Optimization under Generalized Equation Constraints in Asplund Spaces

Seminar

by

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Outline of the Talk

1. Optimization under Generalized Equation Constraints in Asplund Spaces
2. F_2 strictly differentiable and F_1 is continuous
3. F_2 is strictly differentiable and F_1 are strictly Lipschitz continuous
4. F_2 and F_1 are strictly Lipschitz continuous
5. F_2 and F_1 are strictly differentiable

Optimization under Generalized Equation Constraints in Asplund Spaces

$$\min_{x,y} f(x,y) \quad \text{subject to} \quad 0 \in F_1(x,y) + Q(F_2(x,y)), \quad (x,y) \in \Omega,$$

where $F_1 : X \times Y \to W$, $F_2 : X \times Y \to Z$, $Q : Z \rightrightarrows W$, $f : X \times Y \to \mathbb{R}$ and $\Omega \subset X \times Y$

Normal Cone

Let Ω be a non-empty subset of the Asplund space X and let $\bar{x} \in \Omega$. Then the Frechet normal cone to Ω at $\bar{x}$ which is denoted as $\hat{N}(\bar{x}, \Omega)$ and is given as

$$\hat{N}(\bar{x}, \Omega) = \{x^* \in X^* \mid \limsup_{x \in \Omega, x \to \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|u - x\|} \leq 0\}.$$

$$N(\bar{x}, \Omega) = \limsup_{x \to \bar{x}} \hat{N}(x, \Omega).$$

$$N(\bar{x}, \Omega) := \{x^* \in X^* \mid \exists x_n \xrightarrow{\Omega} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \hat{N}(x_n, \Omega)\}$$

Differentiability

$f : X \to Y$ is Fréchet differentiable at $\bar{x}$ if there is a linear operator $\nabla f(\bar{x}) : X \to Y$, called the Fréchet derivative of f at $\bar{x}$, such that

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \nabla f(\bar{x})(x - \bar{x})}{\| x - \bar{x} \|} = 0.$$

Strictly Differentiable

$$\lim_{x \to \bar{x}, u \to \bar{x}} \frac{f(x) - f(u) - \nabla f(\bar{x})(x - u)}{\| x - u \|} = 0.$$

Co-derivative

Given $(x, y) \in X \times Y$, we define the coderivative of F at (x,y) as a multifunction $\hat{D}^* F(x, y) : Y^* \rightrightarrows X^*$ with the values

$$\hat{D}^* F(x, y)(y^*) = \{x^* \in X^* | (x^*, -y^*) \in \hat{N}((x, y), gphF)\}.$$

The normal coderivative of F at $(\bar{x}, \bar{y}) \in gphF$ is a multifunction $D_N^* F(x, y) : Y^* \rightrightarrows X^*$ defined by

$$D_N^* F(\bar{x}, \bar{y})(\bar{y}^*) = \limsup_{\substack{(x,y) \to (\bar{x}, \bar{y}) \\ y^* \xrightarrow{w^*} \bar{y}^*}} \hat{D}^* F(x, y)(y^*).$$

The mixed coderivative of F at $(\bar{x}, \bar{y}) \in gphF$ is a multifunction $D_M^* F(x, y) : Y^* \rightrightarrows X^*$ defined by

$$D_M^* F(\bar{x}, \bar{y})(\bar{y}^*) = \limsup_{\substack{(x,y) \to (\bar{x}, \bar{y}) \\ y^* \to \bar{y}^*}} \hat{D}^* F(x, y)(y^*).$$

Sequential Normal Compactness and Partial Sequential Normal Compactness

Let $\Omega \subset X \times Y$ where X and Y are Asplund spaces. The set Ω is said to be sequentially normally compact (SNC) at $(\bar{x}, \bar{y}) \in \Omega$ if for any sequence $(x_k, y_k, x_k^*, y_k^*) \in \Omega \times X^* \times Y^*$ satisfying $(x_k, y_k) \xrightarrow{\Omega} (\bar{x}, \bar{y})$,

$$(x_k^*, y_k^*) \in \hat{N}((x_k, y_k), \Omega) \tag{1}$$

and $(x_k^*, y_k^*) \xrightarrow{w^*} 0$ then one has $\| (x_k^*, y_k^*) \| \to 0$ as $k \to \infty$. Ω is said to be partially sequentially normally compact (PSNC) if for any sequence $(x_k, y_k) \xrightarrow{\Omega} (\bar{x}, \bar{y})$ satisfying (1) one has the implication $x_k^* \xrightarrow{w^*} 0$, $\| y_k^* \| \to 0$ implies that $\| x_k^* \| \to 0$, as $k \to \infty$.

Necessary Optimality condition

F_2 strictly differentiable and F_1 is continuous

Let $(\bar{x}, \bar{y})$ be a local optimal solution to the problem(P). Assume that f is locally Lipschitz continuous around $(\bar{x}, \bar{y})$ with modulus l_f, that the sets Ω and $gphQ$ are closed sets. Also assume following conditions on F_1 and F_2.

(i) F_2 is strictly differentiable at $(\bar{x}, \bar{y})$.

(ii) F_1 is continuous around $(\bar{x}, \bar{y})$ and either PSNC at $(\bar{x}, \bar{y})$ or Ω is SNC at $(\bar{x}, \bar{y})$.

Let us define the mapping, $\Psi : Z \times W \rightrightarrows X \times Y$ as

$$\Psi(v_1, v_2) = \{(x, y) \in \Omega | (v_1 + F_2(x, y), v_2 - F_1(x, y)) \in gphQ\},$$

which is calm at $(0, 0, \bar{x}, \bar{y})$ with modulus l. Assume further that the constraint qualification(CQ),

$$\{D^*_M F_1(\bar{x}, \bar{y})(0)\} \cap \{-N((\bar{x}, \bar{y}), \Omega)\} = \{0\}$$

holds.

Necessary Optimality condition

F_2 strictly differentiable and F_1 is continuous
Then there exists $(z^*, w^*) \in Z^* \times W^*$ with $\| (z^*, w^*) \| \leq l_f.l$ and $(-z^*, -w^*) \in N((F_2(\bar{x}, \bar{y}), -F_1(\bar{x}, \bar{y})), gphQ)$ satisfying

$$0 \in \partial f(\bar{x}, \bar{y}) + D_N^* F_1(\bar{x}, \bar{y})(w^*) - \nabla F_2(\bar{x}, \bar{y})^*(z^*) + N((\bar{x}, \bar{y}), \Omega)$$

which is equivalent to

$$0 \in \partial f(\bar{x}, \bar{y}) + D_N^* F_1(\bar{x}, \bar{y})(w^*) + \nabla F_2(\bar{x}, \bar{y})^* D_N^* Q(F_2(\bar{x}, \bar{y}), -F_1(\bar{x}, \bar{y}))(w^*) + N((\bar{x}, \bar{y}), \Omega).$$

Strictly Lipschitz function

Let $f : X \to Y$ be a single-valued mapping between Banach spaces. Assume that f is Lipschitzian at $\bar{x}$. Then f is called as strictly Lipschitzian at $\bar{x}$ if there is a neighborhood V of the origin in X such that the sequence

$$y_k = \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in \mathbb{N},$$

contains a norm convergent subsequence whenever $v \in V, x_k \to \bar{x}$ and $t_k \downarrow 0$.

Necessary Optimality condition

F_2 is strictly differentiable and F_1 are strictly Lipschitz continuous

Let $(\bar{x}, \bar{y})$ be a local optimal solution to the problem(P), where F_1, F_2 and Q are mapping between Asplund spaces. Assume that f is locally Lipschitz continuous around $(\bar{x}, \bar{y})$, that the sets Ω and gph Q are closed sets, and that Q is SNC at $(F_2(\bar{x}, \bar{y}), -F_1(\bar{x}, \bar{y}))$. Also assume that F_1 and F_2 are strictly Lipschitz continuous and strictly differentiable at $(\bar{x}, \bar{y})$, respectively, and that relations $(x^*, y^*) \in N_{gphQ}(F_2(\bar{x}, \bar{y}), -F_1(\bar{x}, \bar{y}))$ and

$$(0, 0) \in [\nabla F_2(\bar{x}, \bar{y})^*(x^*) + \partial\langle y^*, -F_1\rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \Omega)],$$

holds only for $x^* = y^* = 0$.

Necessary Optimality condition

F_2 strictly differentiable and F_1 is strictly Lipschitz continuous

Then there is $(z^*, w^*) \in -N(-F(\bar{x}, \bar{y}), gphQ)$ such that the necessary optimality condition

$$0 \in \partial f(\bar{x}, \bar{y}) - \nabla F_2(\bar{x}, \bar{y})^*(z^*) + \partial\langle w^*, F_1\rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \Omega),$$

which is equivalently,

$$0 \in \partial f(\bar{x}, \bar{y}) + \partial\langle w^*, F_1\rangle(\bar{x}, \bar{y}) + \nabla F_2(\bar{x}, \bar{y})^* D_N^* Q(F_2(\bar{x}, \bar{y}), -F_1(\bar{x}, \bar{y}))(w^*) + N((\bar{x}, \bar{y}), \Omega$$

is satisfied.

Necessary Optimality condition

F_2 and F_1 are strictly Lipschitz continuous

Let $(\bar{x}, \bar{y})$ be a local optimal solution to the problem(P). Assume that f is Lipschitz around $(\bar{x}, \bar{y})$, and that sets Ω and $gphQ$ are closed sets. Let us consider F_1 and F_2 are strictly Lipschitz at $(\bar{x}, \bar{y})$. Then there exists $(x^*, y^*) \in N(H(\bar{x}, \bar{y}), gphQ)$ such that

$$0 \in \partial f(\bar{x}, \bar{y}) + \partial\langle x^*, F_2\rangle(\bar{x}, \bar{y}) + \partial\langle y^*, -F_1\rangle(\bar{x}, \bar{y}) + N(\bar{x}, \bar{y}), \Omega),$$

under the assumptions :

(i) If $x_1^* \in \bigcup[\partial\langle(x^*, y^*), -F\rangle(\bar{x}, \bar{y}) | (x^*, y^*) \in N(-F(\bar{x}, \bar{y}), gphQ)], x_2^* \in N((\bar{x}, \bar{y}), \Omega)$, and $x_1^* + x_2* = 0$, then it implies that $x_1^* = x_2^* = 0$.

(ii) $N(-F(\bar{x}, \bar{y}), gphQ) \cap kerD_N^*(-F(\bar{x}, \bar{y})) = \{0\}$.

(iii) Ω is SNC is at $(\bar{x}, \bar{y})$, either $(-F)$ is PSNC at $(\bar{x}, \bar{y})$ and $gphQ$ is SNC at $((\bar{x}, \bar{y}), -F(\bar{x}, \bar{y}))$, or $(-F)$ is SNC at $(\bar{x}, \bar{y})$.

Necessary Optimality condition

F_2 and F_1 are strictly Lipschitz continuous

F_1, F_2 are strictly lipschitz continuous function. Then there exists $(x^*, y^*) \in N((F_2(\bar{x}, \bar{y}), -F_1(\bar{x}, \bar{y})), gphQ)$ such that

$$0 \in \partial f(\bar{x}, \bar{y}) + \partial\langle x^*, F_2\rangle(\bar{x}, \bar{y}) + \partial\langle y^*, -F_1\rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \Omega),$$

holds, under the following qualification condition(CQ), For any $(x^*, y^*) \in N((F_2(\bar{x}, \bar{y}), F_1(\bar{x}, \bar{y})), gphQ)$, if $(0, 0) \in \partial\langle(x^*, y^*), (F_2, -F_1)\rangle(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \Omega)$, then it implies that $x^* = y^* = 0$.

Necessary Optimality condition

F_2 and F_1 are strictly differentiable

Let $(\bar{x}, \bar{y})$ be a local solution of problem(P) and assume that gph$Q$ is closed set and that the constraint qualification(CQ)

$$(\nabla F_2(\bar{x}, \bar{y})^T, -\nabla F_1(\bar{x}, \bar{y})^T)(x, y)^T \in -N((\bar{x}, \bar{y}), \Omega),$$
$$(x, y) \in N_{gphQ}(F_2(\bar{x}, \bar{y}), -F_1((\bar{x}, \bar{y})),$$

implies $(x, y) = (0, 0)$, holds true. Then there exists $\xi \in \partial f(\bar{x}, \bar{y}), \eta \in N_\Omega(\bar{x}, \bar{y})$ and a pair $(z, w) \in N_{ghpQ}(F_2(\bar{x}, \bar{y}), -F_1((\bar{x}, \bar{y}))$ such that

$$0 = \xi + \nabla F_2(\bar{x}, \bar{y})^T(z) - \nabla F_1(\bar{x}, \bar{y})^T(w) + \eta.$$

Bibliography

- Borwein, J M, ; Zhu, Q. J. *Techniques of variational analysis.* CMS Books in Mathematics/Ouvrages de Mathmatiques de la SMC, 20. Springer-Verlag, New York, 2005.
- Burke, J. V.; Lewis, A. S.; Overton, M. L. Approximating subdifferentials by random sampling of gradients. *Math. Oper. Res.* 27 (2002), 567–584.
- Dutta, J. Revisiting the Lagrange multiplier rule. *Pac. J. Optim.* 2 (2006), pp 501–519.
- Durea, M., Dutta, J., Tammer, C., Langrange multiplier for ϵ- Pareto solutions in vector optimization with nonsolid cones in Banach Space, JOTA, (to be published).

Bibliography

- Mordukhovich, B.S.Variational analysis and generalized differentiation. I. Basic theory., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 330. Springer-Verlag ,Berlin,2006.
- Mordukhovich, B.S., Variational analysis and generalized differentiation.II. Applications., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 331. Springer-Verlag,Berlin ,2006.
- Mordukhovich, B.S., Bao, T.Q., Variational principle for set-valued mappings with application to multiobjective optimization,*Control and Cybernetics* vol-36(2007), 1-33.
- Mordukhovich, B.S., Bao, T.Q., Relative Pareto minimizer to multiobjective problems: Existance and optimality conditions,*Preprint*.
- Rockafellar, R. T.,Wets, R.J., Variational Analysis, Springer,Berlin,

Thanks