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# Interaction of Shallow Water Waves

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# Interaction of Shallow Water Waves

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**Abstract**

In this paper, we consider the Riemann problem and interaction of elementary waves for a nonlinear hyperbolic system of conservation laws that arises in shallow water theory. This class of equations includes as a special case the equations of classical shallow water equations. We study the bore and dilatation waves and their properties, and show the existence and uniqueness of the solution to the Riemann problem. Towards the end, we discuss numerical results for different initial data along with all possible interactions of elementary waves. It is noticed that in contrast to the $p$-system, the Riemann problem is solvable for arbitrary initial data, and its solution does not contain vaccum state.

**Keywords**: modified shallow water equations; bore; dilatation wave; Riemann problem; wave interaction

## 1 Introduction

The shallow water equations are a set of hyperbolic equations, which approximate the full free surface gravity flow problem, with viscocity and surface tension effects neglected. These equations being quasilinear and hyperbolic admit discontinuous and piecewise continuous solutions which are called bores and dilatation waves respectively. The motion of a bore over a sloping beach has been studied by Keller, Levine and Whitham [1], whilst an asymptotic analysis of its approach to the shoreline has been performed by Ho and Meyer [2]. They show that forgetfulness will occur to several orders of approximation provided a certain monotonicity condition holds for the incoming motion producing the bore. Barker and Whitham [3] obtained a similarity solution for the asymptotic behavior of a bore as it approaches the shoreline on a sloping beach with an explicit analogy with Guderly's implosion problem in gasdynamics [4]. Radha, Sharma and Jeffrey [5] have described an approximate analytic method for the kinematics of a bore over a sloping beach.

The exact solution to the Riemann problem is of great significance. For instance, it constitutes the basic building block for the construction of solutions to general initial value

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problems using the well known random choice method proposed by Glimm [6]. In the case of Euler equations, the Riemann problem contains the so-called shock-tube problem and for a detailed discussion of this, the reader is referred to the book by Courant and Friedrichs [7]. Lax [8] solved the Riemann problem for the case when the initial data consisting of constant states $U_l$ and $U_r$ are such that $||U_l - U_r||$ is sufficiently small; here $U$ is the vector of unknown variables with $U_l$ to the left of $x = 0$ and $U_r$ to the right of $x = 0$ separated by a discontinuity at $x = 0$. Smoller [9] solved the Riemann problem by considering $U_l$ and $U_r$ to be arbitrary constant vectors; for details and methodologies, the reader is referred to the book by Smoller [10]. Exact solutions of the Riemann problem were proposed by Godunov [11] and Chorin [12]; however, Smoller [10] proposed a rather different approach. Smoller and Temple [13] demonstrated the existence of solutions with shocks for equations describing a perfect fluid in special relativity; this work was generalized by Chen [14] for the general isentropic relativistic gases.Toro [15] presented an efficient solver for computing the exact solution of the Riemann problem for ideal and covolume gases; for detailed methodologies, the reader is referred to the book by Toro [16]. The solutions of Riemann problem for fluid flows in a nozzle with discontinuous cross section can be found in LeFloch and Thanh [17]. Concerning compressible duct flows, and two phase flows, we refer to the papers of Andrianov and Warnecke [18, 19]. For modified shallow water equations, we refer to the work of Karelsky and Petrosyan [20]. Smoller and Johnson [21] have considered shock interactions from a general point of view. For interaction of elementary waves in unsteady one-dimensional Euler equations, we refer to Smoller [10], and Chang & Hsiao [22]. For an illuminating treatment on Riemann problem, we also refer to an article by Liu [23] and the books of Godlewski and Raviart [24], Li−Tsien [25], Dafermos [26], Bressan [27], LeFloch [28] and LeVeque [29].

Karelsky and Petrosyan [20] have derived the modified shallow water equations, which take into account advective transport of horizontal impulse; such flows, which take into consideration the weak vertical inhomogeneities in the initial conditions, are important for environmental applications. Karelsky and Petrosyan [20] have discussed solution of the Riemann problem for the modified model for the case when the arbitrary set of initial conditions, $U_l$ and $U_r$, is given. The main motivation of the present work is to study all possible wave patterns, where we fix $U_l$ and allow $U_r$ to vary, and then discuss all possible elementary wave interactions. The paper is organized as follows; in Section 2, we show that the system is strictly hyperbolic, and that its characteristic fields are genuinely nonlinear. We establish the existence of bores and dilatation waves, and prove the stability conditions for bores, and discuss how velocity and depth vary across bores and dilatation waves. We show that the characteristic speed increases from left to right for dilatation waves. In Section 3, we consider the Riemann problem for arbitrary initial data, and show that it is uniquely solvable. It is noticed that in contrast to the $p$-system, the Riemann problem for the modified shallow water equations is solvable for arbitrary initial data. In Section 4, we discuss numerical results for different initial data along the effects of advective transport of impulse on the solution. In Section 5, we discuss all possible interactions of elementary waves.

## 2 Properties of Bores and Dilatation Waves

The system of equations which governs the one dimensional modified shallow water equations, can be written in conservation form as [20]

$$\begin{aligned} &\frac{\partial}{\partial t}(h) + \frac{\partial}{\partial x}(hu) = 0, \\ &\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(hu^2 + gHh + gh^2/2) = 0, \qquad t > 0,\ x \in \mathbb{R} \end{aligned} \tag{2.1}$$

where $u$ is $x$- component of fluid velocity, $h$ the water depth, $g$ the acceleration due to gravity and $H = (k_0/g)$ the reduced factor characterizing advective transport of impulse. The independent variables $t$ and $x$ denote time and space respectively.

To carry out the characteristic analysis of (2.1), it is convenient to use the primitive variables $U = (h, u)^{tr}$, rather than the vector of conserved variables, where $tr$ denotes transposition. Then for smooth solutions, system (2.1) is equivalent to

$$U_t + AU_x = 0, \tag{2.2}$$

where $A$ is $2 \times 2$ matrix with elements $A_{11} = A_{22} = u$, $A_{21} = c^2/h$ and $A_{21} = h$ with $c = \sqrt{g(h+H)}$ as the speed of propagation of surface disturbance. The eigenvalues of $A$ are $\lambda_1 = u - c$ and $\lambda_2 = u + c$. Thus, the system (2.2) is strictly hyperbolic when $c > 0$. Let $\vec{r}_1 = (h, -c)^{tr}$ and $\vec{r}_2 = (h, c)^{tr}$ be the right eigenvectors corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$ respectively. For the characteristic field $\lambda_1$, we have $\nabla\lambda_1.\vec{r}_1 = (-gh - 2c^2)/2c$ which is negative, therefore, the first characteristic field is genuinely nonlinear. Similarly, the second characteristic field $\lambda_2$ is also genuinely nonlinear. The waves associated with $\vec{r}_1$ and $\vec{r}_2$ characteristic fields will be either bores or dilatation waves. Because the characteristic fields are genuninely nonlinear, we can expect to solve the Riemann problem for (2.1) with bores and dilatation waves.

### 2.1 Bores

Let $h_l$, $u_l = u(h_l)$ and $h$, $u = u(h)$ denote respectively the left and the right hand states of either a bore or a dilatation wave. Here we are going to compute bore curves for the hyperbolic system (2.1). We fix, once and for all, a state $(h_l, u_l)$ in the domain of hyperbolicity, and compute the state $(h, u)$ such that there exists a speed $\sigma$ satisfying the the Rankine$-$Hugoniot jump conditions

$$\sigma[h] = [hu], \tag{2.3}$$

$$\sigma[hu] = [hu^2 + gHh + gh^2/2], \tag{2.4}$$

where $[.]$ denotes the jump across a discontinuity curve $x = x(t)$ and $\sigma = dx/dt$ is the bore speed.

**Lemma 2.1** *Let $B_1$ and $B_2$ respectively denote 1-bore and 2-bore associated with $\lambda_1$ and $\lambda_2$ characteristic fields. Let the states $U_l$ and $U$ satisfy the Rankine–Hugoniot jump conditions (2.3) and (2.4). Then the bore curves satisfy*

$$u = u_l \mp (h - h_l)\phi(h, h_l), \tag{2.5}$$

*where $\phi(h, h_l) = \sqrt{\frac{g(h+h_l+2H)}{2hh_l}}$; indeed, on $B_1$, we have $\frac{du}{dh} < 0$ and $\frac{d^2u}{dh^2} > 0$, whilst on $B_2$ we have $\frac{du}{dh} > 0$ and $\frac{d^2u}{dh^2} < 0$.*

Proof: The $\sigma$-elimination of (2.3) and (2.4) yields (2.5) and then differentiating (2.5) with respect to $h$, we obtain

$$\frac{du}{dh} = \mp(\phi(h, h_l) + (h - h_l)\frac{d\phi(h, h_l)}{dh}). \tag{2.6}$$

It is easy to show using (2.6) that $\frac{du}{dh} < 0$ on $B_1$, and $\frac{du}{dh} > 0$ on $B_2$. Differentiating (2.6) with respect to $h$, we get

$$\frac{d^2u}{dh^2} = \mp(2\frac{d\phi(h, h_l)}{dh} + (h - h_l)\frac{d^2\phi(h, h_l)}{dh^2}). \tag{2.7}$$

Since $2\frac{d\phi(h,h_l)}{dh} + (h - h_l)\frac{d^2\phi(h,h_l)}{dh^2} = -\frac{g^2(h_l+2H)(4hh_l+(h_l+2H)(h+3h_l))}{16h^4h_l^2\phi(h,h_l)^3}$, which is negative for all values of $h$, we obtain, in view of (2.7), that $\frac{d^2u}{dh^2} > 0$ for 1-bore, and $\frac{d^2u}{dh^2} < 0$ for 2-bore. □
We are now going to show that the bores satisfy the Lax stability conditions.

**Lemma 2.2** *Across 1-bore (respectively, 2-bore), $h > h_l$ and $u < u_l$ (respectively, $h < h_l$ and $u < u_l$) if, and only if, the Lax conditions hold, i.e., 1-bore satisfies*

$$\sigma < \lambda_1(U_l), \quad \lambda_1(U) < \sigma < \lambda_2(U), \tag{2.8}$$

*while the 2-bore satisfies*

$$\lambda_1(U_l) < \sigma < \lambda_2(U_l), \quad \lambda_2(U) < \sigma. \tag{2.9}$$

Proof: First let us consider 1-bore and prove $\sigma < \lambda_1(U_l)$. On 1-bore, $h_l < h$, implying thereby that $h_l < (h_l + h)/2$, which means that

$$c_l = \sqrt{g(h_l + H)} < \sqrt{\frac{g(h + h_l + 2H)}{2}}. \tag{2.10}$$

Also, since $h > \sqrt{hh_l}$, it follows from (2.10) that $c_l < h\phi(h, h_l)$, which implies that $-c_l > -h\phi(h, h_l)$; in view of equation (2.5) for 1-bore, we get $-c_l > \frac{h(u-u_l)}{h-h_l}$, implying thereby that

$$\sigma = \frac{hu - h_lu_l}{h - h_l} < u_l - c_l = \lambda_1(U_l). \tag{2.11}$$

Next, since $h_l < h$ on 1-bore, we have $(h + h_l + 2H)/2 < h + H$, which implies that $\sqrt{\frac{g(h+h_l+2H)}{2}} < c$, or equivalently

$$-c < -\sqrt{\frac{g(h + h_l + 2H)}{2}}. \tag{2.12}$$

Also, since $h_l < \sqrt{hh_l}$, which implies that $h_l\phi(h, h_l) < \sqrt{\frac{g(h+h_l+2H)}{2}}$, we have

$$-\sqrt{\frac{g(h + h_l + 2H)}{2}} < -h_l\phi(h, h_l). \tag{2.13}$$

In view of (2.12) and (2.13), we get $-c < -h_l\phi(h, h_l)$; since for 1-bore, $u - u_l = -(h - h_l)\phi(h, h_l)$, it follows that $-c < \frac{h_l(u-u_l)}{h-h_l}$, and hence

$$u - c = \lambda_1(U) < \frac{hu - h_l u_l}{u - u_l}. \tag{2.14}$$

Also, from (2.12) and (2.13), we obtain $-h_l\phi(h, h_l) < c$, which in view of (2.5) yields $\frac{h_l(u-u_l)}{h-h_l} < c$, and hence

$$\frac{hu - h_l u_l}{u - u_l} = \sigma < u + c = \lambda_2(U). \tag{2.15}$$

Therefore 1-bore satisfies Lax conditions; proof for 2-bore follows on similar lines.
Conversely, we assume for 1-bore that the left and right hand states satisfy Lax conditions (2.8), and show that $h > h_l$ and $u < u_l$. Let us define $V = \sigma - u$; then since $\sigma < \lambda_1(U_l)$, it follows that $V_l < -c_l$, implying thereby that $V_l < 0$, and hence $\sigma < u_l$.
Similarly, by using second condition $\lambda_1(U) < \sigma < \lambda_2(U)$, we get $u - c < \sigma < u + c$, which implies that $-c < V < c$, showing thereby that $|V| < c$. From (2.3), we have $hV = h_l V_l$. Since $h$ and $h_l$ are positive, both $V$ and $V_l$ must have the same sign; further, since $V_l < 0$, we have $V < 0$. For 1-bore, the fluid velocity on both sides of bore is greater than the bore velocity, and therefore the particles cross the bore form left to right.
Since for 1-bore we have $V_l^2 > c_l^2$ and $V^2 < c^2$, equation (2.4), namely, $h_l V_l^2 + gHh_l + gh_l^2/2 = hV^2 + gHh + gh^2/2$ implies that
$h_l c_l^2 + gHh_l + gh_l^2/2 < h_l V_l^2 + gHh_l + gh_l^2/2 = hV^2 + gHh + gh^2/2 < hc^2 + gHh + gh^2/2$, showing thereby that $\frac{3}{2}h_l^2 + 2Hh_l < \frac{3}{2}h^2 + 2Hh$; then, it follows that $h_l < h$. From (2.3), we have $V = \frac{h_l V_l}{h}$; since $h_l < h$, it follows that $u - u_l = V_l - V = V_l - \frac{h_l V_l}{h} = V_l(1 - \frac{h_l}{h})$ which is negative and hence $u < u_l$. The corresponding results for 2-bore are proved in a similar way, and we shall not reproduce the details.

## 2.2 Dilatation waves

Here we construct the dilatation wave curves, and recall that an $n$ dilatation wave $(n = 1, 2)$, connecting the states $U_l$ and $U_r$, is a solution to (2.2) of the form

$$U(x,t) = \begin{cases} U_l, & \frac{x}{t} \leq \lambda_n(U_l) \\ U(\frac{x}{t}), & \lambda_n(U_l) \leq \frac{x}{t} \leq \lambda_n(U_r) \\ U_r, & \frac{x}{t} \geq \lambda_n(U_r), \end{cases} \tag{2.16}$$

with $\lambda_n(U_l) \leq \lambda_n(U_r)$, and where $U(\eta)$ with $\eta = \frac{x}{t}$ is a solution to the system of ordinary differential equations $(A - \eta I)(\dot{h}, \dot{u})^{tr} = 0$, where $I$ is $2 \times 2$ identity matrix and an overhead dot denotes differentiation with respect to the variable $\eta$. If $(\dot{h}, \dot{u})^{tr} = (0, 0)$ then $h$ and $u$ are constant; but as we are interested in non-constant solutions, we consider $(\dot{h}, \dot{u})^{tr} \neq (0, 0)$ and then it follows that $(\dot{h}, \dot{u})^{tr}$ is an eigenvector of the matrix $A$ corresponding to the eigenvalue $\eta$. Since the matrix $A$ has two real and distinct eigenvalues $\lambda_1$ and $\lambda_2$, there are two families of dilatation waves, $D_1$ and $D_2$ which denote, respectively, 1-dilatation waves and 2-dilatation waves.

First we consider 1-dilatation waves. Since, $(A - \lambda_1 I)(\dot{h}, \dot{u})^{tr} = 0$ with $\lambda_1 = u - c$, we have, $c\dot{h} + h\dot{u} = 0$, implying thereby that

$$\Pi_1 \equiv u + \psi(h) = constant, \tag{2.17}$$

where $\psi(h) = 2\sqrt{g(h+H)} + \ln\{\frac{\sqrt{g(h+H)} - \sqrt{gH}}{\sqrt{g(h+H)} + \sqrt{gH}}\}$. Equation (2.17) represents $D_1$ curves with $\Pi_1$ as the 1-Riemann invariant. Similarly, 2-dilatation wave curves and given by

$$\Pi_2 \equiv u - \psi(h) = constant, \tag{2.18}$$

and $\Pi_2$ is the 2-Riemann invariant; indeed, the integral curves of the vector fields $\vec{r}_1$ and $\vec{r}_2$ are nothing but the level sets of the Riemann invariants $\Pi_1$ and $\Pi_2$ respectively.

**Theorem 2.1** *On $D_1$ (respectively, $D_2$), the Riemann invariant $\Pi_1$ (respectively, $\Pi_2$) is constant.*

Proof: Let $U$ be an $n$-dilatation wave of the form (2.16), and let $\Pi$ be a $n$-Riemann invariant; here $n = 1, 2$. Since, $U$ is continuous and $\Pi$ is assumed to be smooth, the function $\Pi : (x, t) \to \Pi(U)$ is continuous for $t > 0$. Obviously, $\Pi(U)$ is constant for $\frac{x}{t} \leq \lambda_n(U_l)$ and $\frac{x}{t} \geq \lambda_n(U_r)$. Further, since $\eta = \frac{x}{t}$, we have

$$\frac{d\Pi(U)}{d\eta} = \nabla\Pi(U).\dot{U}. \tag{2.19}$$

As $\dot{U}$ is parallel to $\vec{r}_n$, the right hand side of (2.19) is zero, and this proves the theorem. $\square$

**Theorem 2.2** *The $D_1$ curve is convex and monotonic decreasing while $D_2$ curve is concave and monotonic increasing.*

Proof: The 1-dilatation curve is given by

$$u = u_l + \psi(h_l) - \psi(h), \qquad h \leq h_l \tag{2.20}$$

which on differentiation with respect to $h$, yields $\frac{du}{d\rho} = -\frac{c}{h} < 0$, and subsequently,

$$\frac{d^2u}{d\rho^2} = \frac{c^2 + gH}{2ch^2}. \tag{2.21}$$

Since $c, h, g, H$ are positive, it follows from (2.21) that $\frac{d^2u}{dh^2} > 0$ and, therefore, $u$ is convex with respect to $h$ for 1-dilatation waves. In a similar way, we can prove for 2-dilatation waves.□

**Lemma 2.3** *Across 1-dilatation waves (respectively, 2-dilatation waves), $h \leq h_l$ and $u_l \leq u$ (respectively, $h \geq h_l$ and $u \geq u_l$) if, and only if, the characteristic speed increases from left hand state to right hand state.*

Proof: Since $\frac{dc}{dh} = (g/2c) > 0$, $c$ is an increasing function of $h$; this implies that for 1-dilatation waves, $c(h) \leq c(h_l)$ or equivalently $-c_l \leq -c$. The inequalities $u_l \leq u$ and $-c_l \leq -c$ imply that $\lambda_1(U_l) \leq \lambda_1(U)$. In a similar way, we can prove $\lambda_2(U_l) \leq \lambda_2(U)$ for 2-dilatation waves. Conversely, for 1-dilatation waves, since $\lambda_1(U_l) \leq \lambda_1(U)$, we have

$$c - c_l \leq u - u_l. \tag{2.22}$$

Further, since in 1-dilatation wave region $\Pi_1$ is constant, we have $u - u_l = \psi(h_l) - \psi(h)$, which in view of (2.22) yields $c - c_l \leq \psi(h_l) - \psi(h)$, implying thereby that $h \leq h_l$, and $u - u_l = \psi(h_l) - \psi(h) \geq 0$. Hence, $h \leq h_l$ and $u \geq u_l$. In the same way, one can prove that for 2-dilatation waves $h \geq h_l$ and $u \geq u_l$. □

# 3 The Riemann Problem

In what follows, we consider the Riemann problem for the system (2.1) with piecewise constant initial data consisting of just two constant states, which in terms of primitive variables are $U_l = (h_l, u_l)^{tr}$ to the left of $x = x_0$, and $U_r = (h_r, u_r)^{tr}$ to the right of $x = x_0$, separated by a discontinuity at $x = x_0$, i.e.,

$$U(x, t_0) = \begin{cases} U_l, & if\ x < x_0, \\ U_r, & if\ x > x_0. \end{cases} \tag{3.1}$$

We solve this problem in the class of functions consisting of constant states, separated by either bore or dilatation waves. The solution of the Riemann problem consists of at most three constant states (including $U_l$ and $U_r$), which are separated either by a bore or a dilatation wave.

**Theorem 3.1** *The curves of bore and dilatation waves for 1-family, i.e., $B_1$ and $D_1$ (respectively 2-family, i.e., $B_2$ and $D_2$) have the second order contact at $U_l$.*

Proof: In order to prove $B_1$ and $D_1$ have the second order contact at $U_l$, we have to show that $B_1$ and $D_1$ curves at $h = h_l$, upto second derivatives, are equal.The equation for 1-dilatation wave is given in (2.20), and from (2.21) we obtain

$$u|_{h=h_l} = u_l, \quad \frac{du}{dh}|_{h=h_l} = -\frac{c_l}{h_l}, \quad (\frac{d^2u}{dh^2})|_{h=h_l} = \frac{c_l^2 + gH}{2c_l h_l^2}. \tag{3.2}$$

The equation for 1-bore is given in (2.5) and from (2.6) & (2.7), we get

$$u|_{h=h_l} = u_l, \quad \frac{du}{dh}|_{h=h_l} = -\frac{c_l}{h_l}, \quad (\frac{d^2u}{dh^2})|_{h=h_l} = \frac{c_l^2 + gH}{2c_l h_l^2}. \tag{3.3}$$

Thus $u$, $\frac{du}{dh}$ and $\frac{d^2u}{dh^2}$ at $h = h_l$ have the same value for 1-bore and 1-dilatation wave curve. Therefore, $B_1$ and $D_1$ have the second order contact at $U_l$. Proof for 2-family follows on similar lines. □

When $U_r$ is sufficiently close to $U_l$, the existence and uniqueness of the solution of Riemann problem for system (2.1) in the class of elementary waves follow from the general theorem of Lax, which applies to any system of conservation laws that is strictly hyperbolic and genuinely nonlinear in each characteristic field (see [8], [24]). We will show that the solution of the Riemann problem for system (2.1) exists for any arbitrary initial data.

We consider the physical variables as coordinate system; let us draw the curves $B_1$, $B_2$, $D_1$ and $D_2$, from (2.5), (2.17) and (2.18), respectively, for $g = 1$ and $H = 0.1$, in the $(h, u)$-plane as shown in figure 3a; these curves divide the $(h, u)$-plane into four disjoint open regions $I$, $II$, $III$ and $IV$ for a given left state $U_l$. Indeed, we fix $U_l$ and allow $U_r$ to vary; if $U_r$ lies on any of the above four curves, then we have seen how to solve the problem. We assume that $U_r$ belongs to one of the four open regions $I$, $II$, $III$ and $IV$ as shown in figure 3a.

Following ([10]), we define, for $\hat{U} \in \mathbb{R}^2$, $B_n(\hat{U}) = \{(h, u) : (h, u) \in B_n(\hat{U})\}$, $n = 1, 2$.
$D_n(\hat{U}) = \{(h, u) : (h, u) \in D_n(\hat{U})\}$, and $T_n(\hat{U}) = B_n(\hat{U}) \bigcup D_n(\hat{U})$, $n = 1, 2$.

For fixed $U_l \in \mathbb{R}^2$, we consider the family of curves $\mathcal{S} = \{T_2(\hat{U}) : \hat{U} \in T_1(U_l)\}$. As the $(h, u)$ plane is covered univalently by the family of curves $\mathcal{S}$, i.e., through each point $U_r$, there passes exactly one curve $T_2(\hat{U})$ of $\mathcal{S}$, the solution to the Riemann problem is given as follows; we connect $\hat{U}$ to $U_l$ on the right by a 1-wave (either bore or dilatation wave), and then we connect $U_r$ to $\hat{U}$ on the right by a 2-wave (either $B_2$ or $D_2$). Indeed, depending on the position of $U_r$ we have different wave configurations.

**Theorem 3.2** *Let $U_l$, $U_r \in \mathbb{R}^2$ with $U_l$ fixed, and $U_r$ is allowed to vary then the Riemann problem is solvable.*

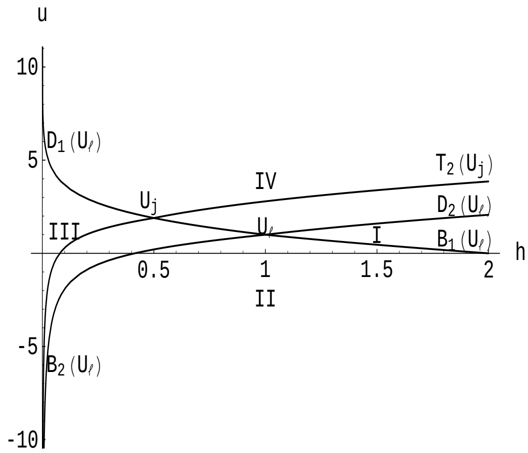


*Fig.3a* : Wave curves in $(h, u)$-plane

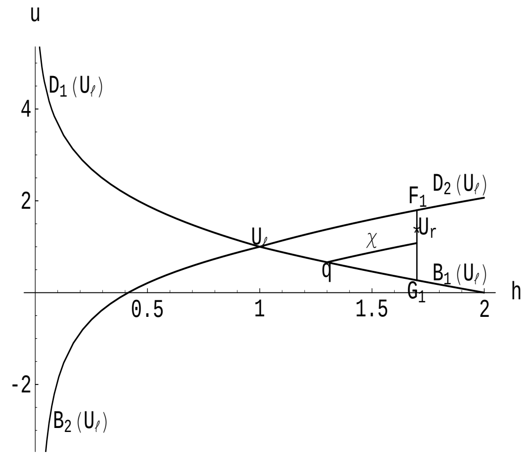


*Fig.3b* : $U_r$ is in region $I$

Proof: We are allowing to vary $U_r \in \mathbb{R}^2$ i.e., $U_r$ is in region $I$, $II$, $III$ or $IV$. If $U_r \in I$, then draw a vertical line $h = h_r$ as shown in figure 3b, which meets $D_2$ and $B_1$ uniquely at $F_1 = (h_1, u_1)$ and $G_1 = (h_2, u_2)$ respectively. We notice that the subfamily of curves in $\mathcal{S}$, consisting of the set $\{T_2(\hat{U}) \equiv T_2(\hat{h}, \hat{u}) : h_l \leq \hat{h} \leq h_r\}$ induces a continuous mapping $q \to \chi(q)$ from the arc $U_l G_1$ to line segment $G_1 F_1$, see [10]; indeed, the region $I$ is covered by curves in $\mathcal{S}$. So, let us suppose that $(h_m, u_m)$ is the point which is mapped to $U_r$. Then

$$u = u_l - (h_m - h_l)\phi(h_m, h_l) - \psi(h_m) + \psi(h_r), \tag{3.4}$$

which on differentiation yields $\frac{du}{dh_m} = -(h_m - h_l)\frac{d\phi(h_m,h_l)}{dh_m} - \phi(h_m, h_l) - \frac{c_m}{h_m} < 0$, implying thereby that $(h_m, u_m)$ is unique. Similarly, we can prove uniqueness if $U_r$ is in region $II$, $III$ or $IV$.

Thus if $U_r \in I$, then the solution to Riemann problem consists of 1-bore and a 2-dilatation wave connecting $U_l$ to $U_r$. Suppose $U_r$ is in region $II$; then the solution consists of bores $B_1$ and $B_2$ joining $U_l$ to $U_r$ . Let $U_r \in III$; then the solution of Riemann problem is as follows; $U_l$ can be connected to $U_r$ by $D_1$ followed by $B_2$. Let $U_r$ lie in region $IV$, then the solution consists of 1-dilatation wave and 2-dilatation wave. Thus the set $\{T_2(\hat{U}) : \hat{U} \in T_1(U_l)\}$ covers the entire half space $h > 0$ in the $(h, u)$-plane, in a 1-1 fashion. Therefore, the solution to the Riemann problem is solvable for any arbitrary initial data. □

Since 1-dilatation wave curve and 2-bore curve defined by (2.17) and (2.5), respectively, diverge to $\infty$, and $-\infty$, as $h \to 0$, and 2-dilatation wave curve and for 1-bore curve defined by (2.18) and (2.5) respectively,, diverge to $\infty$ and $-\infty$, as $h \to \infty$, we can find the solution to the Riemann problem for arbitrary $U_r$; this means that in contrast to the $p$-system, the vacuum state does not occur in this case.

# 4 Numerical solution

For a given left state $U_l$ and a right state $U_r$, Karelsky and Petrosyan [20] have discussed how to to find the unknown state $U_m$ analytically for all the possible cases; here, we give numerical scheme to find the unknown state $U_m$ and discuss the influence of $H$ on the unknown state $U_m$ in the $(x,t)-$plane.

***Case a***: For $h_l < h_m$ and $h_r \geq h_m$, we eliminate $u_m$ from (2.5) and (2.17) to obtain

$$u_r - u_l + \psi(h_m) - \psi(h_r) + (h_m - h_l)\phi(h_m, h_l) = 0. \tag{4.1}$$

***Case b***: For $h_l \geq h_m$ and $h_r \geq h_m$, eliminating $u_m$ from (2.17) and (2.18), we get

$$u_r - u_l - \psi(h_l) - \psi(h_r) + 2\psi(h_m) = 0. \tag{4.2}$$

***Case c***: For $h_l \geq h_m$ and $h_r < h_m$, eliminating $u_m$ from (2.17) and (2.5), we get

$$u_r - u_l + \psi(h_m) - \psi(h_l) - (h_r - h_m)\phi(h_r, h_m) = 0. \tag{4.3}$$

***Case d***: For $h_l < h_m$ and $h_r < h_m$, we obtain from (2.5) that

$$u_r - u_l + (h_m - h_l)\phi(h_m, h_l) - (h_r - h_m)\phi(h_r, h_m) = 0. \tag{4.4}$$

Thus, for all the four possible wave patterns (4.1)-(4.4), we obtain a single nonlinear equation

$$f_r(h_m, U_r) + f_l(h_m, U_l) + u_r - u_l = 0, \tag{4.5}$$

where

$$f_l(\rho_m, U_l) = \begin{cases} (h_m - h_l)\phi(h_m, h_l), & \text{if } h_m > h_l, \\ \psi(h_m) - \psi(h_l), & \text{if } h_m \leq h_l, \end{cases} \tag{4.6}$$

and

$$f_r(\rho_m, U_r) = \begin{cases} (h_m - h_r)\phi(h_m, h_r), & \text{if } h_m > h_r, \\ \psi(h_m) - \psi(h_r), & \text{if } h_m \leq h_r. \end{cases} \tag{4.7}$$

We solve (4.5) for $h_m$ by using Newton-Raphson iterative procedure with a stop criterion when the relative error is less than $10^{-8}$; the initial guess for $h_m$ is taken to be the average value of $h_l$ and $h_r$. Once $h_m$ is known, the solution for the $x$-component of fluid velocity $u_m$ can be obtained from (2.5) or (2.17) (respectively, from (2.5) or (2.18)) depending on whether the 1-wave (respectively, 2-wave) is a bore or a dilatation wave. In case of dilatation waves, we have to find the solution inside the wave region. For 1-dilatation wave, the slope of the characteristic from $(0,0)$ to $(x,t)$ is

$$\frac{dx}{dt} = \frac{x}{t} = u - c, \tag{4.8}$$

where the fluid velocity $u$ and the speed of propagation of surface disturbance $c$ are functions of the unknown $h$.
Since $\Pi_1$ is constant in 1-dilatation wave region we have

$$u = u_l + \psi(h_l) - \psi(h), \tag{4.9}$$

which in view of (4.8) yields

$$u_l + \psi(h_l) - \psi(h) - \frac{x}{t} - c = 0. \tag{4.10}$$

Equation (4.10) is solved for $h$ using Newton-Raphson method and then $u$ is found from (4.9). In a similar way, we find the solution inside the 2-dilatation wave.

When $h_l < h_m$ and $h_r < h_m$, the solution of the Riemann problem consists of 1-bore

| Test | $h_l$ | $u_l$ | $h_m$ | $u_m$ | $h_r$ | $u_r$ | Result |
|---|---|---|---|---|---|---|---|
| 1 | 1.0 | 1.0 | 2.13116 | 0.0 | 1.0 | $-1.0$ | $B_1B_2$ |
| 2 | 1.0 | $-0.5$ | 0.65516 | 0.0 | 1.0 | 0.5 | $D_1D_2$ |
| 3 | 0.8 | 1.1 | 1.32917 | 0.54619 | 1.7 | 0.9 | $B_1D_2$ |
| 4 | 3.0 | 0.0 | 2.20694 | 0.54306 | 1.5 | 0.0 | $D_1B_2$ |

Table 1

| $H = 0$ | | $H = 0.1$ | | $H = 0.2$ | |
|---|---|---|---|---|---|
| $h_m$ | $u_m$ | $h_m$ | $u_m$ | $h_m$ | $u_m$ |
| 2.18076 | 0.51062 | 2.20694 | 0.54306 | 2.20732 | 0.55705 |

Table 2

and 2-bore; indeed, for Test 1 (see, Table 1), the solution profiles are shown in figure 4.1 at time $t = 0.1$. When $h_l \geq h_m$ and $h_r \geq h_m$, the solution consists of 1-dilatation wave and 2-dilatation wave, and the solution profiles for Test 2, at time $t = 0.25$, are shown in figure 4.2. When $h_l < h_m$ and $h_r \geq h_m$, the solution consists of 1-bore and 2-dilatation wave, and the solution profiles for Test 3, at time $t = 0.17$, are shown in figure 4.3. Similarly, when $h_l \geq h_m$ and $h_r < h_m$, the solution of the Riemann problem consists of 1-dilatation wave and 2-bore, and the solution profiles for Test 4, at time $t = 0.35$, are shown in figure 4.4. Table 2 shows how for a fixed $U_l$ and $U_r$ (namely, Test 4), the intermediate state $U_m$ (unknown state) is influenced by the presence of advective transport of impulse ($H$); indeed, an increase in $H$ makes the dilatation wave weaker and bore stronger, see figure 4.4.

## 5 Interaction of Elementary Waves

The interaction of elementary waves, obtained from the Riemann problem (3.1), gives rise to new emerging elementary waves. We define the initial function, with two jump discontinuities at $x_1$ and $x_2$, as follows.

$$U(x,t_0)=\begin{cases} U_l, & if\ -\infty < x \le x_1 \\ U_*, & if\ x_1 < x \le x_2 \\ U_r, & if\ x_2 < x < \infty, \end{cases} \qquad (5.1)$$

with an appropriate choice of $U_*$ and $U_r$ in terms of $U_l$ and arbitrary $x_1$ and $x_2 \in \mathbb{R}$. With the above initial data, we have two Riemann problems locally. An elementary wave of the first Riemann problem may interact with an elementary wave of the second Riemann problem, and a new Riemann problem is formed at the time of interaction. For one dimensional Euler equations, a discussion of the interaction of elementary waves may be found in [7], [12] and [22]; it may be remarked that the analysis presented in [22] does not include the interaction of shock and rarefaction waves from same family, which we discuss here in subsection 5.2.

Here, we use the notation $D_2B_1 \to B_1D_2$, which means that a 2-dilatation wave, $D_2$, of first Riemann problem (connecting $U_l$ to $U_*$) interacts with 1-bore, $B_1$, of second Riemann problem (connecting $U_*$ to $U_r$), and the interaction leads to a new Riemann problem (connecting $U_l$ to $U_r$ via $U_m$), the solution of which consists of 1-bore, $B_1$, and a 2-dilatation wave $D_2$ (i.e., $B_1D_2$). The possible interactions of elementary waves belonging to different families are $B_2B_1$, $B_2D_1$, $D_2D_1$ and $D_2B_1$ while the elementary wave interactions belonging to the same family are $B_2B_2$, $B_1B_1$, $D_1B_1$, $B_1D_1$, $B_2D_2$ and $D_2B_2$.

## 5.1 Interaction of Elementary Waves from Different Families

i) ***Collision of two bores*** ($B_2B_1$):
We consider that $U_l$ is connected to $U_*$ by a 2-bore, $B_2$, of first Riemann problem and $U_*$ is connected to $U_r$ by a 1-bore, $B_1$, of second Riemann problem. In other words, for a given $U_l$, we choose $U_*$ and $U_r$ in such a way that $h_* < h_l$, $u_* = u_l + (h_* - h_l)\phi(h_*, h_l)$ and $h_* < h_r$, $u_r = u_* - (h_r - h_*)\phi(h_r, h_*)$. Since speed of 2-bore of the first Riemann problem is greater than speed of 1-bore of the second Riemann problem, $B_2$ overtakes $B_1$. In order to show that for any arbitrary state $U_l$, the state $U_r$ lies in the region II (see figure 3a), it is sufficient to prove that $(h - h_*)\phi(h, h_*) - (h - h_l)\phi(h, h_l) + (h_l - h_*)\phi(h_l, h_*) > 0$ for $h_* < h_l$ and $h_* < h$.

Since $\phi$ is a decreasing function with respect to second argument and $h_* < h_l$, so $\phi(h, h_*) > \phi(h, h_l)$ which implies that $(h - h_l)\phi(h, h_l) - (h_l - h_*)\phi(h_l, h_*) < (h - h_l)\phi(h, h_l) < (h - h_*)\phi(h, h_*)$. Hence $(h - h_*)\phi(h, h_*) - (h - h_l)\phi(h, h_l) + (h_l - h_*)\phi(h_l, h_*) > 0$, i.e., the curve $B_1(U_*)$ lies below the curve $B_1(U_l)$, and therefore $U_r$ lies in the region II. Thus, in view of the results presented in the preceding section, it follows that the interaction result is $B_2B_1 \to B_1B_2$; the computed results illustrate this case in figure 5.1a.
ii) ***Collision of a bore and dilatation wave***($B_2D_1$):
Here $U_* \in B_2(U_l)$ and $U_r \in D_1(U_*)$. That is, for a given $U_l$, we choose $U_*$ and $U_r$ such that $h_* < h_l$, $u_* = u_l + (h_* - h_l)\phi(h_*, h_l)$ and $h_r \le h_*$, $u_r = u_* + \psi(h_*) - \psi(h_r)$. Since 2-bore

has greater velocity than 1-dilatation wave , it follows that $B_2$ overtakes $D_1$. Moreover, since for any given $U_l$, $\psi(h_l) - \psi(h_*) - (h_* - h_l)\phi(h_*, h_l) > 0$ for $h < h_* < h_l$, it follows that the curve $D_1(U_*)$ lies below the curve $D_1(U_l)$; hence $U_r$ lies in the region III, and subsequently $B_2D_1 \to D_1B_2$. The computed results illustrate this case in figure 5.1b.

iii) ***Collision of two dilatation waves*** ***(**$D_2D_1$**)***:
We consider $U_* \in D_2(U_l)$ and $U_r \in D_1(U_*)$. In other words, for a given $U_l$, we choose $U_*$ and $U_r$ such that $h_l \leq h_*$, $u_* = u_l - \psi(h_l) + \psi(h_*)$ and $h_r \leq h_*$, $u_r = u_* + \psi(h_*) - \psi(h_r)$. Since the trailing end of 2-dilatation wave has a greater velocity (bounded above) in $(x,t)$-plane than that 1-dilatation wave velocity (bounded above), interaction will take place. Since $h_l < h_*$ and $\psi$ is increasing function of $h$, therefore $\psi(h_l) < \psi(h_*)$, it follows that the curve $D_1(U_*)$ lies above the curve $D_1(U_l)$; hence $U_r$ lies in the region IV and the interaction result is $D_2D_1 \to D_1D_2$. The computed results illustrate this case in figure 5.1c.

iv) ***Collision of a dilatation wave and a bore*** ***(**$D_2B_1$**)***:
Here $U_* \in D_2(U_l)$ and $U_r \in B_1(U_*)$, i.e., for a given $U_l$, we choose $U_*$ and $U_r$ such that $h_l \leq h_*$, $u_* = u_l - \psi(h_l) + \psi(h_*)$ and $h_* < h_r$, $u_r = u_* - (h_* - h_r)\phi(h_*, h_r)$. Since 1-bore speed of second Riemann problem is less than the speed of trailing end of 2-dilatation wave of first Riemann problem in $(x,t)$-plane, and therefore $B_1$ penetrates $D_2$. For any given $U_l$, we show that $U_r \in I$; for this, it is enough to show that

$$\psi(h_*) - \psi(h_l) + (h - h_l)\phi(h, h_l) - (h - h_*)\phi(h_*, h) > 0. \tag{5.2}$$

Since $\psi(h)$ is a increasing function of $h$, we have $\psi(h_*) > \psi(h_l)$ for $h_l < h_*$; hence, the inequality (5.2) follows, implying thereby that the curve $B_1(U_*)$ lies above the curve $B_1(U_l)$, and $U_r$ lies in the region I. Thus the interaction result is $D_2B_1 \to B_1D_2$; the computed results illustrate this case in figure 5.1d.

## 5.2 Interaction of Elementary Waves from Same Family

i) ***2-bore overtakes another 2-bore*** ***(**$B_2B_2$**)***:
We consider the situation in which $U_l$ is connected to $U_*$ by a 2-bore of first Riemann problem and $U_*$ is connected to $U_r$ by a 2-bore of second Riemann problem. In other words, for a given left state $U_l$, the intermediate state $U_*$ and the right state $U_r$ are chosen such that $h_* < h_l$ and $u_* = u_l + (h_* - h_l)\phi(h_*, h_l)$ with Lax stability conditions

$$\lambda_1(U_l) < \sigma_2(U_l, U_*) < \lambda_2(U_l), \qquad \lambda_2(U_*) < \sigma_2(U_l, U_*), \tag{5.3}$$

and $h_r < h_*$, $u_r = u_* + (h_r - h_*)\phi(h_r, h_*)$ with Lax stability conditions

$$\lambda_1(U_*) < \sigma_2(U_*, U_r) < \lambda_2(U_*), \qquad \lambda_2(U_r) < \sigma_2(U_*, U_r), \tag{5.4}$$

where $\sigma_2(U_l, U_*)$ is the speed of bore connecting $U_l$ to $U_*$, and similarly $\sigma_2(U_*, U_r)$ is the speed of bore connecting $U_*$ to $U_r$. From (5.3) and (5.4) we obtain $\sigma_2(U_*, U_r) < \sigma_2(U_l, U_*)$, i.e., the 2-bore of second Riemann problem overtakes 2-bore of the first Riemann problem after a finite time, and gives rise to a new Riemann problem with data $U_l$ and $U_r$. In order to solve this problem, we must determine the region in which $U_r$ lies with respect to $U_l$. We claim that $U_r$ lies in region III so that the solution of the new Riemann problem consists of $D_1$ and $B_2$. In other words, to prove our claim, we need to show that $B_2(U_*)$ lies entirely in the region III; to show this we are required to prove that for $h < h_* < h_l$,

$$(h_l - h)\phi(h, h_l) - (h_l - h_*)\phi(h_l, h_*) - (h_* - h)\phi(h, h_*) > 0. \tag{5.5}$$

Let us define a new function $f_1(h) = (h_l - h)\phi(h, h_l) - (h_l - h_*)\phi(h_l, h_*) - (h_* - h)\phi(h, h_*)$, so that $f_1(h_*) = 0$, and differentiate $f_1(h)$ with respect to $h$ to obtain

$$\frac{df_1}{dh} = (h_l - h)\frac{d\phi(h, h_l)}{dh} - \phi(h, h_l) - ((h_* - h)\frac{d\phi(h, h_*)}{dh} - \phi(h, h_*)). \tag{5.6}$$

Now we define

$$f_2(h, h_l) = (h_l - h)\frac{d\phi(h, h_l)}{dh} - \phi(h, h_l). \tag{5.7}$$

Since $f_2(h, h_l)$ is a decreasing function with respect to second variable $h_l$ for $h < h_l$, we have $f_2(h, h_l) < f_2(h, h_*)$, it follows from (5.6) that $f_1(h)$ is a decreasing function of $h$, which implies that $f_1(h) > f_1(h_*) = 0$. Hence, $B_2B_2 \to D_1B_2$; the computed results illustrate this situation in figure 5.2a.

ii) ***1-bore overtakes another 1-bore** ($B_1B_1$)*:

The analytical proof that $U_r$ lies in the region I, so that $B_1B_1 \to B_1D_2$, is similar to the previous case.

iii) ***1-bore overtakes 1-dilatation wave** ($D_1B_1$)*:

In this case, $U_l$ is connected to $U_*$ by 1-dilatation wave of the first Riemann problem and $U_*$ is connected to $U_r$ by 1-bore of the second Riemann problem. That is, for a given $U_l$, we choose $U_*$ and $U_r$ in such a way that $h_* \le h_l$, $u_* = u_l + \psi(h_l) - \psi(h_*)$ and $h_* < h_r$, $u_r = u_* - (h_r - h_*)\phi(h_r, h_*)$.

First we show that $D_1(U_l)$ lies above the curve $B_1(U_*)$ for $h_* < h \le h_l$; in other words, for $h_* < h \le h_l$

$$\psi(h_*) - \psi(h) + (h - h_*)\phi(h, h_*) > 0. \tag{5.8}$$

Let us define $f_3(h) = \psi(h_*) - \psi(h) + (h - h_*)\phi(h, h_*)$ so that $f_3(h_*) = 0$. Differentiating $f_3(h)$ with respect to $h$, we have to show that $f_3'(h) > 0$. Let us assume on contrary that $f_3'(h) \le 0$, which implies that $(h - h_*)\frac{d\phi(h,h_*)}{dh} + \phi(h, h_*) \le \frac{c}{h}$; squaring both sides and simplifying it, we obtain

$$(h - h_*)^2((h_* + 2H)^2 + 4h(h + h_* + 2H)) \le 0, \tag{5.9}$$

which is a contradiction, since the left hand side of the inequality (5.9) is strictly positive. Therefore $f_3'(h) > 0$, implying thereby that $f_3(h) > f_3(h_*) = 0$; hence $D_1(U_l)$ lies above the curve $B_1(U_*)$ for $h_* < h \leq h_l$.

Next we prove that $B_1(U_l)$ lies above the curve $B_1(U_*)$ for $h_l < h$; for this it is sufficient to prove that

$$\psi(h_*) - \psi(h_l) + (h - h_*)\phi(h, h_*) - (h - h_l)\phi(h, h_l) > 0, \quad \forall\ h_l < h. \tag{5.10}$$

Let us define $f_4(h) = \psi(h_*) - \psi(h_l) + (h - h_*)\phi(h, h_*) - (h - h_l)\phi(h, h_l)$ so that $f_4(h_l) = f_3(h_l) > 0$. It may be noticed that $f_4'(h) = \phi(h, h_*) + (h - h_*)\frac{d\phi(h,h_*)}{dh} - \phi(h, h_l) - (h - h_l)\frac{d\phi(h,h_l)}{dh} > 0$, since $\phi(h, h_*) + (h - h_*)\frac{d\phi(h,h_*)}{dh}$ is a decreasing function with respect to $h_*$ for $h_* < h$; thus $f_4'(h) > 0$ for $h_* < h_l < h$, and hence $\psi(h_*) - \psi(h_l) + (h - h_*)\phi(h, h_*) - (h - h_l)\phi(h, h_l) > 0$.

Lastly, we show that $B_2(U_l)$ and $B_1(U_*)$ intersect at some point $(\tilde{h}_1, \tilde{u}_1)$, where $h_* < \tilde{h}_1 < h_l$. To prove this, we define a new function $f_5(h) = \psi(h_l) - \psi(h_*) - (h - h_*)\phi(h, h_*) - (h - h_l)\phi(h, h_l)$ for $h_* \leq h \leq h_l$. Since $f_5(h_l) = -f_3(h_l) < 0$ and $f_5(h_*) > 0$, by virtue of monotonicity and intermediate value property, there exists a unique $\tilde{h}_1$, between $h_*$ and $h_l$, such that $f_5(\tilde{h}_1) = 0$. Thus, the intersection of $B_2(U_l)$ and $B_1(U_*)$ is uniquely determined; the computed results are shown in figure 5.2b. Thus, depending on the value of $h_r$ we distinguish three cases:

a) When $h_r < \tilde{h}_1$, $U_r \in III$ and the interaction result is $D_1B_1 \to D_1B_2$; indeed, 1-bore is weak compared to 1-dilatation wave.

b) When $h_r = \tilde{h}_1$, $U_r$ lies on $B_2(U_l)$ and the interaction result is $D_1B_1 \to B_2$; thus, when two waves of first family interact, they annihilate each other, and give rise to a wave of second family.

c) When $h_r > \tilde{h}_1$, $U_r \in II$ and the interaction result is $D_1B_1 \to B_1B_2$; this means that 1-bore of second Riemann problem, which is strong compared to the 1-dilatation wave of first Riemann problem, overtakes the trailing end of 1-dilatation wave, and a reflected bore $B_2(U_m, U_r)$, connecting a new constant state $U_m$ on the left to the known state $U_r$ on the right, is produced. The transmitted wave, after interaction, is the 1-bore that joins $U_l$ on the left to the state $U_m$ on the right.

iv) ***1-dilatation wave overtakes 1-bore*** $(B_1D_1)$:

Here $U_* \in B_1(U_l)$ and $U_r \in D_1(U_*)$. That is, for a given $U_l$, we choose $U_*$ and $U_r$ such that $h_l < h_*$, $u_* = u_l - (h_* - h_l)\phi(h_*, h_l)$ and $h_r \leq h_*$, $u_r = u_* + \psi(h_*) - \psi(h_r)$. In the $(x, t)$ plane the speed of trailing end of 1-dilatation wave, $\lambda_1(U_*)$, is less than the velocity $\sigma_1(U_l, U_*)$ and therefore 1-dilatation wave from right overtakes 1-bore from left after a finite time.

First we show that $B_1(U_l)$ lies above the curve $D_1(U_*)$ for $h_l < h < h_*$; for this we need to show $\psi(h) - \psi(h_*) - (h - h_l)\phi(h, h_l) + (h_* - h_l)\phi(h_*, h_l) > 0$ for $h_l < h < h_*$. To prove this, we define a new function $f_6(h) = \psi(h) - \psi(h_*) - (h - h_l)\phi(h, h_l) + (h_* - h_l)\phi(h_*, h_l)$ for $h_l < h < h_*$. Then one can show that $f_6'(h) = \frac{c}{h} - \phi(h, h_l) - (h - h_l)\frac{d\phi(h,h_l)}{dh} < 0$ for $h_l < h < h_*$, implying thereby that $f_6(h) > f_6(h_*) = 0$.

Next we show that $D_1(U_*)$ lies below the curve $D_1(U_l)$ for $h \le h_l < h_*$, i.e., $\psi(h_l) - \psi(h_*) + (h_* - h_l)\phi(h_*, h_l) > 0$ for $h \le h_l < h_*$. Since the left hand side of this inequality, for $h \le h_l < h_*$, turns out to be $f_6(h_l)$, which has already been shown to be positive, the conclusion follows.

Lastly, we show that $B_2(U_l)$ and $D_1(U_*)$ intersect uniquely at some point, say, $(\tilde{h}_2, \tilde{u}_2)$; for this, it is enough to show that the equation $\psi(h)-\psi(h_*)+(h-h_l)\phi(h,h_l)+(h_*-h_l)\phi(h_*,h_l) = 0$ has a unique root $\tilde{h}_2$ such that $\tilde{h}_2 < h_l$. To establish this, we define a new function $f_{7(h)} = \psi(h) - \psi(h_*) + (h - h_l)\phi(h, h_l) + (h_* - h_l)\phi(h_*, h_l)$; since $f_7(h_l) > 0$, and $f_7(h)$ takes negative values as $h$ is close to zero, in view of monotonicity and intermediate value property, it follows that the curves $D_1(U_*)$ and $B_2(U_l)$ intersect uniquely; the computed results are shown in figure 5.2c. Here again we distinguish three cases depending on the value of $h_r$:

a) When $h_r > \tilde{h}_2$, $U_r \in II$ and the interaction result is $B_1D_1 \to B_1B_2$, i.e., 1-bore is sufficiently strong compared to 1-dilatation wave which, after interaction, produces a new elementary wave.

b) When $h_r = \tilde{h}_2$, $U_r \in B_2(U_l)$ and the interaction result is $B_1D_1 \to B_2$, i.e., the interaction of elementary waves of first family gives rise to a new elementary wave of second family.

c) When $h_r < \tilde{h}_2$, $U_r \in III$ and the interaction result is $B_1D_1 \to D_1B_2$.

v) ***2-dilatation wave overtakes 2-bore($B_2D_2$)***:

The $B_2D_2$ interaction takes place when $U_* \in B_2(U_l)$ and $U_r \in D_2(U_*)$. In other words, for a given $U_l$, we choose $U_*$ and $U_r$ in such a way that $h_* < h_l$, $u_* = u_l + (h_* - h_l)\phi(h_*, h_l)$ and $h_* \le h_r$, $u_r = u_* - \psi(h_*) + \psi(h_r)$.

First we show that for $h_* < h < h_l$, $B_2(U_l)$ lies above $D_2(U_*)$, i.e.,

$$\psi(h_*) - \psi(h) + (h - h_l)\phi(h, h_l) + (h_l - h_*)\phi(h_*, h_l) > 0, \qquad \forall\ h \in (h_*, h_l]. \tag{5.11}$$

To prove this we define a new function $f_8(h) = \psi(h_*)-\psi(h)+(h-h_l)\phi(h,h_l)+(h_l-h_*)\phi(h_*,h_l)$ so that $f_8(h_*) = 0$. Since $f_8'(h) > 0$, we have $f_8(h) > f_8(h_*)$; it follows that $f_8(h) > 0$, implying thereby that $B_2(U_l)$ lies above $D_2(U_*)$.

Next we show that the curve $D_2(U_l)$ lies above the curve $D_2(U_*)$ for $h_* < h_l \le h$; for this it is enough to prove $\psi(h_*) - \psi(h_l) + (h_l - h_*)\phi(h_*, h_l) > 0$ for $h_* < h_l \le h$. We notice that the left hand side of this inequality is $f_8(h_l)$ which has already been shown to be positive, and hence the curve $B_2(U_l)$ lies above the curve $D_2(U_*)$ for $h_* < h_l \le h$.

Lastly, we show that $D_2(U_*)$ and $B_1(U_l)$ intersect uniquely, say, at $(\tilde{h}_3, \tilde{u}_3)$ for $h_* < h_l < \tilde{h}_3$. Now we define $f_9(h) = \psi(h) - \psi(h_*) + (h - h_l)\phi(h, h_l) + (h_* - h_l)\phi(h_*, h_l)$ for $h_* < h_l \le h$ so that $f_9(h_l) < 0$, and we can choose a constant $K > 0$ such that $f_9(h) > 0$ for all $h > K$. Then, there exists a $\tilde{h}_3$ such that $f_9(\tilde{h}_3) = 0$. Thus $B_2(U_*)$ and $B_1(U_l)$ intersect uniquely at $(\tilde{h}_3, \tilde{u}_3)$ as $D_2(U_*)$ and $B_1(U_l)$ are monotone; the computed results are shown in figure 5.2d. Here again the following cases arise:

a) When $h_r < \tilde{h}_3$, $U_r \in II$ and the interaction result is $B_2D_2 \to B_1B_2$; this means that the strength of $D_2$ is small compared to the elementary wave $B_2$, and $B_2$ annihilates $D_2$ in a finite time. The strength of reflected $B_1$ wave is small compared to the incident waves $B_2$ and $D_2$.

b) When $h_r = \tilde{h}_3$, $U_r \in B_1(U_l)$ and the interaction result is $B_2D_2 \to B_1$, showing thereby that the reflected shock $B_1$ is weak compared to incident waves $D_2$ and $B_2$.

c) When $h_r > \tilde{h}_3$, $U_r \in I$ and the interaction result is $B_2D_2 \to B_1D_2$, implying thereby that, $D_2$ is stronger than $B_2$.

vi) ***2-bore overtakes 2-dilatation wave** ($D_2B_2$)*;

Here $U_* \in D_2(U_l)$ and $U_r \in B_2(U_*)$. Thus, for a given $U_l$, we choose $U_*$ and $U_r$ such that $h_l \leq h_*$, $u_* = u_l - \psi(h_l) + \psi(h_*)$ and $h_r < h_*$, $u_r = u_* + (h_r - h_*)\phi(h_r, h_*)$.

Now we show that $D_2(U_l)$ lies above $B_2(U_*)$ for $h_l \leq h < h_*$, i.e.,

$$\psi(h) - \psi(h_*) - (h - h_*)\phi(h, h_*) > 0, \quad \forall\, h_l \leq h < h_*. \tag{5.12}$$

To prove this we define a new function $f_{10}(h) = \psi(h) - \psi(h_*) - (h - h_*)\phi(h, h_*)$ for $h_l \leq h \leq h_*$ so that $f_{10}(h_*) = 0$. This, in view of the expressions for $c(h)$ and $\phi(h, h_*)$, yields $\frac{df_{10}(h)}{dh} = \frac{-[(h-h_*)^2((h_*+2H)^2+4h(h+h_*+2H))]}{4h^2h_*\phi(h,h_*)} < 0$, implying thereby that $f_{10}(h) > f_{10}(h_*) = 0$. Hence, the result.

Next we show that $B_2(U_l)$ lies above the curve $B_2(U_*)$ for $h < h_l < h_*$; for this it is sufficient to prove $\psi(h_l) - \psi(h_*) - (h - h_*)\phi(h, h_*) + (h - h_l)\phi(h, h_l) > 0$ for $h < h_l < h_*$. In order to prove this inequality we define a new function $f_{11}(h) = \psi(h_l) - \psi(h_*) - (h - h_*)\phi(h, h_*) + (h - h_l)\phi(h, h_l)$ for $h \leq h_l < h_*$ and $f_{11}(h_l) = f_{10}(h_l) > 0$. Since $\phi(h, h_l) + (h - h_l)\phi(h, h_l)$ is an increasing function with respect to second argument for $h_l > h$, it follows that $\frac{df_{11}(h)}{dh} < 0$, implying thereby that $f_{11}(h) > f_{11}(h_l) > 0$.

Lastly, we show that $B_1(U_l)$ and $B_2(U_*)$ intersect uniquely at a point, say, $(\tilde{h}_4, \tilde{u}_4)$, where $h_l < \tilde{h}_4 < h_*$. The proof for this follows on similar lines as discussed earlier; the computed results are shown in figure 5.2e. Here also we encounter three possibilities:

a) When $h_r > \tilde{h}_4$, $U_r \in I$ and the interaction result is $D_2B_2 \to B_1D_2$; this means that $D_2$ is strong compared to the elementary wave $B_2$, and the strength of reflected $B_1$ is small compared to the incident waves $B_2$ and $D_2$.

b) When $h_r = \tilde{h}_4$, $U_r \in B_1(U_l)$ and the interaction result is $D_2B_2 \to B_1$.

c) When $h_r < \tilde{h}_4$, $U_r \in II$ and the interaction result is $D_2B_2 \to B_1B_2$, implying thereby that the elementary wave $B_2$ is strong compared to $D_2$.

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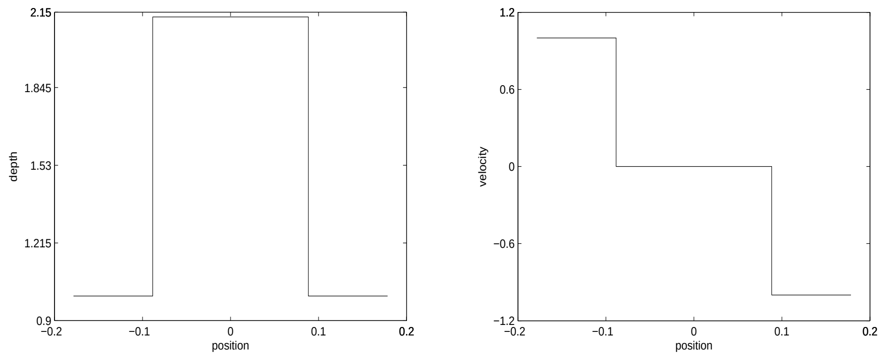
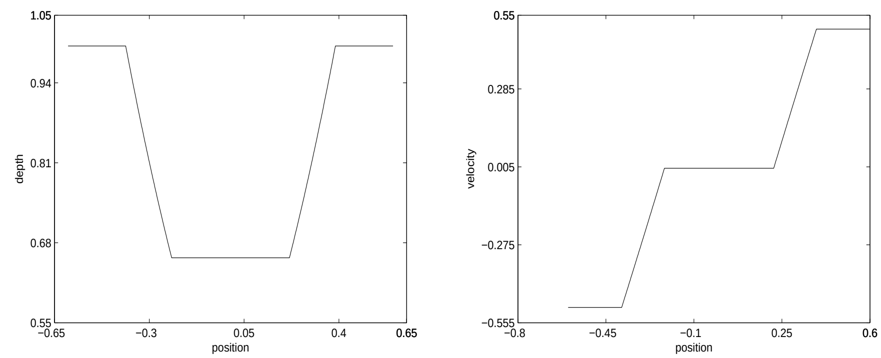
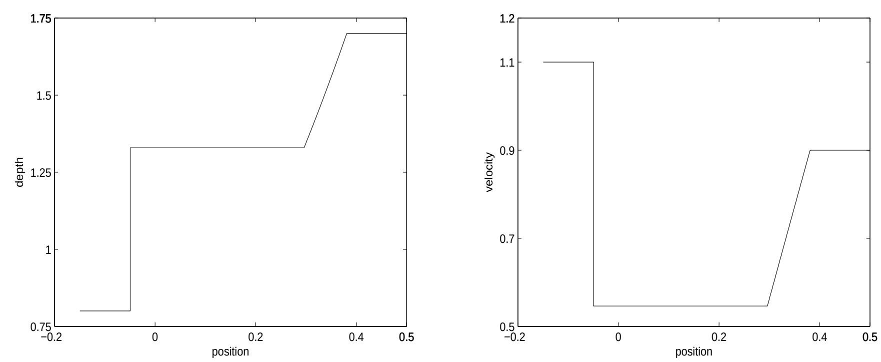


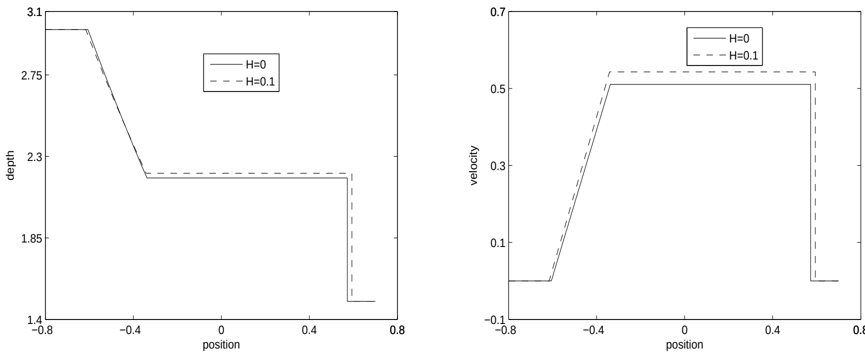

$Fig.4.1$ : Exact solution for depth and velocity at $t = 0.1$



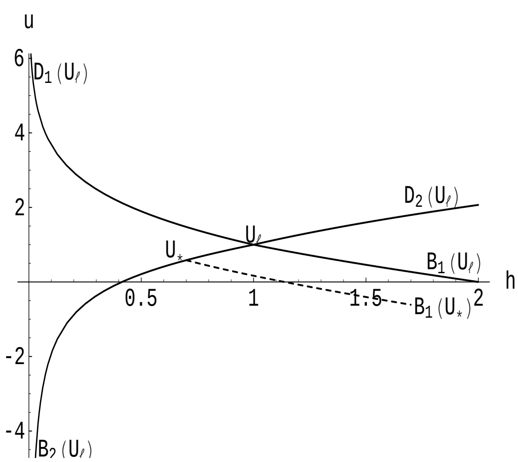

$Fig.4.2$ : Exact solution for depth and velocity at $t = 0.25$



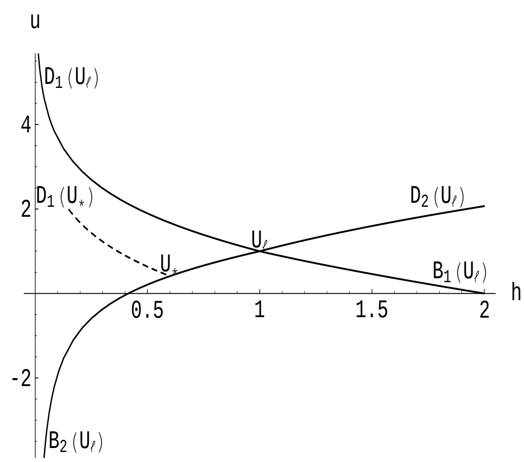

$Fig.4.3$ : Exact solution for depth and velocity at $t = 0.17$



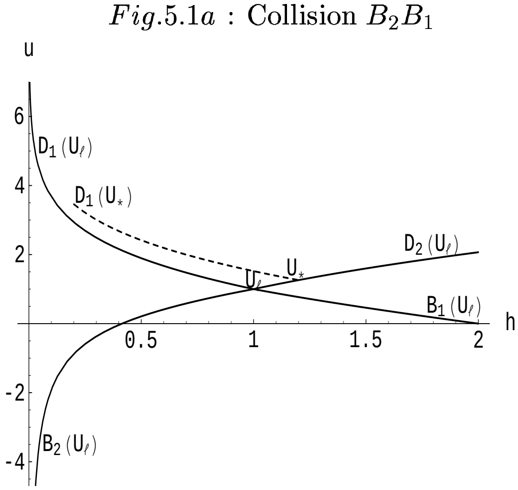

$Fig.4.4$ : Exact solution for depth and velocity at $t = 0.35$



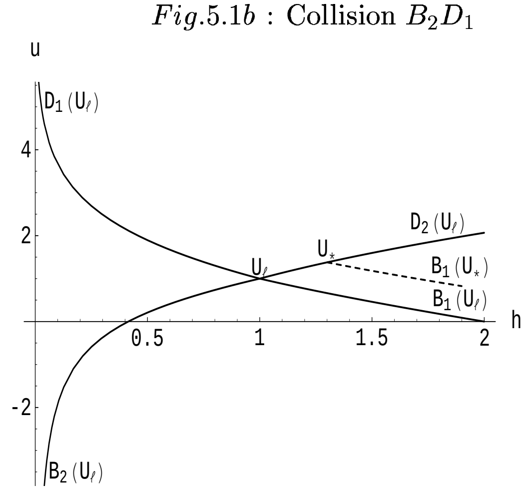

$Fig.5.1a$ : Collision $B_2B_1$



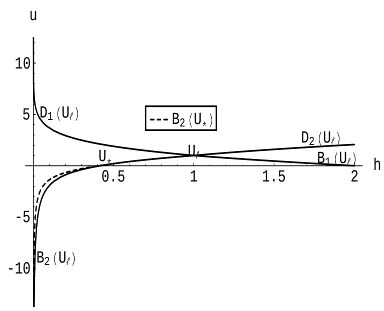

$Fig.5.1b$ : Collision $B_2D_1$



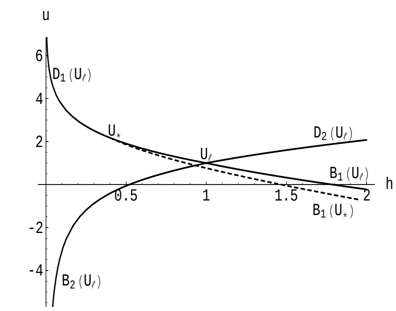

$Fig.5.1c$ : Collision $D_2D_1$



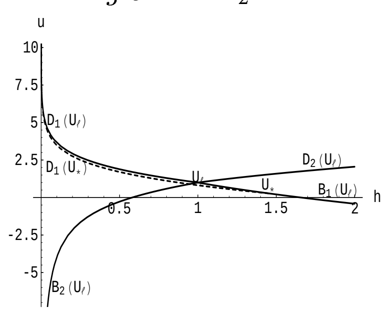

$Fig.5.1d$ : Collision $D_2B_1$



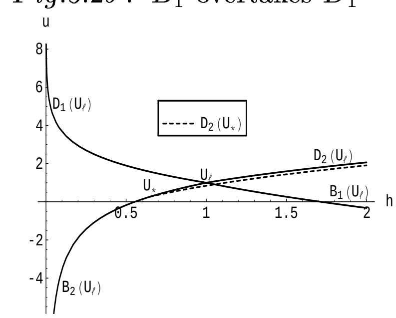

$Fig.5.2a:$ $B_2$ overtakes $B_2$



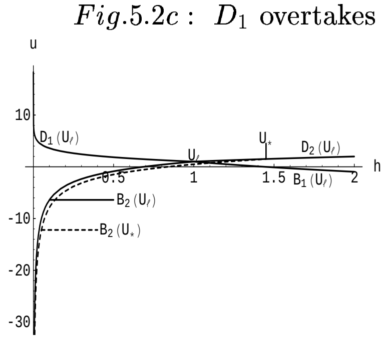

$Fig.5.2b:$ $B_1$ overtakes $D_1$




$Fig.5.2c:$ $D_1$ overtakes $B_1$




$Fig.5.2d:$ $D_2$ overtakes $B_2$




$Fig.5.2e:$ $B_2$ overtakes $D_2$