

# NARMAX Modeling of a Two-Link Flexible Robot

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**Abstract**— Modeling of a two-link flexible robot as a nonlinear autoregressive, moving average with exogenous input (NARMAX) model is considered in this paper. The advantage of using NARMAX model over functional series representation for complex nonlinear systems is well established due to less number of parameters involved in the former. Next, the real-time identification for the NARMAX model parameters is acquired by using the recursive extended least square (RELS) algorithm. Model validation for a two-link flexible robot in real time is established by operating the robot under variable payload conditions. The experimental results show that the proposed NARMAX method provides better identification of the TLFR dynamics compared to ARMAX model.

**Keywords**- NARMAX, Nonlinear Adaptive Control, Recursive Extended Least Square, Two-Link Flexible Robot, Variable Payload

## I. INTRODUCTION

Flexible-link robots are more suitable for some specific applications such as space robots compared to rigid-link robots due to their low inertia, increased payload carrying capacity and high maneuvering speed. However, exact modeling of a flexible-link robot is difficult owing to distributed link flexibility. So, in order to control and predict its behavior under different operating conditions, accurate identification of the flexible-link robot is necessary.

Different identification methods and models have been applied to estimate a flexible robot dynamics. Yurkovich et.al presented an identification and control of a single-link flexible robot using on-line frequency domain linear model in [1], and an ARMA model with weighted recursive least square (RLS) algorithm for parameter adaptation in [2]. Yazdizadeh et.al proposed a dynamic neural network using dynamic neurons for identification of a two-link flexible robot in [3]. The main disadvantages of the above methods are that linear model is considered, but due to distributed link flexibility a flexible-link robot is inherently infinite dimensional one.

Recent works by Kukreja et.al [4] and Tsai et.al [5] using NARMAX model based identification for nonlinear complex systems show outstanding results. Also there have no approach till now to identify a TLFR using NARMAX modeling real-time. Motivate by the above mentioned reasons, an attempt has been made to represent the complex dynamics of a flexible robot in an accurate mathematical

form using NARMAX model proposed by Chen and Billings in [6]. To verify the nonlinearity handling capacity of the NARMAX model the results are also compared with an ARMAX model under similar variable load operation by the TLFR.

## II. DYNAMICS OF THE TWO-LINK FLEXIBLE ROBOT

Using the Euler-Bernoulli beam theory, a partial differential equation for deformation in the link due to flexibility can be written as [7]

$$(EI)_i \frac{\partial^4 d_i(x_i, t)}{\partial x_i^4} + (\rho)_i \frac{\partial^2 d_i(x_i, t)}{\partial t^2} = 0, \quad i = 1, 2 \quad (1)$$

where  $(EI)_i$  is the flexural rigidity of the  $i^{\text{th}}$  link,  $\rho_i$  is uniform density of the  $i^{\text{th}}$  link. The deflection along the link  $I$  is given by

$$d_i(x_i, t) = \sum_{i=1}^2 \psi_i(x_i) \delta_i(t), \quad i = 1, 2 \quad (2)$$

$\delta_i(t)$  is the time varying flexible mode of  $i^{\text{th}}$  link and  $\psi_i(x_i)$  is the eigen-function along the length of link  $i$ . Using assumed mode method and Lagrangian dynamics, a state space representation of the dynamics of the TLFR is given by [7].

$$M(\theta_i, \delta_i) \begin{bmatrix} \ddot{\theta}_i \\ \ddot{\delta}_i \end{bmatrix} + \begin{bmatrix} \mathbf{c}_1(\theta_i, \delta_i, \dot{\theta}_i, \dot{\delta}_i) \\ \mathbf{c}_2(\theta_i, \delta_i, \dot{\theta}_i, \dot{\delta}_i) \end{bmatrix} + K \begin{bmatrix} 0 \\ \delta_i \end{bmatrix} + D \begin{bmatrix} 0 \\ \dot{\delta}_i \end{bmatrix} = \begin{bmatrix} \tau_i \\ 0 \end{bmatrix} \quad (3)$$

where  $\theta_i$  is the joint angle of the  $i^{\text{th}}$  joint,  $M$  is the positive-definite symmetric inertia matrix,  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are the vectors consists of Coriolis and Centrifugal forces,  $K$  is the stiffness matrix and  $D$  is the damping matrix. Matrix  $M$  and vector  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are skew symmetric [7]. If the output is taken as tip position, the overall manipulator system becomes non-minimum phase [8]. Hence, a new output position is proposed in [8] to make the system minimum phase. To define the new output as close as possible to the tip position, the sum of joint angle and a scaling of the tip elastic deflection is taken as the output for control of each link [8] where  $l_i$  length of the  $i^{\text{th}}$  link and  $-1 <$

$$y_{pi} = \theta_i + \alpha_i \left[ \frac{d_i(l_i, t)}{l_i} \right] \quad (4)$$

$\alpha_i < 1$ , for different values of  $\alpha_i$  corresponds to different points along link  $i$ . TLFR dynamics using the redefined output stated in eq. (3), the state space representation of the TLFR is made in following form

$$\dot{x} = f(x) + g(x)u \quad (5)$$

where  $x$  is a state vector defined as  $x = [\theta_i, \dot{\theta}_i, \delta_i, \dot{\delta}_i]$  and

$$f(x) = M(\theta_i, \delta_i)^{-1} \left( - \begin{bmatrix} \mathbf{c}_1(\theta_i, \delta_i, \dot{\theta}_i, \dot{\delta}_i) \\ \mathbf{c}_2(\theta_i, \delta_i, \dot{\theta}_i, \dot{\delta}_i) \end{bmatrix} - K \begin{bmatrix} 0 \\ \delta_i \end{bmatrix} - D \begin{bmatrix} 0 \\ \dot{\delta}_i \end{bmatrix} \right)$$

$$g(x) = M(\theta_i, \delta_i)^{-1} \text{ and } u = \begin{bmatrix} \tau_i \\ 0 \end{bmatrix}$$

using the expression for  $d_i(x_i, t)$  from eq. (2) in (4) one can obtain

$$y_{pi} = \theta_i + \alpha_i \left[ \frac{\psi_i(l_i) \delta_i(t)}{l_i} \right] \quad (6)$$

using eq. (6) and rewriting eq. (5) with redefined output vector  $\tilde{x} = [y_{pi}, \dot{y}_{pi}]$  as

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})u \quad (7)$$

where

$$\tilde{f}(\tilde{x}) = f_1(x) + \alpha_i \left[ \frac{\psi_i \times f_2(x)}{l_i} \right],$$

$$\tilde{g}(\tilde{x}) = g_1(x) + \alpha_i \left[ \frac{\psi_i \times g_2(x)}{l_i} \right],$$

$$f_1(x) = M(\theta_i, \delta_i)^{-1} \left( - \begin{bmatrix} \mathbf{c}_1(\theta_i, \delta_i, \dot{\theta}_i, \dot{\delta}_i) \end{bmatrix} \right),$$

$$f_2(x) = M(\theta_i, \delta_i)^{-1} \left( - \begin{bmatrix} \mathbf{c}_2(\theta_i, \delta_i, \dot{\theta}_i, \dot{\delta}_i) \end{bmatrix} - K \delta_i - D \dot{\delta}_i \right),$$

$$g_1(x) = M(\theta_i, \delta_i)^{-1} \tau_i,$$

$$g_2(x) = M(\theta_i, \delta_i)^{-1} \tau_2,$$

and  $u = \tau_i$  Eq. (7) can be written as

$$\tilde{g}(\tilde{x})^{-1} [\dot{\tilde{x}} - \tilde{f}(\tilde{x})] = \tau_i \quad (8)$$

### III. RESENTATION OF THE TLFR DYNAMICS AS A NARMAX MODEL

Fig. 1 shows the open loop identification of the nonlinear TLFR using a NARMAX model. The ARMAX model is extended for nonlinear systems to give a NARMAX, which is defined as [4]:

$$y(k) = f^l(y_1(k-1), \dots, y_1(k-ny), u_1(k-1), \dots, u_1(k-nu), \xi_1(k-1), \dots, \xi_1(k-ne)) + e(k) \quad (9)$$

where  $u(k)$  and  $y(k)$  denote the input and output at time step  $k$  ( $k=0,1,\dots$ ),  $\xi(k)$  is the noise term in the form of residual scalar. The nonlinear state space representation of the TLFR dynamics using the redefined output given in eq. (7) is a two-input two-output MIMO system. The dynamic nonlinear rational NARMAX model of the MIMO system in general for  $m$ -input and  $p$ -output is given by [6]

$$y_i(k) = \frac{f_{i1} [y_1(k-1), \dots, y_1(k-d_{ny}), \dots, y_p(k-d_{ny}), \dots, y_p(k-1), \dots, y_p(k-1-d_{dy}), \dots, y_p(k-1-d_{dy})]}{f_{i2} [y_1(k-1), \dots, y_1(k-d_{dy}), \dots, y_p(k-d_{dy}), \dots, y_p(k-1), \dots, y_p(k-1-d_{du}), \dots, y_p(k-1-d_{du})]} \times \frac{u_1(k-1), \dots, u_1(k-d_{nu}), \dots, u_m(k-1), \dots, u_m(k-1-d_{du})}{u_1(k-1), \dots, u_1(k-d_{du}), \dots, u_m(k-1), \dots, u_m(k-1-d_{du})} \times \frac{\xi_{i1}(k-1), \dots, \xi_{i1}(k-d_{ne}), \dots, \xi_{ip}(k-1), \dots, \xi_{ip}(k-d_{ne})}{\xi_{i1}(k-1), \dots, \xi_{i1}(k-d_{de}), \dots, \xi_{ip}(k-1), \dots, \xi_{ip}(k-d_{de})} \quad (10)$$

Eq. (10) is a nonlinear equation with nonlinear polynomials  $f_{i1}$  and  $f_{i2}$ . The multivariable output NARMAX model can be written in parametric form as

$$y_i(k) = \frac{\sum_{j=1}^{N_{num}} \theta_{nj} \phi_{nj}(k)}{\sum_{j=1}^{N_{den}} \theta_{dj} \phi_{dj}(k)}, \quad i=1,2,\dots,p. \quad (11)$$

Eq. (11) is nonlinear in terms of parameters, the standard Recursive Extended Least Square (RELS) algorithm cannot

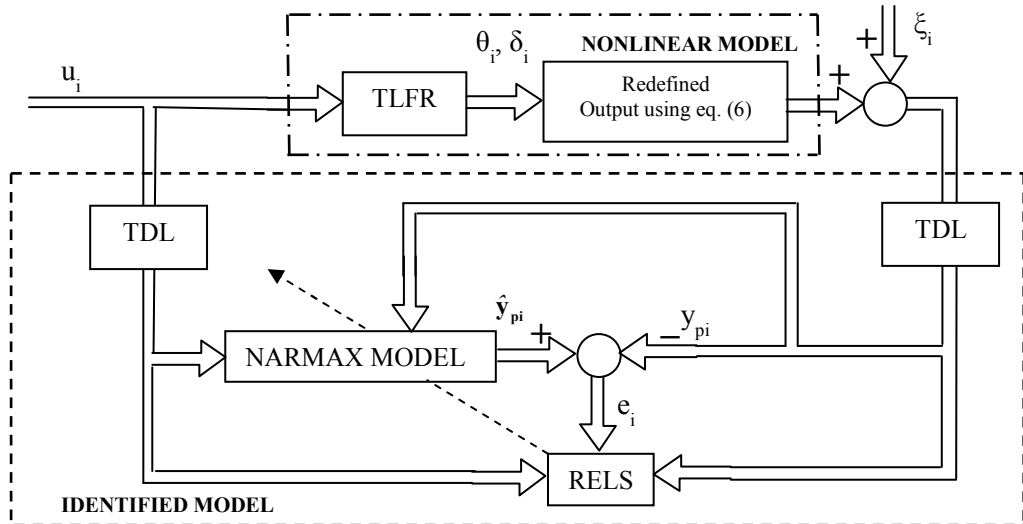


Fig.1 Real-time identification of the TLFR using NARMAX model

be used directly. So in order to estimate the NARMAX parameters  $\theta_{ij}$  and  $\theta_{dj}$  on-line, eq. (11) is multiplied out to yield a linear in-the-parameter expression [6].

$$\begin{aligned} y_i(k) &= f_{i1}(\bullet) - y_i(k) [f_{i2}(\bullet) - \theta_{id} \phi_{id}(k)] \\ &= \sum_{j=1}^{N_{num}} \theta_{inj} \phi_{inj}(k) - \sum_{j=1}^{N_{den}} \theta_{dj} y_i(k) \phi_{dj}(k) = \phi_i^T(k) \theta_i \end{aligned} \quad (12)$$

$$y_i(k) = \phi_i^T(k) \theta_i(k-1), \quad i = 1, 2, \dots, p, \quad (13)$$

where

$$\begin{aligned} \theta_{id1} = \phi_{id}(k) = 1, \quad \theta_i &= [\theta_{in1}, \dots, \theta_{innum}, \theta_{id2}, \dots, \theta_{idnden}]^T, \\ \phi_i(k) &= \begin{bmatrix} \phi_{in1}(k) \dots \phi_{innum}(k), -y_i(k) \phi_{id2}(k), \dots \\ -y_i(k) \phi_{idnden}(k) \end{bmatrix}^T \\ f_{i2}(\bullet) &= \sum_{j=1}^{N_{den}} \theta_{idj} \phi_{idj}(k) = \phi_{id}^T(k) \theta_{id}. \end{aligned}$$

The principle of least square is to estimate  $\theta_i$  defined in eq. (13) to the true parameter such that it minimizes the sum of squared errors  $J(\theta_i)$  [9]:

$$J(\theta) = \sum_{k=1}^N e_i^2(k, \theta) \quad (14)$$

The NARMAX parameters are estimated using the RELS algorithm given as [9]

$$\hat{\theta}_i(k) = \hat{\theta}_i(k-1) + \frac{P_i(k-1) \phi_i(k)}{\lambda(k) + \phi_i^T(k) P_i(k-1) \phi_i(k)} e_i(k), \quad (15a)$$

$$P_i(k) = \frac{1}{\lambda(k)} \left\{ \begin{array}{c} P_i(k-1) \phi_i(k) \phi_i^T(k) \\ P_i(k-1) - \frac{P_i(k-1) \phi_i(k) \phi_i^T(k)}{\lambda(k) + \phi_i^T(k) P_i(k-1) \phi_i(k)} \\ \phi_i(k) \end{array} \right\}, \quad (15b)$$

$$y_i(k) = \phi_i^T(k) \hat{\theta}_i(k-1) + e_i(k), \quad i = 1, 2, \dots, p, \quad (15c)$$

where  $P_i(k) \in \mathcal{R}^{(nnum+nden-1) \times (nnum+nden-1)}$  with  $\lambda(k) \leq 1$  is the forgetting factor,  $P_i(0) = \Gamma_i I_{nnum+nden-1}$  and  $\Gamma$  is large value.

To be able to estimate the system parameters uniquely the number  $N$  in eq. (14) must not be less than the number of unknown parameters. The result of the estimation based on  $N^{\text{th}}$  iteration the recursive parameter estimate given by eq. (15) with any initial condition is equal to the optimal least squares estimation based on the data provided i.e.  $P_i(0) = \Gamma_i I$ .

## I. EXPERIMENTAL RESULTS AND DISCUSSION

### A. Experimental set-up

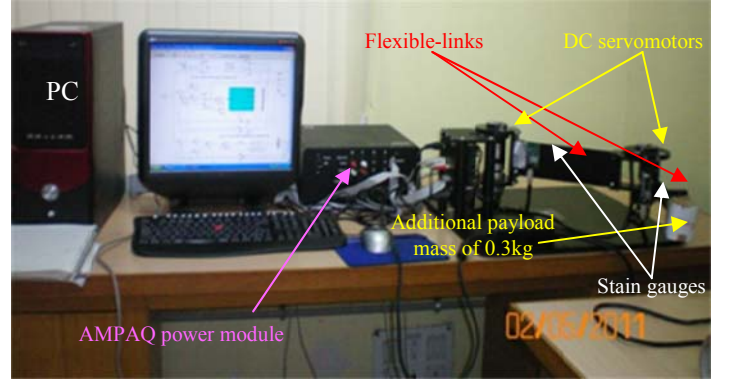


Fig.2 Photograph of the experimental set-up

Fig.2 shows the experimental set-up. It is a serial link flexible manipulator. The links are steel beams with stain gauges mounted at the base of the two links respectively. The gauges are calibrated to output as 1 volt/inch of deflection. TLFR robot that consists of two DC servomotors and flexible links attached to them. The drive for link-1 offers zero. The servomotor is a voltage-driven SRV02-ET model equipped with angular speed and position sensors. The amplifier signal is provided by a two channel amplifier package (AMPAQ) power module. MATLAB tool box is used to generate the target-logic C-code. CORE(TM) 2 Duo processor is used using real-time win target to run the compiled code in real time. Analog to digital and digital to analog signal is processed using an inbuilt hardware-in-the-loop (HIL) board. Both the controllers have been implemented using MATLAB/SIMULINK. The physical parameters of the TLFR are given in Table 1.

Table 1: Physical parameter of the TLFR

Parameter	Symbol	Value
Link length	$L_1, L_2$	0.201m, 0.2m
Elasticity	$E_1 = E_2$	$2.0684 \times 10^{11}$ (Pascal)
Rotor moment of Inertia	$K_{s1}, K_{s2}$	$6.28 \times 10^{-6}, 1.03 \times 10^{-6}$ (kg m <sup>2</sup> )
Drive moment of Inertia	$J_{11}, J_{21}$	$7.361 \times 10^{-4}, 44.55 \times 10^{-6}$ (kg m <sup>2</sup> )
Link moment of Inertia	$J_{21}, J_{22}$	0.17043, 0.0064387 (kg m <sup>2</sup> )
Gear ratio	$N_1, N_2$	100, 50
Maximum Rotation	$R_1, R_2$	+/-90, +/-90
Drive Torque constant	$K_{t1}, K_{t2}$	0.119; 0.0234 (Nm/A)

Backlashes for a gear ratio of 100:1 and for link-2 50:1. The DC motors are fed with 42 watts at  $\pm 15$ VDC power supply. The experiments for the identification of the open loop TLFR is carried out with the initial NARMAX parameter values  $\theta_i(0) = [I_2 0_{2 \times 9} \mid I_2 0_{2 \times 10} \mid 0_{2 \times 12} \mid I_2 0_{2 \times 3}]$ . Also set  $\lambda(0) = 0.9$ , and

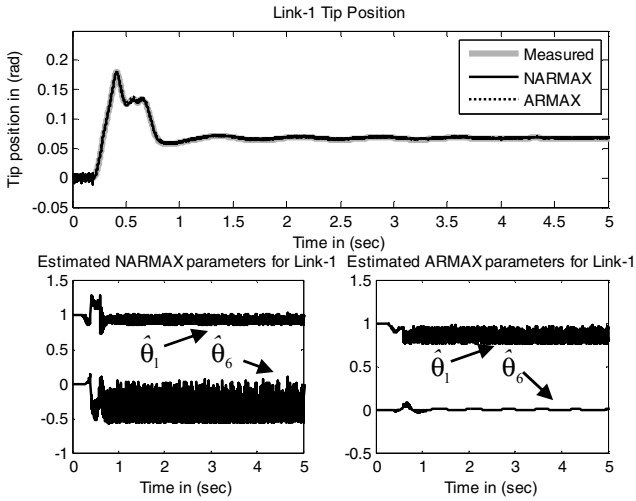


Fig.3 Estimation of the link-1 TLFR dynamics under 0.157kg payload

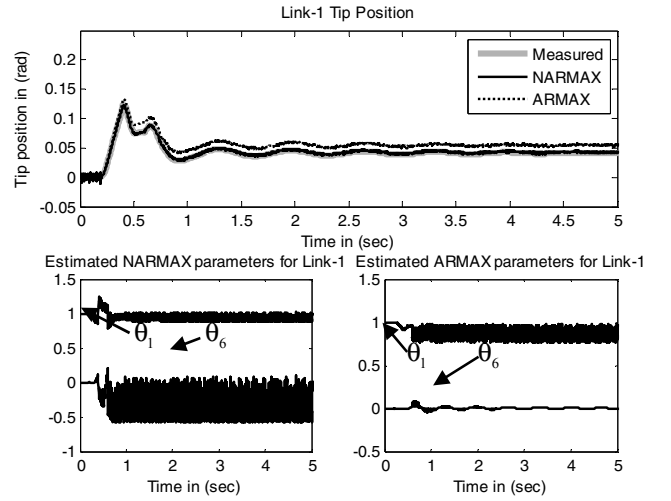


Fig.5 Estimation of the link-1 TLFR dynamics under 0.457kg payload

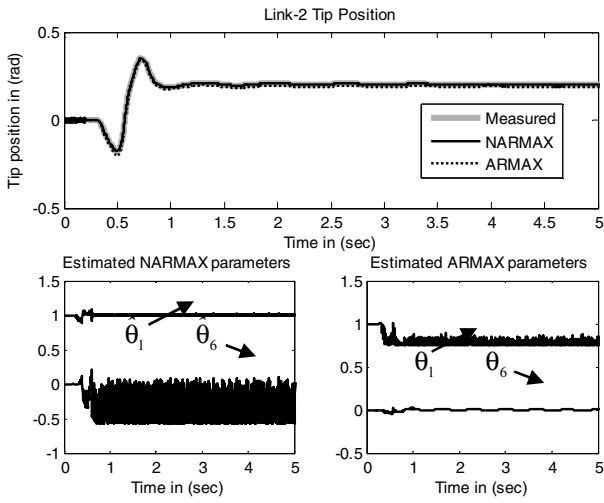


Fig.4 Estimation of the link-2 TLFR dynamics under 0.157kg payload

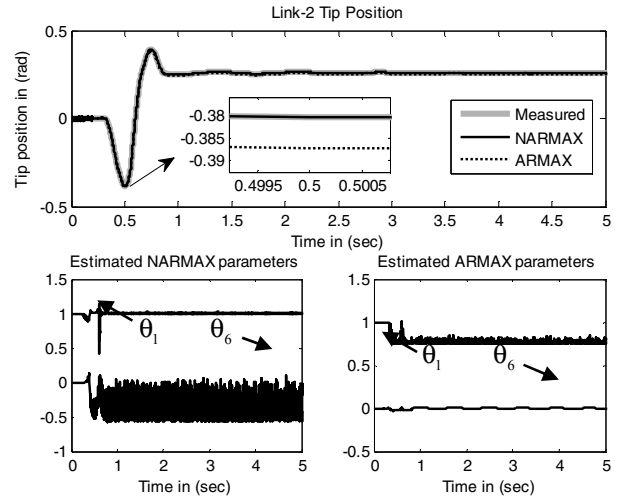


Fig.6 Estimation of the link-2 TLFR dynamics under 0.457kg payload

$P_i(0) = \Gamma_i I$  set  $\Gamma_i = 10^3$ . The experimental results are performed for different operating conditions for payload of 0.157kg and 0.457kg.

#### B. Experimental results with nominal payload of 0.157kg

Fig.3 and Fig.4 show the open loop tip position identification for link-1 and link-2 with nominal payload of 0.157kg respectively. A bang-bang torque of 0.11 rad is supplied to the actuators of link-1 to excite the open loop dynamics and data are processed with 0.1 sec sampling time. The results show that the NARMAX model approximate the TLFR tip positions for link-1 and link-2 excellently compared to the ARMAX model. Figs. (3-6) also show the NARMAX and ARMAX parameter estimated using the RELS algorithm. The numbers of parameters used under NARMAX model are 40 in numbers whereas the ARMAX uses only 10 parameters to estimate the link dynamics.

#### C. Experimental results with nominal payload of 0.457kg

Next in order to further verify the effectiveness of the NARMAX model the system is subjected to different operating conditions, by adding an additional payload of 0.3kg. Fig.5 and Fig.6 show the link-1 and link-2 tip position respectively. The results verify the effectiveness of the NARMAX model under additional payload of 0.3kg. The NARMAX and ARMAX parameters estimated via the RELS algorithm converged to a constant value by the modification of the model structure not the adaptation algorithm.

## II. CONCLUSION

This work proposed a NARMAX model based open loop identification of a TLFR in real-time, in conjunction with a RELS parameter adaptation algorithm. It offers following

advantages over a linear ARMAX model: (i) Figs. 3-6 shows that NARMAX based identification procedure provides an accurate representation of the complex nonlinear dynamics of the TLFR, with almost constant parameters that can be used to design an adaptive controller in real-time. (ii) Also Fig 3 to Fig. 6 it can be verified that under different operating conditions i.e. change in payload from 0.157kg to 0.457kg the NARMAX based model gives exact approximation of the link dynamics compared to the ARMAX model.

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