})$ *is less than or equal to the risk of* $\underline{d}_{\underline{c}*}$ *for* $\xi \leq 0$ *and* $0 \leq \alpha^+ \leq 1$. *This is a direct verification of Theorem* 3.1.

The Mixed Estimators

- **Theorem 3.2** *The estimator* $\underline{\delta}_{\alpha^+}$ *is inadmissible for* $\alpha^+ > \frac{1}{2}$ *and admissible for* $\alpha^+ \leq \frac{1}{2}$ *among the class of mixed estimators.*
- Next, we introduce a new class of estimators following Kumar and Sharma (1988). Consider

$$\begin{aligned} \underline{\delta}^*_\beta &= \underline{T} + \underline{c}^* S, && \text{if } \bar{X}_1 \leq \bar{X}_2, \\ &= (\beta \bar{X}_1 + (1-\beta)\bar{X}_2)\underline{e} + \underline{c}^* S, && \text{if } \bar{X}_1 > \bar{X}_2. \end{aligned} \tag{3.3}$$

- The risk of $\underline{\delta}^*_\beta$ can be evaluated as

$$\begin{aligned} R(\underline{\mu}, \sigma, \underline{\delta}^*_\beta) &= \frac{1}{\sigma^2}\left[\int_{\bar{X}_1 \leq \bar{X}_2} \|\underline{T} + \underline{c}^* S - \underline{\mu} - \eta\sigma\underline{e}\|^2 dP \right. \\ &\quad \left. + \int_{\bar{X}_1 > \bar{X}_2} \|(\beta\bar{X}_1 + (1-\beta)\bar{X}_2)\underline{e} + \underline{c}^* S - \underline{\mu} - \eta\sigma\underline{e}\|^2 dP \right], \end{aligned}$$

where dP denotes the integration with respect to the probability measures of $\bar{X}_1$, $\bar{X}_2$ and S.

The Mixed Estimators

- After some mathematical calculations and simplifications we get,

$$
\begin{aligned}
R(\underline{\mu}, \sigma, \underline{\delta}_\beta^*) &= \frac{2}{n}[1 + (\beta^2 + (1-\beta)^2))g(\xi) - \Phi(\xi)] + 2\eta^2[1 - 2(n-1)b_{2n-1}^2] \\
&\quad + \frac{2\sqrt{2}\eta}{\sqrt{n}}(1-2\beta)(1-2(n-1)b_{2n-1}^2)(\phi(\xi) + \xi\Phi(\xi)). \qquad (3.4)
\end{aligned}
$$

- In the next theorem we obtain a class of minimax estimators for quantiles $\underline{\theta}$.

Theorem 3.3 *Consider the estimators of the form $\underline{\delta}_\beta^*$, given by (3.3) for estimating quantiles $\underline{\theta} = (\theta_1, \theta_2)$ with respect to the loss function (2.1) when $\mu_1 \leq \mu_2$. (i) When $\eta \geq 0$, $\{\underline{\delta}_\beta^* : \frac{1}{2} \leq \beta \leq 1\}$ is a class of minimax estimators. (ii) When $\eta < 0$, $\{\underline{\delta}_\beta^* : 0 \leq \beta \leq \frac{1}{2}\}$ is a class of minimax estimators.*

Proof: Note that the first term on the right hand side of (3.4) achieves its maximum at $\beta = 0$ and $\beta = 1$, which is smaller than $\frac{2}{n}$. Using the fact that $\phi(\xi) + \xi\Phi(\xi) \geq 0$ for all ξ, we get the result.

The Mixed Estimators

- Next we derive an essentially complete class of estimators among estimators of the form $\underline{\delta}^*_\beta$. Define $\beta_0 = \frac{1}{2} + \eta(1 - 2(n-1)b^2_{2n-1})\sqrt{\frac{n}{\pi}}$.
- **Theorem 3.4** (i) *When $\eta \geq 0$, the estimator $\underline{\delta}^*_\beta$ is inadmissible for $\beta < \beta_0$ and admissible for $\beta \geq \beta_0$ among the class of estimators of the form $\underline{\delta}^*_\beta$. (ii) When $\eta < 0$, the estimator $\underline{\delta}^*_\beta$ is inadmissible for $\beta > \beta_0$ and admissible for $\beta \leq \beta_0$ among the class of estimators of the form $\underline{\delta}^*_\beta$.*
- **Remark 3.3** *Note that the mixed estimator $\underline{\delta}_{\alpha^+}(\underline{d}_{\underline{c}*})$ for $\alpha^+ = \frac{1}{2}$ and the estimator $\underline{\delta}^*_\beta$ for $\beta = \frac{1}{2}$ are the same. An application of Theorem 3.4 then proves the following corollary.*
- **Corrolary 3.1** *The mixed estimator $\underline{\delta}_{\frac{1}{2}}(\underline{d}_{\underline{c}*})$ is inadmissible, when $\eta \neq 0$.*

  Proof: Note that $\beta_0 > \frac{1}{2}$ when $\eta > 0$ and $\beta_0 < \frac{1}{2}$ when $\eta < 0$. Theorem 3.4 then yields that $\underline{\delta}_{\frac{1}{2}}(\underline{d}_{\underline{c}*}) = \underline{\delta}^*_{\frac{1}{2}}$ is improved by $\underline{\delta}^*_{\beta_0}$.

4 A Generalized Bayes Estimator

- Let $\underline{X} \sim N(\underline{\mu}, I)$, $\underline{\mu} \in C \subseteq \mathcal{R}^k$, where C is a closed convex set with nonempty interior. Consider estimation of $\underline{\mu}$ with respect to the loss function as the sum of squared errors. Let $\underline{\delta}_J$ denote the generalized Bayes estimator of $\underline{\mu}$ with respect to the non-informative prior (Jeffrey's prior) over C. Hartingan (2004) showed that $\underline{\delta}_J$ improves $\underline{X}$ when $\underline{\mu} \in C$. Motivated by this, we in this section, consider the Jeffrey's improper prior and derive the generalized Bayes estimator of quantiles $\underline{\theta} = (\theta_1, \theta_2)$ with respect to the loss function sum of squared errors.
- The Jeffrey's prior for $(\underline{\mu}, \sigma)$ is obtained as

$$h_1(\underline{\mu}, \sigma) = \frac{1}{\sigma^3}, \quad \mu_1 \leq \mu_2, \; \sigma > 0. \tag{4.1}$$

- The generalized Bayes estimator of $\underline{\theta} = (\theta_1, \theta_2)$ with respect to the loss function as the sum of squared errors is the posterior expectation of $\underline{\theta}$ and is given by

$$\underline{\delta}_J = E(\underline{\theta} | \bar{x}_1, \bar{x}_2, s). \tag{4.2}$$

A Generalized Bayes Estimator

- After some simplifications, for the case $k = 2$, the components of generalized Bayes estimator are given by

$$\delta_{J1} = \frac{\int_0^\infty \{(\bar{x}_1 + \eta\sigma)\Phi(v) - \frac{\sigma}{\sqrt{2n}}\phi(v)\}\frac{1}{\sigma^{2n+1}} e^{-\frac{s^2}{2\sigma^2}}\,d\sigma}{\int_0^\infty \Phi(v)\frac{1}{\sigma^{2n+1}} e^{-\frac{s^2}{2\sigma^2}}\,d\sigma}, \tag{4.3}$$

and

$$\delta_{J2} = \frac{\int_0^\infty \{(\bar{x}_2 + \eta\sigma)\Phi(v) + \frac{\sigma}{\sqrt{2n}}\phi(v)\}\frac{1}{\sigma^{2n+1}} e^{-\frac{s^2}{2\sigma^2}}\,d\sigma}{\int_0^\infty \Phi(v)\frac{1}{\sigma^{2n+1}} e^{-\frac{s^2}{2\sigma^2}}\,d\sigma}, \tag{4.4}$$

where

$$v = \sqrt{\frac{n}{2}}\left(\frac{\bar{x}_2 - \bar{x}_1}{\sigma}\right). \tag{4.5}$$

- **Remark 4.1** *The risk of $\underline{\delta}_J$ has been evaluated using simulations. In some regions of the parameter space it is seen to have satisfactory performance as compared to other estimators considered in this paper.*

5 Some Heuristic Estimators

- Let the prior density function for (μ_1, μ_2) be

$$g_1(\mu_1, \mu_2) = 1, \ \ \mu_1 \leq \mu_2, \tag{5.1}$$

and the variance σ^2 is assumed to be known. Now the sufficient statistics for this problem is $\underline{T} = (\bar{X}_1, \bar{X}_2)$. The generalized Bayes estimator of $\underline{\mu} = (\mu_1, \mu_2)$ is

$$\underline{\delta}_U = E(\underline{\mu} | \bar{x}_1, \bar{x}_2). \tag{5.2}$$

- The components of $\underline{\delta}_U = (\delta_{U1}, \delta_{U2})$ are obtained as

$$\delta_{U1} = \bar{x}_1 - \frac{\sigma}{\sqrt{2n}} \Psi(v), \tag{5.3}$$

and

$$\delta_{U2} = \bar{x}_2 + \frac{\sigma}{\sqrt{2n}} \Psi(v), \ \text{where } \Psi(v) = \frac{\phi(v)}{\Phi(v)}. \tag{5.4}$$

Some Heuristic Estimators

- Now we construct some heuristic estimators for quantiles $\underline{\theta} = (\theta_1, \theta_2)$ when $\mu_1 \leq \mu_2$ by replacing in (5.3) and (5.4), the standard deviation σ by its estimators. If we replace σ by $\frac{S}{\sqrt{2n}}$, we get a heuristic estimator

(5.5) $$\underline{\delta}^1_{HU} = (\delta^1_{HU1}, \delta^1_{HU2})$$

given by

(5.6) $$\delta^1_{HU1} = \bar{X}_1 - \frac{S}{2n}\Psi(v_1) + \eta\frac{S}{\sqrt{2n}},$$

and

(5.7) $$\delta^1_{HU2} = \bar{X}_2 + \frac{S}{2n}\Psi(v_1) + \eta\frac{S}{\sqrt{2n}},$$

where $v_1 = \frac{n(\bar{X}_2 - \bar{X}_1)}{S}$.

Some Heuristic Estimators

- If we replace σ by S_* where

(5.8) $$S_*^2 = \frac{S^2}{2n} + \frac{(\bar{X}_1 - \bar{X}_2)^2}{4},$$

we get the estimator

(5.9) $$\underline{\delta}_{HU}^2 = (\delta_{HU1}^2, \delta_{HU2}^2)$$

given by

(5.10) $$\delta_{HU1}^2 = \bar{X}_1 - \frac{S_*}{\sqrt{2n}} \Psi(v_2) + \eta S_*,$$

and

(5.11) $$\delta_{HU2}^2 = \bar{X}_2 + \frac{S_*}{\sqrt{2n}} \Psi(v_2) + \eta S_*,$$

where $v_2 = \sqrt{\frac{n}{2}}(\frac{\bar{X}_2 - \bar{X}_1}{S_*})$.

Some Heuristic Estimators

- Further, we may use restricted MLE of σ as $\tilde{\sigma}$ to get another heuristic estimator $\underline{\delta}^3_{HU} = (\delta^3_{HU1}, \delta^3_{HU2})$, where

$$\begin{aligned} \underline{\delta}^3_{HUi} &= \underline{\delta}^1_{HUi}, \text{ if } \bar{X}_1 \leq \bar{X}_2 \\ &= \underline{\delta}^2_{HUi}, \text{ if } \bar{X}_1 > \bar{X}_2, \quad i = 1, 2. \end{aligned} \tag{5.12}$$

6 Numerical Comparison

- In this section the risk functions of various estimators derived in previous sections have been tabulated when the loss is given by (2.1). The risk functions of the best affine equivariant estimator $\underline{d}_{\underline{c}^*}$ and the MLE (when there are no restrictions on μ_is) are constant and depend on $|\eta|$ and n only. The risk functions of the mixed estimators $\underline{\delta}_{\alpha+}$ and the estimators $\underline{\delta}^*_{\beta}$ are functions of n, $|\eta|$ and $\tau = \frac{\mu_2-\mu_1}{\sigma}$ with respect to the loss function (2.1). All the risk functions are functions of τ. For numerical evaluations of various risks functions 20000 random samples of size n each were generated from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ populations respectively. For evaluation of integrals in various estimators numerical integration has been used. In Tables 6.1 to 6.4 risk values of estimators $\underline{d}_{\underline{c}^*}$, MLE (with no restrictions) $\underline{\hat{\theta}}$, MLE (with restrictions) $\underline{\tilde{\theta}}$, the mixed estimator $\underline{\delta}_{1/2}$, the estimator $\underline{\delta}^*_{\beta_0}$,the generalized Bayes estimator $\underline{\delta}_J$ and the heuristic estimators $\underline{\delta}^1_{HU}$, $\underline{\delta}^2_{HU}$, $\underline{\delta}^3_{HU}$ have been tabulated for values of $n = 4, 8, 12, 16, 20, 24$; $\eta = 1, 2$ and different values of τ.

Numerical Comparison

The following conclusions can be drawn from Tables 6.1 to 6.4.

(i) The risk functions of all the estimators are increasing in $|\eta|$ and decreasing in n.

(ii) The risk functions of all estimators converge to a certain value as τ increases except for the estimator $\underline{\delta}_{HU}^2$. The risk functions of $\underline{d}_{\underline{c}^*}$ and $\underline{\hat{\theta}}$ are constant with respect to τ. The risk functions of $\underline{\delta}_{1/2}$, $\underline{\delta}_{\beta_0}^*$, $\underline{\tilde{\theta}}$ increases and converge towards a certain value. The risk functions of $\underline{\delta}_J$, $\underline{\delta}_{HU}^1$, $\underline{\delta}_{HU}^3$ first decrease and then increase and converge towards a fixed value as τ increases. The risk function of $\underline{\delta}_{HU}^2$ does not converge.

(iii) The restricted MLE $\underline{\tilde{\theta}}$ uniformly dominates the usual MLE $\underline{\hat{\theta}}$. We conjecture that $\underline{\tilde{\theta}}$ theoretically improves $\underline{\hat{\theta}}$.

(iv) Among estimators $\underline{\hat{\theta}}$, $\underline{d}_{\underline{c}^*}$, $\underline{\delta}_{1/2}$, $\underline{\delta}_{\beta_0}^*$, the estimator $\underline{\delta}_{\beta_0}^*$ was shown to be the best and it is observed in tables also.

Numerical Comparison

(v) The estimator $\underline{\delta}^*_{\beta_0}$ is seen to improve the restricted MLE $\underline{\tilde{\theta}}$ also except for a couple of values of n and τ. We further conjecture that $\underline{\delta}^*_{\beta_0}$ theoretically dominates $\underline{\tilde{\theta}}$.

(vi) The risk performance of the heuristic estimators $\underline{\delta}^1_{HU}$, $\underline{\delta}^2_{HU}$ and $\underline{\delta}^3_{HU}$ is good for some moderate values of τ. In fact for this region they are better than $\underline{\delta}^*_{\beta_0}$ also.

(vii) We recommend using estimator $\underline{\delta}^*_{\beta_0}$ as its performance seems to be the best for all values of n, η and τ. Only for some very specific region of τ, heuristic estimators may be used.

(viii) Similar observations were made for some other values of n and η and we omit the tables here.

List of References

[1] Barlow, R. E., Bartholomew, D.J., Bremner, J.M. and Brunk, H. D.(1972). Statistical Inference Under Order Restrictions, Wiley, New York.

[2] Blumenthal, S. and Cohen, A. (1968). Estimation of two ordered translation parameters. *Ann. Math. Statist.* Vol.39, No.2, 517-530.

[3] Brewster, J. F. and Zidek, J. V. (1974). Improving on equivariant estimators. *Ann. Statist.* Vol. 2, 21-38.

[4] Cohen, A. and Sackrowitz, H. (1970). Estimation of the last mean of a monotone sequence. *Ann. Math. Statist.* Vol. 41, No.6, 2021-2034.

[5] Dudewicz, E. J. and Koo, J. O.(1982). Complete Catagorized Guide to Statistical Selection and Ranking Procedures, *American Sciences Inc.*, Columbus, Ohio.

[6] Elfessi, A. and Pal, N. (1992). A note on the common mean of two normal populations with order restricted variances, *Commun. Statist. Theo. Meth.* Vol.21, No.11, 3177-3184.

[7] Graybill, F. A. and Deal, R. B. (1959). Combining unbiased estimators. *Biometrics* Vol. 15, 543-550.

[8] Gupta, R. D. and Singh, H. (1992). Pitman nearness comparisons of estimates of two ordered normal means. *Austral. J. Statist.* Vol. 34, No. 3, 407-414.

[9] Hartigan, J.A. (2004). Uniform priors on convex sets improve risk. *Statist. Prob. Lett.* Vol. 67, 285-288.

[10] Kumar, S. (1988). Some problems of estimation in restricted parameter spaces. *Ph.D. dissertation, IIT Kanpur, India.*

[11] Kumar, S. and Sharma, D. (1988). Simultaneous estimation of order parameters. *Commun. Statist. Theor. Meth.* Vol. 17, No. 12, 4315-4336.

[12] Kumar, S. and Sharma, D. (1989). On the Pitman estimator of ordered normal means. *Commun. Statist. Theor. Meth.* Vol. 18, No. 11, 4163-4175.

[13] Kumar, S. and Sharma, D. (1993). Minimaxity of the Pitman estimator of ordered normal means when the variances are unequal. *J. Ind. Soc. Ag. Statist.* Vol. 45, No.2, 230-234.

[14] Kumar, S. and Tripathi, Y. M. (2005). Estimating components of a normal mean vector under order restrictions. *Int. J. Appl. Math. Statist.* Vol. 3, No. J05, 82-96.

[15] Kumar, S., Kumar, A. and Tripathi, Y. M. (2005a). A note on the Pitman estimator of ordered normal means when the variances are unequal. *Commun. Statist. Theo. Meth.* Vol. 34, 2115-2122.

[16] Kumar, S., Tripathi, Y. M. and Misra, N. (2005b). James-Stein type estimators for ordered normal means. *J. Statist. Comp. Simul.* Vol. 75, No. 7, 501-511.

[17] Lecam, L. (1955). An extension of Wald's theory of statistical decision functions. *Ann. Math. Statist.* Vol. 26, 69-81.

[18] Lee,C.I.C. (1981). The quadratic loss of isotonic regression under normality. *Ann. Statist.* Vol. 9, 686-688.

[19] Misra, N and van der Meulen, E. C. (1997). On estimation of the common mean of k (≥ 2) normal populations with order restricted variances. *Statist. Prob. Let.* Vol. 36, 261-267.

[20] Robertson, T., Wright, F. T. and Dykstra, R. L.(1988). Order Restricted Statistical Inference (New York: Wiley).

[21] Sackrowitz, H. (1970). Estimation for monotone parameter sequences: The discrete case. *The Ann. Math. Statist.* Vol. 41, No. 2, 609-620.

[22] Sackrowitz, H.(1982). Procedures for improving upon the MLE of ordered binomial parameters. *J. Statist. Plann. Inf.* Vol. 6, 287-296.

[23] Sackrowitz, H. and Strawderman, W. (1974). On the admissibility of the MLE for ordered binomial parameters. *The Ann. Statist.* Vol. 2, No. 4, 822-828.

[24] Zidek, J.V. (1971). Inadmissibility of a class of estimators of a normal quantile. *The Ann. Math. Statist.* Vol. 42, No. 4, 1444-1447.

THANK YOU