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Evolution of weak discontinuities in shallow water equations

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Abstract

In this paper, we determine the critical time, when a weak discontinuity in the shallow water equations culminates into a bore. Invariance group properties of the governing system of partial differential equations (PDEs), admitting Lie group of point transformations with commuting infinitesimal operators, are presented. Some appropriate canonical variables are characterized that transform equations at hand to an equivalent form, which admits non-constant solutions. The propagation of weak discontinuities is studied in the medium characterized by the particular solution of the governing system.
Keywords: Shallow water equations: Group theoretic method; Exact solution; Weak discontinuity

1 Introduction

For nonlinear systems, we do not have the luxury of complete exact solutions; for analytic work we have to rely on some approximate analytical or numerical methods which may be useful to set the scene and provide useful information towards our understanding of the complete physical phenomena involved. Special exact solutions of a system of nonlinear PDEs are of great interest; these solutions play a major role in designing, analyzing and testing of numerical methods for solving special initial / or boundary value problems. One of the most powerful methods in order to determine particular solutions to PDEs is based upon the study of their invariance with respect to one parameter Lie group of point transformations [1]. Indeed the invariance reduces the number of independent variables by one. Besides similarity methods, another use of Lie point symmetries admitted by a given system of PDEs consists in introducing some invertible point transformations that map the original system to an equivalent one admitting special solutions [2]. A different approach has been described by Oliveri and Speciale [3-4] for unsteady equations of perfect gases and ideal magnetogasdynamic equations using substitution principles. Ames and Donato [5] obtained solutions for the problem of elastic-plastic deformation generated by torque , and analyzed the

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evolution of weak discontinuities in a state characterized by invariant solutions. In the present paper, we consider the propagation of weak discontinuities in a medium characterized by the particular solution, and determine the critical time when a weak discontinuity culminates into a bore.

2 Lie Group Analysis

The system of equations which governs the one dimensional shallow water equations, can be written as [6]

$$\begin{aligned} u_t + uu_x + 2cc_x &= 0, \\ c_t + (c/2)u_x + uc_x &= 0, \end{aligned} \tag{2.1}$$

where u is the x-component of fluid velocity and $c = \sqrt{gh}$ is the speed of propagation of surface disturbance in water of variable depth h subjected to an acceleration due to gravity g. The independent variables t and x denote time and space respectively.

Using a straight forward analysis, it is found that the system (2.1) admits the group $Y = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$, where α_1, α_2 and α_3 are arbitrary constants and $X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + c\frac{\partial}{\partial c}$, $X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$ and $X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - c\frac{\partial}{\partial c}$. We introduce canonical variables τ, ξ, U and C such that $Y\tau = 1, Y\xi = 0, YU = 0$, and $YC = 0$. This implies that when $\alpha_3 \neq 0$ and $\alpha_1 \neq \alpha_3$, we have

$$\begin{aligned} &\tau = \frac{1}{\alpha_3}\ln(t), \quad \xi = (x - \frac{\alpha_2 t}{\alpha_3 - \alpha_1})t^{-\alpha_1/\alpha_3}, \quad C(\tau,\xi) = ct^{(\alpha_3-\alpha_1)/\alpha_3}, \\ &U(\tau,\xi) = (u - \frac{\alpha_2}{\alpha_3 - \alpha_1})t^{(\alpha_3-\alpha_1)/\alpha_3}. \end{aligned} \tag{2.2}$$

In terms of these new canonical variables, the system (2.1) becomes

$$\begin{aligned} \frac{\partial U}{\partial \tau} + \alpha_3(U - \frac{\alpha_1}{\alpha_3}\xi)\frac{\partial U}{\partial \xi} + 2\alpha_3 C\frac{\partial C}{\partial \xi} + (\alpha_1 - \alpha_3)U = 0, \\ \frac{\partial C}{\partial \tau} + \alpha_3(U - \frac{\alpha_1}{\alpha_3}\xi)\frac{\partial C}{\partial \xi} + \frac{\alpha_3 C}{2}\frac{\partial U}{\partial \xi} + (\alpha_1 - \alpha_3)C = 0. \end{aligned} \tag{2.3}$$

The system of equations (2.3) satisfies a particular solution of the form

$$U = \frac{2\xi}{3}, \quad C = \frac{\xi}{3}. \tag{2.4}$$

Thus, in view of (2.4) and (2.2), the solution of the system (2.1) is as follows;

$$u = \frac{2x}{3t} + \frac{\alpha_2}{3(\alpha_3 - \alpha_1)}, \quad c = \frac{1}{3}(\frac{x}{t} - \frac{\alpha_2}{(\alpha_3 - \alpha_1)}). \tag{2.5}$$

It is interesting to observe that the above solution, in particular for $\alpha_2 = 0$, is identical with the solution given in Akyildiz [7] using a different approach. It may be remarked that the state such as this, where the particle velocity exhibits linear dependence on the spatial coordinate, has been discussed by Pert [8], Sharma et al. [9] and Clarke [10]; Pert has shown that such a form of velocity distribution is useful in modelling the free expansion of polytropic fluids, and is attained in the large time limit.

3 Evolution of Weak Discontinuities

The governing system (2.1) can be written in matrix form

$$W_t + AW_x = 0, \tag{3.1}$$

where $W = (h, u)^T$ is a column vector with superscript T denoting transposition, while A is a 2×2 matrix with elements $A_{11} = A_{22} = u$, $A_{12} = 2c$ and $A_{21} = c/2$. The matrix A has the eigenvalues

$$\lambda^{(1)} = u - c, \qquad \lambda^{(2)} = u + c \tag{3.2}$$

with the corresponding left and right eigenvectors

$$\begin{aligned} l^{(1)} &= (1, -2), \quad r^{(1)} = (2, -1)^T, \\ l^{(2)} &= (1, 2), \quad r^{(2)} = (2, 1)^T. \end{aligned} \tag{3.3}$$

The evolution of weak discontinuity for a hyperbolic quasilinear system of equations satisfying the Bernoulli's law has been studied quite extensively in the literature (see, [11-13]). The transport equation for the weak discontinuities across the i^{th} characteristic of a hyperbolic system of n equations of the type (3.1) is given by (see, [13])

$$\begin{aligned} l_0^{(i,k)}\Big(\frac{d\Lambda_i}{dt} + (W_{0x} + \Lambda_i)(\nabla\lambda^i)_0\Lambda) + ((\nabla l^{(i,k)})_0\Lambda_i)^T\frac{dW_0}{dt} \\ +(l_0^{(i,k)}\Lambda_i)((\nabla\lambda^i)_0 W_{0x} + \lambda^i_{0x}) = 0, \end{aligned} \tag{3.4}$$

where the coefficient matrix possesses q distinct eigenvalues λ^i, $i = 1, 2, 3, ..., q$, assumed to be ordered so that $\lambda^{(1)} < \lambda^{(2)} < \lambda^{(3)} < .. < \lambda^{(q-1)} < \lambda^{(q)}$ with multiplicities m_i, such that $\sum_{i=1}^{q} = n$, together with n linearly independent left and right eigenvectors $l^{(i,k)}$ and $r^{(i,k)}$, $k = 1, 2, 3, .., m_i$, corresponding to the eigenvalues $\lambda^{(i)}$. Here, the subscript 0 refers to the state ahead of the i^{th} characteristic curve, and

$$\Lambda_i = \sum_{k=1}^{m_i} \alpha_k^i(t) r_0^{(i,k)} \tag{3.5}$$

is the jump in W_x across the i^{th} characteristic curve with $\alpha_k^{(i)}$ being the amplitude of the weak discontinuity wave propagating along $dx/dt = \lambda^{(i)}$. For the system under consideration, $\Lambda_2 = \alpha r^{(2)}$ denotes the jump in W_x across the weak discontinuity wave with amplitude α, propagating along the curve determined by $dx/dt = \lambda^{(2)}$ originating from the point (x_0, t_0). Then, on using (3.1), (3.3) and (3.5) in (3.4), we obtain the following Bernoulli type equation for the amplitude α

$$\frac{d\alpha}{dt} + 3\alpha^2 + (\frac{11}{4}u_x + \frac{7}{2}c_x)\alpha = 0. \tag{3.6}$$

In view of the particular solution (2.5), the above equation becomes

$$\frac{d\alpha}{dt} + 3\alpha^2 + \frac{3}{t}\alpha = 0, \tag{3.7}$$

which yields on integration

$$\alpha = \frac{\alpha_0}{\eta((1+k)\eta^2 - k)}, \tag{3.8}$$

where α_0 is the value of α at $t = t_0$, $k = (3t_0\alpha_0)/2$ and $\eta = t/t_0$. Eqn (3.8) shows that if $\alpha_0 > 0$ then $\alpha \to 0$ as $\eta \to \infty$, implying thereby that the wave decays and dies out eventually; the corresponding situation is illustrated by the curve in Fig.1. However, if $\alpha_0 < 0$ it follows from (3.8) that there are two possibilities:

(a) Let $|\alpha_0| \leq \frac{2}{3t_0}$. Then α is finite and nonzero for $\eta < \infty$ and $\alpha \to 0$ as $\eta \to \infty$ implying

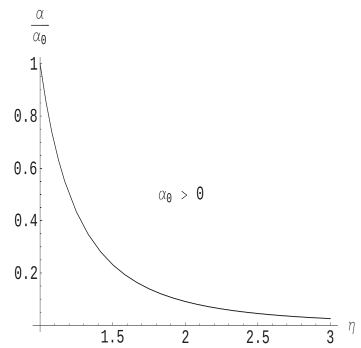

*Fig.*1

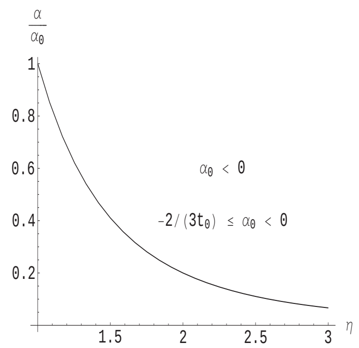

*Fig.*2

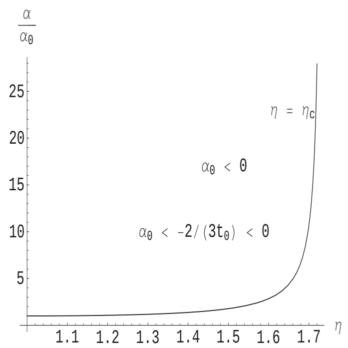

*Fig.*3

Figs. (1-3): Variation of amplitude with η for different values of α_0

thereby that the wave decays, the corresponding situation is illustrated by the curve in Fig.2 with $-\frac{2}{3t_0} \leq \alpha_0 < 0$.

(b) Let $|\alpha_0| > \frac{2}{3t_0}$. Then there exists a finite time $\eta_c > 1$, given by $\eta_c = \sqrt{\frac{k}{1+k}}$, such that α is finite, nonzero and continuous on $[1, \eta_c)$ and $|\alpha| \to \infty$ as $\eta \to \eta_c$. This signifies the appearance of a bore at an instant η_c; indeed, weak discontinuity wave culminates into a bore in a finite time only when the initial discontinuity associated with the wave exceeds a critical value. The corresponding situation is illustrated by the curve in Fig.3 with $\alpha_0 < -\frac{2}{3t_0} < 0$.

4 Conclusions

Lie group analysis is used to obtain an exact solution of partial differential equations that describe one dimensional shallow water equations. The particle velocity described by the exact solution is useful in modelling the free expansion of polytropic fluids, and is attained in the large time limit (see, [8-10]). The evolution of weak discontinuities in a state characterized by the exact solution is studied. It is shown that a weak discontinuity wave culminates into a bore after a finite time, only if the initial discontinuity associated with it exceeds a critical value *i.e.*, $\alpha_0 < -\frac{2}{3t_0} < 0$ (see, Fig. 3). However, when $-\frac{2}{3t_0} < \alpha_0 < 0$ or $\alpha_0 > 0$, in both the cases the wave decays eventually (see, Figs. 1-2).

To our knowledge, an analytical description towards achieving a detailed comprehension of the wave interaction problem involving bores and weak discontinuity waves has not been studied. In order to study this problem, we need to know an exact solution of the system (2.1) satisfying the Rankine-Hugoniot jump conditions that hold on the bore; a search for such a solution using the approach outlined in this paper is currently under way.

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