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Thermal analysis of an infinite slab during quenching

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SUMMARY

This paper deals with finite-element solution of two-dimensional convection–diffusion equation in an infinite domain, arising out of quenching of an infinite slab. The solution gives the quench front temperature as a function of various model parameters, such as Biot number and Peclet number. The results show good agreement with available closed-form solutions, thus validates the numerical procedure adopted. It is therefore expected that the present method of solution may be extended to quenching problems involving heat generation and precursory cooling, etc., in various other geometries.

KEY WORDS: heat transfer; quenching; upwind FEM; infinite domain

INTRODUCTION

The process of quenching of hot surfaces is of practical importance in Nuclear, Chemical and Metallurgical industries. For example, a hypothetical loss-of-coolant accident (LOCA) in water-cooled nuclear reactors may result in rapid heating of the fuel channels. In order to prevent the fuel from reaching a metallurgical prohibitive temperature, an emergency core cooling system is activated to reflood the core. The time delay in re-establishing the effective cooling may result in a rise of cladding temperature significantly above the saturation temperature. If the cladding temperature rises above the rewetting temperature, a stable vapour blanket will prevent immediate return to liquid–solid contact. Rewetting is the re-establishment of liquid contact with a hot cladding surface and thereby bringing to an acceptable cladding temperature. Such quenching phenomena also exists in numerous industrial applications, such as metallurgical quenching, drying out of steam generators or quenching of a container wall during filling with cryogenic fluid. The cooldown process during the quenching is characterized by the formation of a wet patch on the hot surface, which eventually develops into a steadily moving quench front. As the quench front progresses along the hot solid, the upstream end of the solid is cooled by convection to the contacting liquid, while its downstream end is cooled by heat transfer to a mixture of vapour and entrained liquid droplets, called precursory cooling.

Estimation of rewetting (quench front) temperature is essential in predicting the rate at which the coolant quenches a hot surface. The rewetting models for two-dimensional quasi-static heat transfer with a step change in heat transfer coefficient at the quench front have been solved for the single slab [1] and for the composite slab [2]. The one-dimensional transient rewetting equation with a constant boundary heat flux has been solved by Chan and Zhang [3]. The common solution methods employed are either separation of variables or Wiener–Hopf technique. The main difficulty in solving the governing equation numerically is due to infinite domain of the slab and prescription of the spatial boundary conditions at infinity. This problem can be alleviated by mapping the infinite physical domain to a finite one. One of the approach in infinite–finite mapping involves using infinite elements with an appropriate singular shape function [4–5]. In the present paper an alternative method has been suggested, in which the governing equation is transformed by a suitable mapping function. The value of the stretching parameter associated with such a transformation has been found by minimizing the overall heat balance. The numerical procedure proposed herewith involves Galerkin weighted residual method incorporating an upwinded weighting function and then solving the simultaneous algebraic equations iteratively. The numerical model is validated by comparing the results with known analytical solutions.

MATHEMATICAL MODEL

The two-dimensional transient heat conduction equation for the slab is

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\rho C\frac{\partial T}{\partial t}, \quad -\delta<x<\delta \text{ and } 0<y<L,\ L\to\infty \tag{1}$$

where L is the length of the slab and 2δ is its thickness. The density, specific heat and thermal conductivity of the slab material are ρ, C and k, respectively. The origin of the co-ordinate frame is at bottom mid-point of the slab. Because of symmetry about y-axis, one-half of the slab (of thickness δ) is considered in the analysis. The plane of symmetry then becomes equivalent to an adiabatic wall. To convert this transient equation into quasi-steady-state form, the following transformation is used:

$$\bar{x}=x, \quad \bar{y}=y-ut$$

where u is the constant rewetting velocity and $\bar{x}$ and $\bar{y}$ are normal and axial co-ordinates, respectively (Figure 1). Thus the transformed heat conduction equation in a co-ordinate system moving with the quench front at this velocity is

$$\frac{\partial^2 T}{\partial \bar{x}^2}+\frac{\partial^2 T}{\partial \bar{y}^2}+\frac{\rho C u}{k}\frac{\partial T}{\partial \bar{y}}=0, \quad 0<\bar{x}<\delta, \quad -\infty<\bar{y}<\infty \tag{2}$$

In conduction-controlled rewetting analysis, it is believed that the advance of the quench front is attributed to axial conduction from the dry region ahead of the quench front to the wet region. Since only axial conduction is considered, the effect of coolant mass flux, coolant inlet subcooling and its pressure gradient, etc., are not accounted for explicitly, but only implicitly in terms of wet-region heat transfer coefficient. In the present analysis, the heat transfer coefficient h is assumed to be constant over entire wet region. The coolant temperature is taken to be equal to its saturation temperature T_s. On the dry side of the slab, the wall is cooled by the surrounding vapor, which

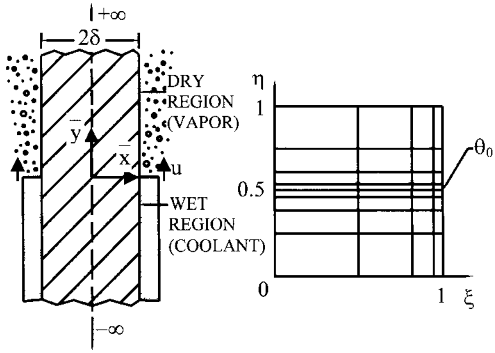

Figure 1. Physical and computational domain of infinite slab.

may be assumed to be adiabatic. Moreover it is assumed that the far upstream of the quench front (at $\bar{y} \to -\infty$), the wet region has been quenched to a temperature T_s, while the far prequenched zone (at $\bar{y} \to +\infty$) is still at the initial slab temperature T_w. Equation (2) can be expressed in the following dimensionless form:

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} + Pe\frac{\partial\theta}{\partial Y} = 0, \quad 0 < X < 1, \quad -\infty < Y < \infty \tag{3}$$

The associated boundary conditions are

$$\begin{aligned}
&\frac{\partial\theta}{\partial X} = 0 \quad \text{at } X = 0, \quad -\infty < Y < \infty \\
&\frac{\partial\theta}{\partial X} + Bi\,\theta = 0 \quad \text{at } X = 1, \quad Y < 0 \\
&\frac{\partial\theta}{\partial X} = 0 \quad \text{at } X = 1, \quad Y > 0 \\
&\theta = 0 \quad \text{at } Y \to -\infty \\
&\theta = 1 \quad \text{at } Y \to +\infty
\end{aligned} \tag{4}$$

The dimensionless quantities used above are

$$X = \frac{\bar{x}}{\delta}, \quad Y = \frac{\bar{y}}{\delta}, \quad \theta = \frac{T - T_s}{T_w - T_s}, \quad Bi = \frac{h\delta}{k} \quad \text{and} \quad Pe = \frac{\rho C u \delta}{k}$$

The main interest of the present analysis is to solve for the quench front temperature T_0 for the given values of Biot (Bi) and Peclet (Pe) numbers. The dimensionless quench front temperature is defined by

$$\theta_0 = \frac{T_0 - T_s}{T_w - T_s} = \theta(1, 0)$$

The infinite physical domain ($-\infty < Y < +\infty$) is then mapped to a finite computational domain (Figure 1) by the following infinite–finite transformation:

$$\xi = X, \quad \eta = 0.5(1 + \tanh \beta Y)$$

where β is the stretching parameter. The rationale of such a transformation is that the analytical boundary conditions at infinity can be used in the discretized equations. The convection–diffusion Eq. (3) is thus transformed to

$$\frac{\partial}{\partial \xi}\left(\frac{1}{\eta_y}\frac{\partial \theta}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\eta_y\frac{\partial \theta}{\partial \eta}+Pe\,\theta\right)=0 \tag{5}$$

where $\eta_y=\partial\eta/\partial Y=2\beta\eta(1-\eta)$. The transformed boundary conditions are

$$\begin{aligned}
&\frac{\partial \theta}{\partial \xi}=0 \quad \text{at } \xi=0, \quad 0<\eta<1\\
&\frac{\partial \theta}{\partial \xi}+Bi\,\theta=0 \quad \text{at } \xi=1, \quad \eta<0.5\\
&\frac{\partial \theta}{\partial \xi}=0 \quad \text{at } \xi=1, \quad \eta>0.5\\
&\theta=0 \quad \text{at } \eta=0\\
&\theta=1 \quad \text{at } \eta=1\\
&\theta=\theta_0 \quad \text{at } \xi=1, \quad \eta=0.5
\end{aligned} \tag{6}$$

NUMERICAL SOLUTION

The elliptic scalar field Equation (5) can be solved by the conventional Galerkin weighted residual method (weighting and shape function being identical). However, serious difficulties may be encountered with dominant convective (first derivative) terms [6]. Finite element solution of Equation (5) may be oscillatory if the element size exceeds a critical value and at high Peclet numbers acceptable answers can only be obtained by excessive reduction in the element size. Heinrich *et al.* [6] showed that with particular choice of weighting functions, with orders higher than that of shape functions, it is possible to generate the upwind scheme. Employing Galerkin upwind scheme (weighting and shape function being different), Equation (5) can be integrated to

$$\begin{aligned}
&\int_A\left(D_x\frac{\partial[W]^{\mathrm{T}}}{\partial \xi}\,\frac{\partial[N]}{\partial \xi}\right)\mathrm{d}A\{\theta^{(e)}\}+\int_A\left(D_y\frac{\partial[W]^{\mathrm{T}}}{\partial \eta}\,\frac{\partial[N]}{\partial \eta}\right)\mathrm{d}A\{\theta^{(e)}\}\\
&-Pe\int_A\left([W]^{\mathrm{T}}\frac{\partial[N]}{\partial \eta}\right)\mathrm{d}A\{\theta^{(e)}\}-\int_\Gamma\left(D_x[W]^{\mathrm{T}}\frac{\partial \theta}{\partial \xi}\cos\phi+D_y[W]^{\mathrm{T}}\frac{\partial \theta}{\partial \eta}\sin\phi\right)\mathrm{d}\Gamma=0
\end{aligned} \tag{7}$$

where $D_x=1/\eta_y$ and $D_y=\eta_y$. The element stiffness matrices are then developed by integrating Equation (7), using bilinear four-noded isoparametric rectangular elements in a natural coordinate system. Within the element, temperature can be interpolated in terms of nodal temperatures by $\theta=\sum N_i\theta_i$ (for $i=1,\ldots,4$); where θ is the temperature at any point in the element and θ_i is its value at ith node of that particular element. The following definitions for Weighting function (W_i) and Shape function (N_i) in a natural coordinate system (r, s) are adopted from Reference [6]:

$$N_i = \tfrac{1}{4}(1 + rr_i)(1 + ss_i)$$

$$W_i = \frac{(1 + rr_i)}{2}\left[\frac{(1 + ss_i)}{2} - \frac{3}{4}\alpha s_i(1 - s^2)\right]$$

where i is the number of functions with $r_i = -1, 1, 1, -1$ and $s_i = -1, -1, 1, 1$ for $i = 1, \ldots, 4$, respectively. The upwinding parameters are defined as

$$\alpha = \coth(\gamma/2) - (2/\gamma)$$

$$\gamma = Pe\, b/D_y$$

where a and b are sides of the rectangular element in ξ and η direction, respectively. The non-dimensional co-ordinates of the isoparametric elements for nodes 1–4 are $(-1, -1), (1, -1), (1, 1)$ and $(-1, 1)$, respectively. The natural co-ordinates are defined as

$$r = 2(\xi - \xi_c)/a$$

$$s = 2(\eta - \eta_c)/b$$

where ξ_c, η_c are co-ordinates of the centroid of each element. The element stiffness matrices are thus obtained as

$$[K_x] = \left(\frac{D_x b}{6a}\right)\begin{bmatrix} 2+1.5\alpha & -(2+1.5\alpha) & -(1+1.5\alpha) & 1+1.5\alpha \\ -(2+1.5\alpha) & 2+1.5\alpha & 1+1.5\alpha & -(1+1.5\alpha) \\ -(1-1.5\alpha) & 1-1.5\alpha & 2-1.5\alpha & -(2-1.5\alpha) \\ 1-1.5\alpha & -(1-1.5\alpha) & -(2-1.5\alpha) & 2-1.5\alpha \end{bmatrix}$$

$$[K_y] = \left(\frac{D_y a}{6b}\right)\begin{bmatrix} 2+\gamma(1+\alpha) & 1+\frac{\gamma}{2}(1+\alpha) & -1-\frac{\gamma}{2}(1+\alpha) & -2-\gamma(1+\alpha) \\ 1+\frac{\gamma}{2}(1+\alpha) & 2+\gamma(1+\alpha) & -2-\gamma(1+\alpha) & -1-\frac{\gamma}{2}(1+\alpha) \\ -1+\frac{\gamma}{2}(1-\alpha) & -2+\gamma(1-\alpha) & 2-\gamma(1-\alpha) & 1-\frac{\gamma}{2}(1-\alpha) \\ -2+\gamma(1-\alpha) & -1+\frac{\gamma}{2}(1-\alpha) & 1-\frac{\gamma}{2}(1-\alpha) & 2-\gamma(1-\alpha) \end{bmatrix}$$

$$[K_M] = \left(\frac{Bi\, D_x b}{6}\right)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2+1.5\alpha & 1+1.5\alpha & 0 \\ 0 & 1-1.5\alpha & 2-1.5\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For internal elements as well as boundary elements on the dry side, $([K_x] + [K_y])\{\theta^{(e)}\} = 0$. For the boundary elements on the wet side, $([K_x] + [K_y] + [K_M])\{\theta^{(e)}\} = 0$.

Although standard elimination routines (e.g. Frontal or Skyline) do exist to solve a set of simultaneous equations with an unsymmetric matrix, an iterative solution has been adopted in the present case. The advantage of an iterative solution may be recognized in case of re-solution, where the previous solution can be a good starting matrix for the next solution. The nine-point representation of elliptic scalar field Equation (5) can be written in the general form as

$$A^9_{i,j}\theta_{i,j} = A^1_{i,j}\theta_{i,j+1} + A^2_{i,j}\theta_{i+1,j+1} + A^3_{i,j}\theta_{i+1,j} + A^4_{i,j}\theta_{i+1,j-1} + A^5_{i,j}\theta_{i,j-1} + A^6_{i,j}\theta_{i-1,j-1} + A^7_{i,j}\theta_{i-1,j} + A^8_{i,j}\theta_{i-1,j+1} \quad (8)$$

The coefficients $A_{i,j}$ of Equation (8) have been found by assembling the element stiffness matrices at each individual element level and thus the formation of a global temperature matrix has been avoided. The simultaneous algebraic equations thus formed are then solved by modified strongly implicit procedure [7], which employs the strategy of $[L][U]$ decomposition because of the sparseness of the resulting matrix. The present analysis adopts the above block iterative procedure as it seems to be computationally faster than the point iterative schemes. All computations have been carried out using a non-uniform structured mesh system with 20×160 elements. Since steep temperature gradients are encountered near the quench front, a mesh system has been adopted with finer elements near the quench front and progressively coarser elements away from it (Figure 1). A convergence criterion of 0.01 per cent change in θ at all nodes has been selected to test the convergence of the iterative scheme. Sample calculations were also carried out by doubling the mesh size to ensure that the results are independent of mesh system.

Since the main objective of the present study is to estimate the quench front temperature as correctly as possible, it is essential that the temperature field satisfies the heat balance. This is done by integrating Equation (5) over the entire computational domain, that gives

$$Pe = Bi \int_0^{0.5} \frac{\theta}{2\beta\eta(1-\eta)} \, d\eta \quad (9)$$

The heat balance Equation (9) has been derived with the assumption that the temperature field is sufficiently flat at far upstream ($\eta = 0$) and downstream ($\eta = 1$) of the quench front, so that the far-field temperature gradient in the η direction may be neglected. The integral on the right-hand side of Equation (9) has been calculated by Simpson's 1/3 rule. Since the integral becomes improper at $\eta = 0$, the indeterminate form at this location has been avoided by applying L'Hospital rule. The absolute difference between the right- and left-hand sides of the above equation is first divided by minimum of the two values and then multiplied by 100 to get the percentage difference.

If the percentage heat balance so obtained is assumed to be objective function, then the stretching parameter associated with mapping can be treated as an independent variable. Thus, starting from an arbitrary base point ($\beta > 0$), the variable can be moved towards an optimum based on sequential minimization of the objective function. To reduce the number of function evaluations, an optimization technique (Golden Section Search) is used that does not require the derivative of the function. A tolerance limit of 0.01 per cent change of the function value has been selected, below which the search process is terminated. The heat balance, achieved through minimization as above, establishes the accuracy of the temperature field obtained and determines the optimal value of the stretching parameter, β.

RESULTS AND DISCUSSION

The dependence of quench front temperature on Peclet number for various values of Biot number is shown in Figure 2. θ_0 is found to increase with increase in Pe and with decrease in Bi, reflecting the

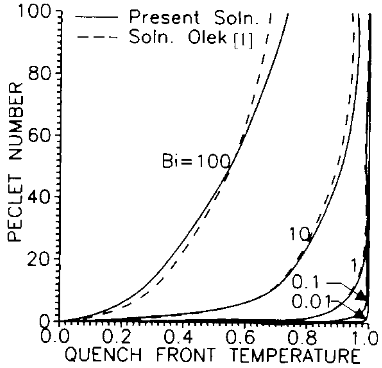

Figure 2. Variation of quench front temperature for various Biot and Peclet numbers.

fact that a quench front progresses more easily when the heat transfer to the coolant is increased. The solution has been compared with the analytical solution of Olek [1]. The numerical results are in good agreement with the analytical ones for low Biot numbers, while the accuracy deteriorates as the Biot number becomes large. This is probably due to the existence of the mismatch boundary conditions (normal temperature gradient, i.e. $\partial\theta/\partial\xi$) at the quench front as evident from Equation (6). Apparently, the strength of the discontinuity increases with increase in Biot number and the accuracy of the solution deteriorates with increase in *Bi*. The present finite element solution may be beneficial in case of solving the non-linear rewetting equations arising due to temperature-dependent thermo-physical properties, whereas a plausible analytical solution may not exist.

CONCLUSION

A numerical solution for solving infinite domain problems arising out of rewetting analysis has been suggested. The infinite physical domain can be mapped to a finite computational domain by transforming the governing equation. The value of the stretching parameter used for infinite–finite transformation can be obtained by minimizing the heat balance. Quench front temperature is observed to increase with increase in Peclet number and with decrease in Biot number. It is felt that the present solution procedure, in principle, may be extended to other infinite domain rewetting problems in various other geometries.

APPENDIX: NOTATION

a, b	element sides
Bi	Biot number
C	specific heat
h	heat transfer coefficient
k	thermal conductivity
$[K]$	stiffness matrix
L	length of the slab

N shape function
Pe Peclet number
r, s natural co-ordinate system
t time
T temperature
u quench front velocity
W weighting function
x, y physical co-ordinates
$\bar{x}, \bar{y}$ co-ordinates in quasi-steady state
X, Y dimensionless co-ordinates in quasi-steady state

Greek symbols

α, γ upwind parameter
β stretching parameter
δ half-thickness of the slab
ρ density
θ dimensionless temperature
ξ, η co-ordinates after infinite-finite transformation

Subscripts

s saturation condition
w wall condition
0 quench front

REFERENCES

1. Olek S. On the two-region rewetting model with a step change in heat transfer coefficient. *Nuclear Engineering Design* 1988; **108**:315–322.
2. Olek S. Quenching of a composite slab. *International Communications on Heat Mass Transfer* 1994; **21**:333–344.
3. Chan SH, Zhang W. Rewetting theory and the dry out heat flux for smooth and grooved plates with a uniform heating. *Transactions of the ASME. Journal of Heat Transfer* 1994; **116**:173–179.
4. Bettess P. Infinite Elements. *International Journal for Numerical Methods in Engineering* 1977; **11**:53–64.
5. Zienkiewicz OC, Morgan K. *Finite Elements and Approximation.* Wiley: New York, 1983; 220.
6. Heinrich JC, Huyakorn PS, Zeinkiewicz OC. An upwind finite element scheme for two-dimensional convective transport equation. *International Journal for Numerical Methods in Engineering* 1977; **11**:131–143.
7. Schneider GE, Zedan M. A modified strongly implicit procedure for the numerical solution of field problems. *Numerical Heat Transfer* 1981; **4**:1–19.