Self similar solutions in shallow water equations

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Infinitesimal transformations

Infinitesimal transformations: Consider a one-parameter Lie group of transformations $x^* = X(x;\epsilon)$ with identity $\epsilon = 0$ and law of composition ϕ. If we expand $x^* = X(x;\epsilon)$ about $\epsilon = 0$ we get

$$x^* = x + \epsilon\left(\frac{\partial X}{\partial \epsilon}\right)_{\epsilon=0} + \frac{\epsilon^2}{2}\left(\frac{\partial^2 X}{\partial \epsilon^2}\right)_{\epsilon=0} + \cdots$$

$$x^* = x + \epsilon\left(\frac{\partial X}{\partial \epsilon}\right)_{\epsilon=0} + O(\epsilon^2)$$

$$= x + \epsilon\xi(x)$$

where $\xi(x) = \left(\frac{\partial X}{\partial \epsilon}\right)_{\epsilon=0}$. This is called infinitesimal transformation of $x^* = X(x;\epsilon)$ and the components of $\xi(x)$ are called infinitesimals of $x^* = X(x;\epsilon)$.

Infinitesimal generator

Infinitesimal generator: The infinitesimal generator of the one-parameter Lie group of transformations $x^* = X(x;\epsilon)$ is the operator

$$X = X(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial x_i}$$

where

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \cdots\cdots \frac{\partial}{\partial x_n})$$

For any differentiable function $F(x) = F(x_1, x_2, x_3, \cdots\cdots x_n)$

$$XF(x) = \xi(x).\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial x_i}$$

Problem analysis

We consider the system of equations which governs the one dimensional modified shallow water equations as follows [**?**]

$$
\begin{aligned}
h_t + hu_x + uh_x &= 0, \\
u_t + \frac{g(h+H)}{h} h_x + uu_x &= 0,
\end{aligned}
\tag{1}
$$

where x, t are the independent variables denoting the space and time respectively and

$u =$ x-component of fluid velocity,
$h =$ variable depth
$g =$ acceleration due to gravity, $H = k_0/g$.

Firstly, we consider Lie group of transformations with independent variables x,t and dependent variables u, h for the problem

$$\begin{aligned}\tilde{x} &= \tilde{x}(x,t,h,u;\epsilon),\\ \tilde{t} &= \tilde{t}(x,t,h,u;\epsilon),\\ \tilde{u} &= \tilde{u}(x,t,h,u;\epsilon),\\ \tilde{h} &= \tilde{h}(x,t,h,u;\epsilon).\end{aligned} \tag{2}$$

where ϵ is the group parameter. The infinitesimal generator of the group (2) can be expressed in the following vector form

$$V = \xi^{x}\frac{\partial}{\partial x} + \xi^{t}\frac{\partial}{\partial t} + \eta^{u}\frac{\partial}{\partial u} + \eta^{h}\frac{\partial}{\partial h}$$

in which ξ^{x}, ξ^{t}, η^{u}, η^{h} are infinitesimal functions of the group variables.Then the corresponding one-parameter Lie group of transformations is given by

$$\begin{aligned}
\tilde{x} &= x + \epsilon\xi^{x}(x,t,h,u) + O(\epsilon^{2}),\\
\tilde{t} &= t + \epsilon\xi^{t}(x,t,h,u) + O(\epsilon^{2}),\\
\tilde{u} &= u + \epsilon\eta^{u}(x,t,h,u) + O(\epsilon^{2}),\\
\tilde{h} &= h + \epsilon\eta^{h}(x,t,h,u) + O(\epsilon^{2}).
\end{aligned}$$

Since the system of one-layer shallow-water equations has at most first-order derivatives, the first prolongation of the generator should be considered in the form:

$$Pr' V = V + \tau_x^u \frac{\partial}{\partial u_x} + \tau_t^u \frac{\partial}{\partial u_t} + \tau_x^h \frac{\partial}{\partial h_x} + \tau_t^h \frac{\partial}{\partial h_t} \tag{3}$$

where

$$\tau_t^u = \eta_t^u + \eta_u^u u_t + \eta_h^u h_t - u_x(\xi_t^x + \xi_u^x u_t + \xi_h^x h_t) - u_t(\xi_t^t + \xi_u^t u_t + \xi_h^t h_t)$$
$$\tau_x^u = \eta_x^u + \eta_u^u u_x + \eta_h^u h_x - u_x(\xi_x^x + \xi_u^x u_x + \xi_h^x h_x) - u_t(\xi_x^t + \xi_u^t u_x + \xi_h^t h_x)$$
$$\tau_t^h = \eta_t^h + \eta_u^h u_t + \eta_h^h h_t - h_x(\xi_t^x + \xi_u^x u_t + \xi_h^x h_t) - h_t(\xi_t^t + \xi_u^t u_t + \xi_h^t h_t)$$
$$\tau_x^h = \eta_x^h + \eta_u^h u_x + \eta_h^h h_x - h_x(\xi_x^x + \xi_u^x u_x + \xi_h^x h_x) - h_t(\xi_x^t + \xi_u^t u_x + \xi_h^t h_x).$$

if we apply the first prolongation of the infinitesimal generator (3) to the system of partial differential equations (1)

$$Pr^{'}V(h_t + hu_x + uh_x)_{h_t=-uh_x-hu_x} = 0,$$
$$Pr^{'}V(u_t + \frac{g(h+H)}{h}h_x + uu_x)_{u_t=-uu_x-\frac{g(h+H)}{h}h_x} = 0.$$

then we obtained the following system of equations

$$\eta^u h_x + \eta^h u_x + \tau^h_t + u\tau^h_x + h\tau^u_x = 0,$$
$$-\frac{gH}{h^2}\eta^h + \eta^u u_x + g\tau^h_x + \tau^u_t + u\tau^u_x = 0.$$

which gives us the following determining equations

Power-series method

Firstly, we choose the first order of power-series of the infinitesimals which are given by

$$\begin{aligned}
\xi^x &= a_0 + a_1 x + a_2 t + a_3 h + a_4 u \\
\xi^t &= b_0 + b_1 x + b_2 t + b_3 h + b_4 u \\
\eta^u &= c_0 + c_1 x + c_2 t + c_3 h + c_4 u \\
\eta^h &= d_0 + d_1 x + d_2 t + d_3 h + d_4 u
\end{aligned}$$

where a_i, b_i, c_i, a_i, $(i = 0, 1, 2, 3, 4)$ are constant coefficients.

Then substituting these power-series forms into the determining equations and straightforward calculations for the first order of power-series forms, we find three-parameter Lie group of transformations of one-layer shallow-water equations as follows

$$
\begin{aligned}
\xi^t &= a_1 t + a_4, \\
\xi^x &= a_1 x + a_2 t + a_3, \\
\eta^u &= a_2, \\
\eta^h &= 0.
\end{aligned}
$$

These transformations provide the following three Lie point generators:

$$
\begin{aligned}
X_1 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\\
X_2 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
X_3 &= \frac{\partial}{\partial x}.
\end{aligned}
$$

For $a_1 \neq 0$,$b_2 \neq 0$ and $b_0 \neq 0$ respectively.

Consider the infinitesimal generators V_A, V_B defined by

$$\begin{aligned} V_A &= \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \\ &= \alpha_1 t \frac{\partial}{\partial t} + (\alpha_1 x + \alpha_2 t + \alpha_3)\frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial u}, \end{aligned}$$

and

$$\begin{aligned} V_B &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \\ &= \beta_1 t \frac{\partial}{\partial t} + (\beta_1 x + \beta_2 t + \beta_3)\frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial u}, \end{aligned}$$

$\alpha_i, \beta_i \in R$.

Compatible Condition

For the compatible condition we consider the following relation

$$[V_A, V_B] = V_A V_B - V_B V_A = 0$$

which yields

$$\alpha_3 \beta_1 - \alpha_1 \beta_3 = 0.$$

Uses of similarity variables

Since the system is invariant under the group generated by V_A, we introduce a set of canonical variables defined by,

$$V_A\bar{\tau} = 1, \qquad V_A\bar{\xi} = 0, \qquad V_A\bar{U} = 0, \qquad V_A\bar{P} = 0,$$

allowing one to express V_A as a translation with respect to $\bar{\tau}$, the characteristic conditions are

$$\frac{dt}{\alpha_1 t} = \frac{dx}{(\alpha_1 x + \alpha_2 t + \alpha_3)} = \frac{du}{\alpha_2} = \frac{d\bar{\tau}}{1}, \tag{4}$$

where α_1, α_2, and α_3 are non-zero constants. Hence equation (4) yield the following transformation of variables

$$
\begin{aligned}
\bar{\tau} &= \frac{1}{\alpha_1} \ln t, \\
\bar{\xi} &= t^{-1} e^{\frac{(\alpha_1 x + \alpha_3)}{\alpha_2 t}}, \\
\bar{U} &= e^{u} t^{\frac{-\alpha_2}{\alpha_1}}, \\
\bar{P} &= h.
\end{aligned}
$$

Now we can express V_B using the new variables as

$$\begin{aligned}\bar{V}_B &= V_B\bar{\tau}\frac{\partial}{\partial\bar{\tau}} + V_B\bar{\xi}\frac{\partial}{\partial\bar{\xi}} + V_B\bar{U}\frac{\partial}{\partial\bar{U}} + V_B\bar{P}\frac{\partial}{\partial\bar{P}}, \\ &= \frac{\beta_1}{\alpha_1}\frac{\partial}{\partial\bar{\tau}} + \frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha_1}\bar{\xi}\frac{\partial}{\partial\bar{\xi}} + \frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha_1}\bar{U}\frac{\partial}{\partial\bar{U}}.\end{aligned}$$

In a similar manner, we choose a second set of canonical variables allowing $\bar{V}_B$ to be written as translation with respect to ξ, *i.e.*,

$$\bar{V}_B\tau = 0, \qquad \bar{V}_B\xi = 1, \qquad \bar{V}_B U = 0, \qquad \bar{V}_B P = 0. \qquad (5)$$

the characteristic conditions associated with (5) yield the following transformation of variables

$$\frac{\alpha_1 d\bar{\tau}}{\beta_1} = \frac{\alpha_1 d\bar{\xi}}{(\alpha_1\beta_2 - \alpha_2\beta_1)\bar{\xi}} = \frac{\alpha_1 d\bar{U}}{(\alpha_1\beta_2 - \alpha_2\beta_1)\bar{U}} = \frac{d\xi}{1}.$$

the characteristic conditions yield the following transformation of variables

$$\tau = \ln t^{\frac{\alpha_2\beta_1 K + 1}{\alpha_1}} - \frac{\beta_1 K\left(x + \frac{\alpha_3}{\alpha_1}\right)}{t}, \tag{6}$$

$$\xi = \ln t^{-\alpha_2 K} + \frac{\alpha_1 K\left(x + \frac{\alpha_3}{\alpha_1}\right)}{t}, \tag{7}$$

$$u = \ln U + \frac{\left(x + \frac{\alpha_3}{\alpha_1}\right)}{t}, \tag{8}$$

$$h = P, \tag{9}$$

where $K = \frac{1}{(\alpha_1\beta_2 - \alpha_2\beta_1)}$.

Using the above transformation in the governing system (1), we get the following system of PDEs

$$\left(\frac{\alpha_2\beta_1 K+1}{\alpha_1}-\beta_1 K\ln U\right)\frac{\partial P}{\partial\tau}+(\alpha_1 K\ln U-\alpha_2 K)\frac{\partial P}{\partial\xi} -\frac{\beta_1 KP}{U}\frac{\partial U}{\partial\tau}+\frac{\alpha_1 KP}{U}\frac{\partial U}{\partial\xi}+P=0, \tag{10}$$

$$\left(\frac{\alpha_2\beta_1 K+1}{\alpha_1}-\beta_1 K\ln U\right)\frac{\partial U}{\partial\tau}+(\alpha_1 K\ln U-\alpha_2 K)\frac{\partial U}{\partial\xi}- \frac{\beta_1 Kg(P+H)}{P}U\frac{\partial P}{\partial\tau}+\frac{\alpha_1 Kg(P+H)}{P}U\frac{\partial P}{\partial\xi}+U\ln U=0.$$

By considering $U = 1$, the above system of PDEs can be reduced as follows

$$\begin{aligned} -\beta_1 \frac{\partial P}{\partial \tau} + \alpha_1 \frac{\partial P}{\partial \xi} &= 0, \\ \frac{\partial P}{\partial \xi} - \frac{\beta_1}{\alpha_1} \frac{\partial P}{\partial \tau} &= 0. \end{aligned} \qquad (11)$$

Equation (11) can be solved as

$$P(\xi, \tau) = P_1(\eta) \qquad (12)$$

where $\eta = \tau + \frac{\beta_1}{\alpha_1}\xi$. Using (12) in the equation (10), we obtained

$$\frac{dP_1}{d\eta} + \alpha_1 P_1 = 0 \tag{13}$$

equation (13) can be solved

$$P_1 = Ce^{-\alpha_1 \eta} \tag{14}$$

where C is an arbitrary constant and thus, in view of the equations (6), (12) and (14) the solution of the system (1) can be expressed as follows

$$h = \frac{C}{t}, \qquad u = \frac{x + \frac{\alpha_3}{\alpha_1}}{t}. \tag{15}$$

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Thank You.