J. Phys. B: At. Mol. Opt. Phys. 41 (2008) 025401 (9pp) doi:10.1088/0953-4075/41/2/025401

Bright solitons on a cnoidal wave background for the inhomogeneous nonlinear Schrödinger equation

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Received 6 September 2007, in final form 23 November 2007
Published 8 January 2008
Online at stacks.iop.org/JPhysB/41/025401

Abstract

In this paper, we investigate the bright solitons on a cnoidal wave background train of the inhomogeneous nonlinear Schrödinger equation, which may be applicable to many physically realizable systems such as Bose–Einstein condensation media and plasma, etc. We use well-known methods to reduce the inhomogeneous nonlinear Schrödinger equation to a standard nonlinear Schrödinger equation by using the combination of Husimi's and Lens-type transformations. We study the superposed configuration of soliton with a cnoidal wave solution of the underlying equation. Finally, we discuss the dynamics of soliton on a cnoidal wave background in Bose–Einstein condensation trapped in linear density and harmonic density profiles separately.

1. Introduction and model

It is well established that the shape of a soliton remains the same during propagation by establishing a dynamical balance of the spreading from its dispersion with nonlinear interactions [1]. Initially, this feature has been applied in soliton-based fibre optic communications. In the recent past, it has been proved beyond doubt that solitons do exist not only in optics but also in many other areas of science namely, fluids [2], plasmas [3], magnetic films [4] and recently in quantum superfluids of atomic Bose–Einstein condensates (BECs) [5]. Recently, researchers have generated BECs by bringing various dilute gases to extremely low temperatures with the help of the laser and evaporative cooling methods. In general, the Gross–Pitaevskii (GP) equation, or the nonlinear Schrödinger equation (NLS) has been successfully used to model a condensate in weak interactions [6–9]. It is of great interest to mention that the BECs were used to study a number of diverse phenomena, for instance, phase coherence [10], matterwave diffraction [11], quantum logic [12], etc. Recently, BEC transport has also been reported in periodic washboard potentials with linear Stark force [13]. However, in this paper, we study the dynamical evolution of BECs in terms of bright solitons on a cnoidal wave background under the influence of both linear and harmonic trapping potentials.

Cnoidal waves are periodic waves with sharp crests separated by wide flat troughs. Here, the wave characteristics are described in the parametric form in terms of the modulus k, over the range 0 and 1, of the elliptic integrals. Thus, there are two known limits to the cnoidal waves. The first one is the solitary wave theory which occurs when the period of the Jacobian elliptic function is infinite ($k = 1$). The second limit is the linear wave theory which occurs for $k = 0$ wherein the cnoidal wave approaches the sinusoidal wave [14]. In this paper, we utilize both cnoidal and solitary wave theories to investigate the dynamical evolution of BECs.

The dynamics of an adiabatic N-soliton train confined to external fields (quadratic, periodic, and tilted potentials) in the framework of the perturbed complex Toda chain model was investigated, based on the nonlinear Schrödinger equation, both analytically and numerically [15]. The adiabatic N-soliton interactions in weak external potentials were discussed too. Using a parametric field theory approach, the formation of coherent molecular soliton has been investigated in molecular BECs [16]. The condition of discrete solitons generated in BECs trapped in optical lattices has been analysed for the case of both positive and negative atomic scattering lengths [17]. In [18], the dynamical phase diagram of a dilute BEC trapped in a periodic potential was discussed. The existence of localized excitations, discrete solitons and breathers for repulsive interaction BECs was also discussed.

Modulational instability and the nonlinear dynamics of multiple solitary wave formation in two-component BECs that depend mainly on the sign and magnitudes of the scattering lengths have been demonstrated numerically [19]. Recently, Rapti *et al* [20] examined modulational and parametric instabilities arising in a non-autonomous discrete NLS equation in the context of BECs trapped in deep optical lattices. The instability and bright solitons of the cylindrical BECs in optical lattices have been analysed both analytically and numerically [21]. The properties of lattice solitons in BECs were analysed, for the case of either attractive or repulsive atomic interactions, by exactly solving the mean-field GP equation in the presence of a periodic potential [22]. The formation of bright, dark, ring and matter wave solitons has been widely discussed in an inhomogeneous BEC [23–27]. Recently, a new precise time-dependent criterion for the instability of a trapped BEC has been elaborated both analytically and numerically with the help of lens-type transformation [28, 29].

As discussed above, this paper is devoted to explore the dynamical evolution of the BEC in terms of bright solitons on a cnoidal wave background under the influence of both linear and harmonic trapping potentials. In order to study the dynamical evolution of the BEC under these potentials, we consider the inhomogeneous nonlinear Schrödinger (INLS) equation with the linear and harmonic density profiles including damping term as follows [30–32]

$$\mathrm{i}\psi_t + \psi_{zz} + 2|\psi|^2\psi - (\alpha z - (\beta' z)^2)\psi + \mathrm{i}\beta\psi = 0, \tag{1}$$

where $\psi_t = \frac{\partial\psi}{\partial t}$ and $\psi_{zz} = \frac{\partial^2\psi}{\partial z^2}$. Here αz and $(\beta' z)^2$ are the linear and harmonic density profiles respectively. The sixth term represents the gain/loss. Note that the non-conservative parameter (sixth term) β pertains to two different physical systems depending on the sign of β . That is $\beta > 0$ corresponds to the damping term, which is applicable to a system of plasmas [31] and $\beta < 0$ represents the feeding of the condensates from the non-equilibrium thermal clouds [33]. It is to be noted that the INLS equation is an integrable system only when $\beta' = \beta$ and hence this condition allows us to construct exact solutions of equation (1) [31, 32]. Analytically, this integrable condition will be identified later (see equation (9)). The explicit form of the one-soliton solution has been derived from the Darboux transformation (DT) method for the system of coupled INLS equations [30]. Recently, based on Husimi's and Lens-type transformations, the bright solitons on a continuous wave background were constructed in plasmas described by the INLS equation. When the continuous wave background approaches to zero, the bright soliton solutions on a continuous wave background reduce to well-known one-bright soliton solutions [31]. In [34], the exact two-soliton solution has been derived for the above INLS equation using the inverse scattering transform method.

As discussed in the previous paragraph, equation (1) has two different density profiles, namely linear and harmonic. Here, we intend to investigate both density profiles separately. First, we study the linear density profile and secondly we shall analyse the harmonic density profile. Now, if we consider the parameter $\beta' = \beta = 0$, equation (1) reduced to standard NLS equation with linear density profile is as follows

$$\mathrm{i}\psi_t + \psi_{zz} + 2|\psi|^2\psi - \alpha z\psi = 0. \tag{2}$$

Analytically, the nonlinear wave or Langmuir wave propagation in an inhomogeneous medium in the model of NLS equation with a linear time-independent density was discussed by Chen and Liu [35, 36]. Recently, the one- and two-soliton solutions have been investigated for the NLS equation with an arbitrary time-dependent linear potential which denotes the dynamics of a quasi-one-dimensional BEC [37]. In [38], a number of Jacobian elliptic function solutions has been discussed in the mean-field model of a quasi-1D BEC trapped in the time-dependent linear potential. In the liming case, when modulus equals to 1 and 0, Jacobian elliptic function solutions lead to various localized solutions (dark and bright solitons) and trigonometric functions, respectively.

In the second case, we study the harmonic density profile. For this purpose, we assume the linear density parameter $\alpha = 0$. Under this condition, equation (1) takes the form

$$\mathrm{i}\psi_t + \psi_{zz} + 2|\psi|^2\psi + (\beta' z)^2\psi + \mathrm{i}\beta\psi = 0. \tag{3}$$

As mentioned earlier, in BECs, the last term related to the feeding of the condensates from the non-equilibrium thermal cloud when $\beta < 0$. In this case, the condensate density will grow exponentially as $\mathrm{e}^{-\beta t}$ due to the collisional effect between the atoms in the thermal clouds of the condensates. The exact bright soliton solution has been constructed in a system of quasi-one-dimensional BECs, which is described by the above equation (3) [32].

In this paper, our aim is to establish the bright solitons on a cnoidal wave train background in the context of BECs wherein we discuss the two different potentials namely linear and harmonic. The paper is laid out as follows. In section 2, we reduce equation (1) to the standard NLS equation using the Husimi's and Lens-type transformations. Then, we obtain the bright solitons on a cnoidal wave train background for equation (1). In section 3, the superposed configuration of the soliton and cnoidal wave solution is discussed in BEC for two different cases as follows: (i) linear density profile and (ii) harmonic density profile. Finally, conclusion is presented in section 4.

2. Soliton solution on a cnoidal wave background

In this section, we study the superposed configuration of soliton and cnoidal wave solution of equation (1). To proceed further, we use the following combination of Husimi's and Lens-type transformations

$$\psi(z,t) = \frac{1}{\ell(t)} \times \exp\left[\mathrm{i}\left(a(t)z^2 + b(t)z - \int_0^t b(t)^2\,\mathrm{d}t\right)\right]\phi(Z,T), \tag{4}$$

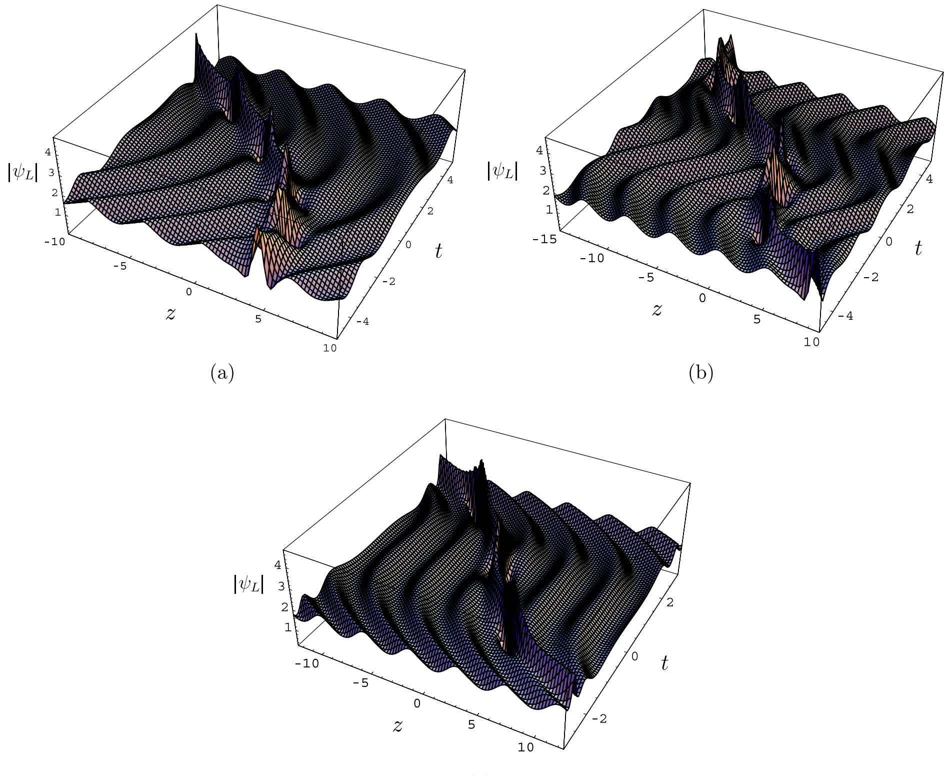


Figure 1. Dynamics of the soliton moving crossing the cnoidal wave in the presence of $\alpha(t)$. Parameters are $\beta = 0$, $C_1 = 0.1$, $C_2 = 0$, $v = 0$, $k = 0.75$, $p = 1$, $u = -0.28 + 0.43\mathrm{i}$, $M = 1.4$ for (a) $\alpha = 0.25$, (b) $\alpha = \cos(t)$ and (c) $\alpha = t$.

where $a(t)$ and $b(t)$ are real functions, $Z = \frac{z-\rho(t)}{\ell(t)}$ and $T = T(t)$. Substituting equation (4) in equation (1), we obtain the following standard INLS equation

$$\mathrm{i}\phi_T + \phi_{ZZ} + 2|\phi|^2\phi = \mathrm{i}\varepsilon(t)\phi, \tag{5}$$

under the following conditions

$$\begin{gathered} a_t + 4a^2 - (\beta')^2 = 0, \\ \ell_t - 4a\ell = 0, \\ \ell^2 T_t - 1 = 0, \\ b_t + 4ba + \alpha = 0, \\ \rho_t - 2b - 4a\rho = 0, \end{gathered} \tag{6}$$

where

$$\varepsilon(t) = (2a - \beta)\ell^2. \tag{7}$$

From equation (7), it is well known that equation (5) is integrable only when $a = \frac{\beta}{2}$. It is to be noticed that the linear density profile parameter α and harmonic density profile parameter β' have widely been discussed [31]. According to this integrability condition, equation (5) takes the following form of the standard NLS equation

$$\mathrm{i}\phi_T + \phi_{ZZ} + 2|\phi|^2\phi = 0, \tag{8}$$

The integrability condition $a = \frac{\beta}{2}$ is a constant. According to this condition and solving equation (6), we have

$$\begin{gathered} \beta' = \pm\beta, \\ \ell(t) = \mathrm{e}^{2\beta t}, \\ T(t) = \frac{1 - \mathrm{e}^{-4\beta t}}{4\beta}, \\ b(t) = \left[-\int_0^t \alpha(t)\,\mathrm{e}^{2\beta t}\,\mathrm{d}t + C_1\right]\mathrm{e}^{-2\beta t}, \\ \rho(t) = \left[2\int_0^t b(t)\,\mathrm{e}^{-2\beta t}\,\mathrm{d}t + C_2\right]\mathrm{e}^{2\beta t}. \end{gathered} \tag{9}$$

From the first relation of equation (9), it is clear that the INLS equation is an integrable system and hence this condition allows to construct exact solutions of equation (1).

The soliton solution on a cnoidal wave train background has been discussed for the NLS equation (8) [39]. The soliton solution which propagates on a cnoidal wave background for the INLS equation (1) under the integrable condition $a = \beta/2$ is given by

$$\begin{aligned} \psi(z,t) = {} & \left[p\,\mathrm{dn}(\chi,k)\,\mathrm{e}^{\mathrm{i}\xi} + 2\mathrm{i}(\sigma - \sigma^*)\frac{q_1 q_2^*}{\sum_{m=1}^{2}|q_m|^2}\right] \\ & \times \exp\left[\mathrm{i}\left(az^2 + b(t)z - \int_0^t b(t)^2\,\mathrm{d}t\right) - 2\beta t\right]. \end{aligned} \tag{10}$$

where $\chi = p(Z - vT)$, $\xi = \left[\frac{vZ}{2} + p^2(2 - k^3)T - \frac{v^2 T}{4}\right]$ with $Z = [z - \rho(t)]/\ell(t)$ and $T = (1 - \mathrm{e}^{-4\beta t})/4\beta$. Here dn, cn

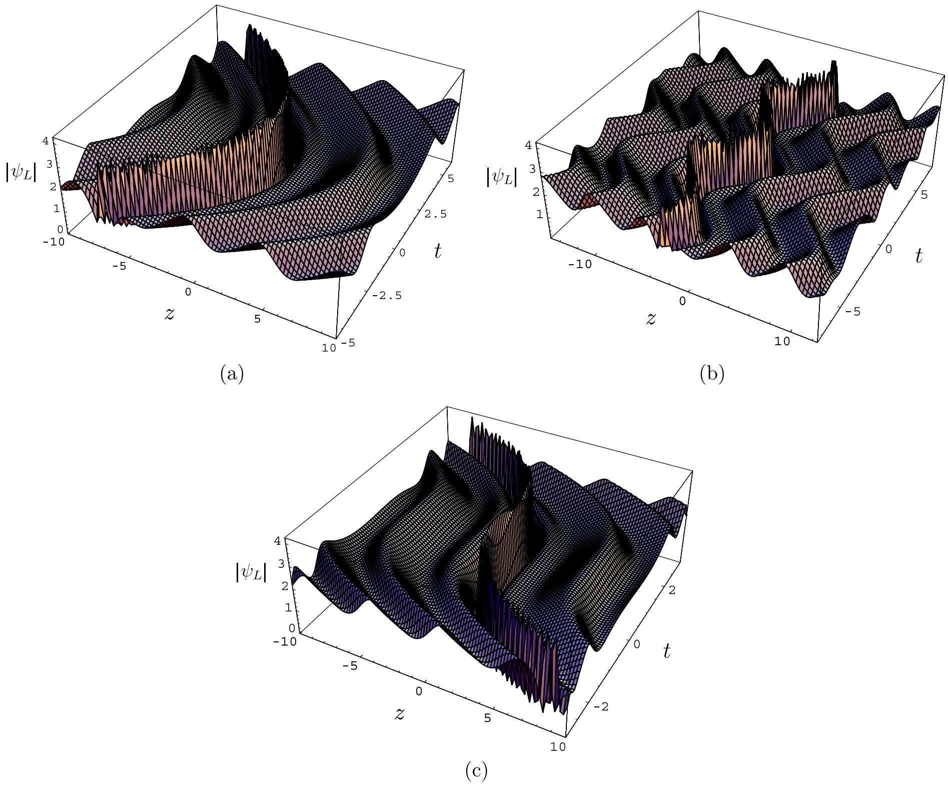


Figure 2. Dynamics of the soliton moving parallel on the trough of the cnoidal wave in the presence of $\alpha(t)$. Parameters are $\beta = 0$, $C_1 = 0.1$, $C_2 = 0$, $v = 0$, $k = 0.9$, $p = 1$, $u = 0.95\mathrm{i}$, $M = 1.5$ for (a) $\alpha = 0.25$, (b) $\alpha = \cos(t)$ and (c) $\alpha = t$.

and sn are the standard Jacobian elliptic functions with the modulus k $(0 < k < 1))$ and p is an arbitrary constant. The parameter v is the velocity of the cnoidal wave. Finally, the parameters q_1 and q_2 are defined by

$$q_1 = \frac{\mathrm{e}^{\mathrm{i}\xi/2}}{\theta_0\left(\frac{\chi}{2K}\right)}\Big[\mathrm{e}^{\mathrm{i}\Delta}\theta_2(-w_1)\theta_0(w_2) - M\,\mathrm{e}^{-\mathrm{i}\Delta}\theta_1(-w_1)\theta_3(w_3)\Big],$$

$$q_2 = \frac{\mathrm{e}^{-\mathrm{i}\xi/2}}{\theta_0\left(\frac{\chi}{2K}\right)}\Big[-\mathrm{e}^{\mathrm{i}\Delta}\theta_1(-w_1)\theta_3(w_2) + M\,\mathrm{e}^{-\mathrm{i}\Delta}\theta_2(-w_1)\theta_0(w_3)\Big], \tag{11}$$

with $w_1 = \frac{iu}{2K}$, $w_2 = \frac{\chi+iu}{2K}$ and $w_3 = \frac{\chi-iu}{2K}$. Here M and u are the arbitrary number and complex parameter, respectively. The parameters $\theta_0, \theta_1, \theta_2$ and θ_3 are the Jacobian theta functions and $\Delta = p\delta Z + (\gamma - p\delta v)T$ with

$$\delta = \mathrm{i}E[\sin^{-1}[\mathrm{sn}(\mathrm{i}u, k)]] + \frac{E}{K}u + \frac{\mathrm{dn}(u, k')(1 + \mathrm{sn}^2(u, k'))}{2\,\mathrm{cn}(u, k')\,\mathrm{sn}(u, k')},$$
$$\gamma = -\frac{p^2}{2}\left[\mathrm{dn}^2(u, k') + \frac{\mathrm{cn}^2(u, k')}{\mathrm{sn}^2(u, k')}\right], \tag{12}$$

and K and E are the complete elliptic integrals of the first and second kind, respectively. The DT parameter σ is written in terms of the complex parameter u as follows

$$\sigma = \frac{v}{4} + \frac{p}{2}\frac{\mathrm{dn}(u, k')\,\mathrm{cn}(u, k')}{\mathrm{sn}(u, k')}. \tag{13}$$

3. Results and discussion

In this section, we intend to investigate the influence of the linear density α and harmonic density β profiles on the dynamics of soliton with a cnoidal wave background.

3.1. Linear density profile ($\alpha \neq 0, \beta = 0$)

In order to discuss the role of linear density profile, we consider $\beta = 0$. Under this condition, the INLS equation is reduced to equation (2), which also describes a system of BEC trapped in linear potential. According to above physical condition, the parameters are $a = 0, \ell(t) = 1, T(t) = t, b(t) = -\int_0^t \alpha(t)\,\mathrm{d}t + C_1$ and $\rho(t) = 2\int_0^t b(t)\,\mathrm{d}t + C_2$. Then from equation (10), we get the soliton solution on a cnoidal wave background of equation (2) and is given by

$$\psi_L(z, t) = \left[\phi_c(Z, T) + 2\mathrm{i}(\sigma - \sigma^*)\frac{q_1 q_2^*}{\sum_{m=1}^{2}|q_m|^2}\right] \times \exp\left[\mathrm{i}\left(b(t)z - \int_0^t b(t)^2\,\mathrm{d}t\right)\right], \tag{14}$$

where $\phi_c(Z, T) = p\mathrm{dn}(\chi, k)\,\mathrm{e}^{\mathrm{i}\xi}$ and subscript L in $\psi_L(z, t)$ indicates the linear density profile case. Thus, equation (14)

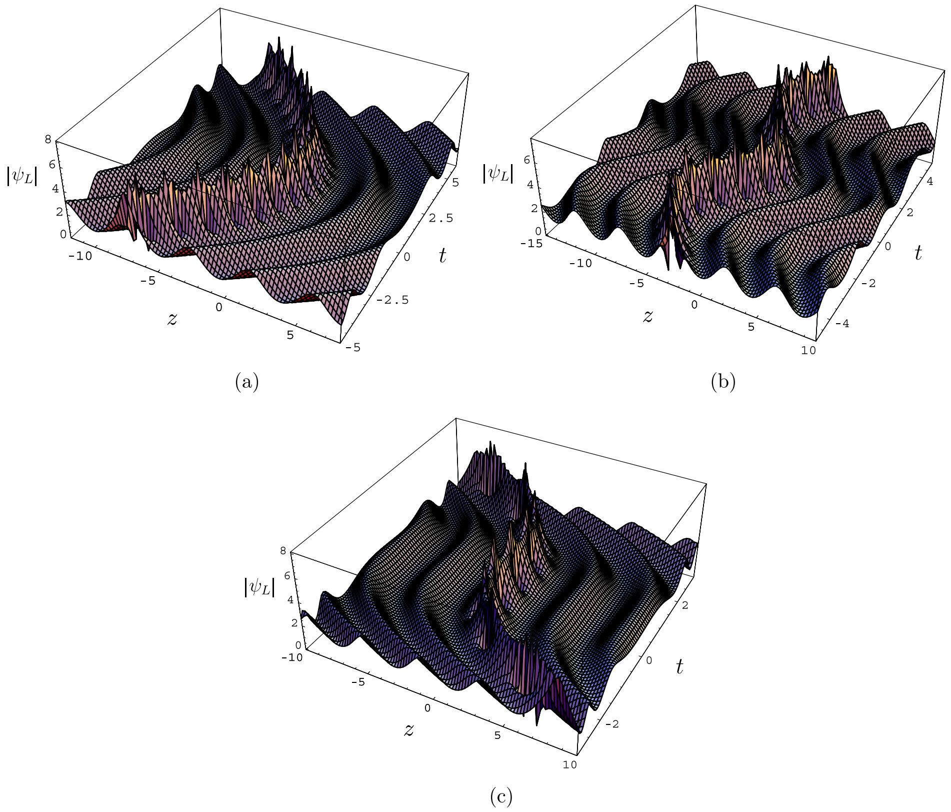


Figure 3. Dynamics of the soliton moving parallel on the crest of the cnoidal wave in the presence of $\alpha(t)$. Parameters are $\beta = 0$, $C_1 = 0.1$, $C_2 = 0$, $v = 0$, $k = 0.9$, $p = 1.2$, $u = 0.38\mathrm{i}$, $M = 1.5$ for (a) $\alpha = 0.25$, (b) $\alpha = \cos(t)$ and (c) $\alpha = t$.

explains the BEC soliton which has been formed in the presence of linear potential. The condensate density is calculated by

$$N_L = \int_{-\infty}^{\infty} (|\psi_L(z,t)|^2 - |\psi_L(\pm\infty, t)|^2)\,\mathrm{d}z. \tag{15}$$

From equations (11) and (14), the trajectory of the superposed configuration of the soliton in the cnoidal wave for the linear density profile case is given by

$$\mathrm{Im}(\Delta) = \mathrm{Im}[p\delta(z - \rho(t)) + (\gamma - p\delta v)t] = \text{constant}. \tag{16}$$

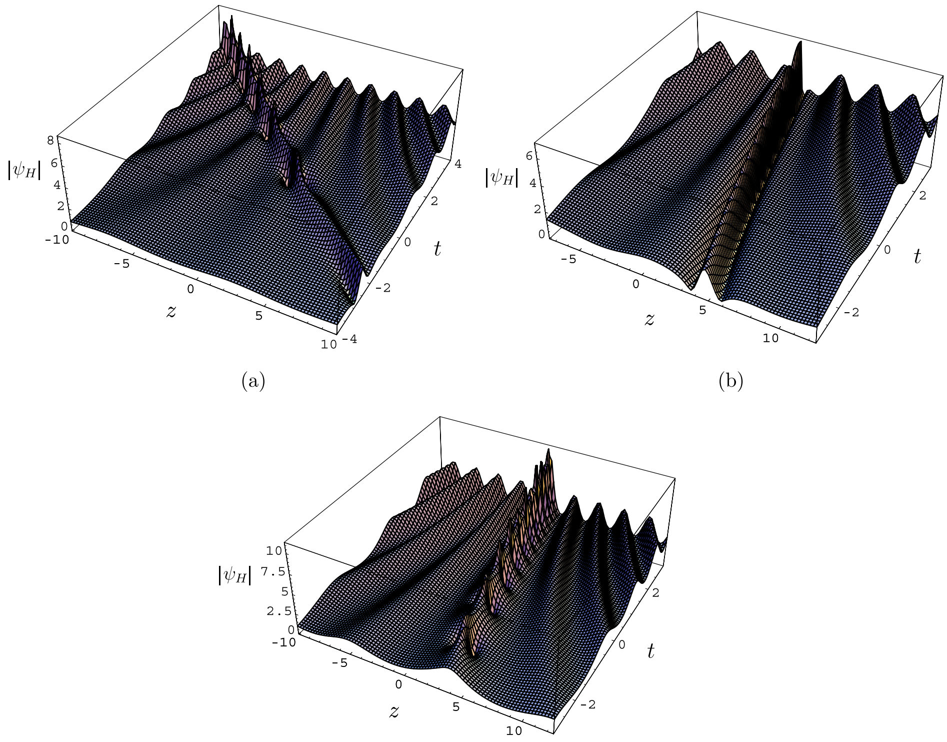
From this relation, the trajectory of the soliton in the cnoidal wave strongly depends on the linear density profile parameter $\alpha(t)$. Then the velocity of the soliton of linear density profile case is defined by

$$v_{sL} = -\left[\frac{\mathrm{Im}(\gamma)}{\mathrm{Im}(p\delta)} + 2\int_0^t \alpha(t)\,\mathrm{d}t - 2C_1 - v\right]. \tag{17}$$

In what follows, we discuss the different forms of superposed configuration of the BEC soliton and cnoidal wave solution of equation (14) for different forms of α as follows.

3.1.1. Solitons crossing the cnoidal wave. In this subsection, we examine the BEC soliton on a cnoidal wave background of equation (2). Before exploring the dynamics of a soliton with cnoidal wave, it is interesting to mention about the physical parameter u which plays an indispensable role. In general, the physical parameter u can be complex. If u is considered to be in the complex form, then resulting soliton pulse interacts with cnoidal wave. In other words, we can also say that the soliton crosses the cnoidal wave. Contrary to the above, the soliton also travels in parallel with the cnoidal wave only when the u parameter is chosen to be imaginary form. Further, it should be noted that the soliton could travel in parallel with either crest, (for relatively higher values of u), or trough (for relatively lower values of u) of a cnoidal wave. Finally, another important point to be noted is that the soliton disappears and the cnoidal wave only appears if the parameter u is real.

Now, we turn to explore the dynamics of a soliton with a cnoidal wave background. To start with, we consider the first situation wherein the physical parameter u is complex. Therefore, in the entire span, the soliton will always interact with the cnoidal wave or the soliton crosses the cnoidal wave. Figure 1 portrays the superposed configuration of the BEC soliton and cnoidal wave solution of equation (14) for three different forms of $\alpha(t)$, i.e. $\alpha(t) = \alpha(\text{constant})$, $\alpha(t) = \cos(t)$ and $\alpha(t) = t$. Now, we consider the first case, $\alpha(t) = \alpha(\text{constant})$ the trajectory of both the BEC soliton and cnoidal wave will be a parabola as seen in figure 1(a). In the second case, $\alpha(t) = \cos(\omega t)$, the trajectory of the soliton and cnoidal wave will become to oscillate with the period of $2\pi/\omega$ as seen in figure 1(b) for $\omega = 1$. Finally, we consider $\alpha(t) = t$, the trajectory of the soliton and cnoidal wave becomes inverse S-type, which is depicted in figure 1(c). In general,



(a) (b) (c)

Figure 4. The soliton as a cnoidal wave background in the absence of $\alpha(t)$. The parameters are $\alpha = 0$, $\beta = -0.1$, $C_1 = 0$, $C_2 = 2$, $v = 0$ for (a) soliton moving crossing the cnoidal wave when $k = 0.75$, $p = 1$, $u = -0.28 + 0.43\mathrm{i}$, $M = 1.4$; (b) soliton moving parallel on the through of the cnoidal wave $k = 0.9$, $p = 1$, $u = 0.95\mathrm{i}$, $M = 1.5$; and (c) soliton moving parallel on the crest of the cnoidal wave $k = 0.9$, $p = 1.2$, $u = 0.38\mathrm{i}$, $M = 1.5$.

figures 1(a)–(c) represent the BEC soliton crossing the cnoidal wave for the following physical parameter values: $\beta = 0, C_1 = 0.1, C_2 = 0, v = 0, k = 0.75, p = 1, u = -0.28 + 0.43\mathrm{i}, M = 1.4$ with the following three different forms of $\alpha(t)$ (0.25, $\cos(t)$ and t).

Further, we calculate the crest of the cnoidal wave background shifts constantly across the bright soliton. Let us consider a region $\mathrm{i}\Delta \to \infty$ then $\mathrm{e}^{-\mathrm{i}\Delta} \to 0$, we find

$$\frac{q_1 q_2^*}{\sum_{m=1}^{2}|q_m|^2} = -\mathrm{e}^{\mathrm{i}\xi}\left[\frac{\mathrm{sn}(u_I,k)\,\mathrm{cn}(u_I,k)\,\mathrm{dn}(\chi - u_I,k)}{1 - k^2\,\mathrm{sn}^2(u_I,k)\,\mathrm{sn}^2(\chi - u_I,k)}\right], \tag{18}$$

and

$$\sigma - \sigma^* = -\mathrm{i}p\frac{\mathrm{dn}(u_I,k)}{\mathrm{sn}(u_I,k)\,\mathrm{cn}(u_I,k)}, \tag{19}$$

where u_I is the imaginary part of the complex parameter $u(u_R + \mathrm{i}u_I)$. Using equations (18) and (19), equation (14) can be written as

$$\psi_L(z,t) = \left[p\,\mathrm{dn}(\chi,k) - 2p\frac{\mathrm{dn}(u_I,k)\,\mathrm{dn}(\chi - u_I,k)}{1 - k^2\,\mathrm{sn}^2(u_I,k)\,\mathrm{sn}^2(\chi - u_I,k)}\right] \times \exp\left[\mathrm{i}\left(\xi + b(t)z - \int_0^t b(t)^2\mathrm{d}t\right)\right]. \tag{20}$$

Using the addition theorem of Jacobian elliptic functions, equation (20) can be written as

$$|\psi_L(z,t)| = -p\,\mathrm{dn}(\chi - 2u_I,k), \tag{21}$$

which is a cnoidal wave. Similarly, in the region $\mathrm{i}\Delta \to -\infty$ after that $\mathrm{e}^{\mathrm{i}\Delta} \to 0$, we obtain

$$\frac{q_1 q_2^*}{\sum_{m=1}^{2}|q_m|^2} = -\mathrm{e}^{\mathrm{i}\xi}\left[\frac{\mathrm{sn}(u_I,k)\,\mathrm{cn}(u_I,k)\,\mathrm{dn}(\chi + u_I,k)}{1 - k^2\,\mathrm{sn}^2(u_I,k)\,\mathrm{sn}^2(\chi + u_I,k)}\right]. \tag{22}$$

Using equations (19) and (22), equation (14) can be written as

$$|\psi_L(z,t)| = -p\,\mathrm{dn}(\chi + 2u_I,k). \tag{23}$$

From equations (21) and (23), it is clear that crests of the cnoidal wave are shifted by $4u_I$ across the soliton. From figure 1(a), the shift of crests of the cnoidal wave is found to be $4u_I = 1.72$.

3.1.2. Soliton travels in parallel with the cnoidal wave. In contrast to the above studies, in this sub-section, we consider another physical situation wherein the soliton always travels in parallel with the cnoidal wave. This case is possible only when the u parameter is purely imaginary. As has been mentioned in the previous paragraph, we address the two special cases for a soliton which travels in parallel with the crest and trough of a cnoidal wave. Now, we investigate the soliton travels in parallel with the trough of the cnoidal wave of equation (2). Figure 2 shows the BEC soliton moving parallel on the trough of the cnoidal wave when the parameter values are $\beta = 0, C_1 = 0.1, C_2 = 0, v = 0, k = 0.9, p = 1, u = 0.95\mathrm{i}$,

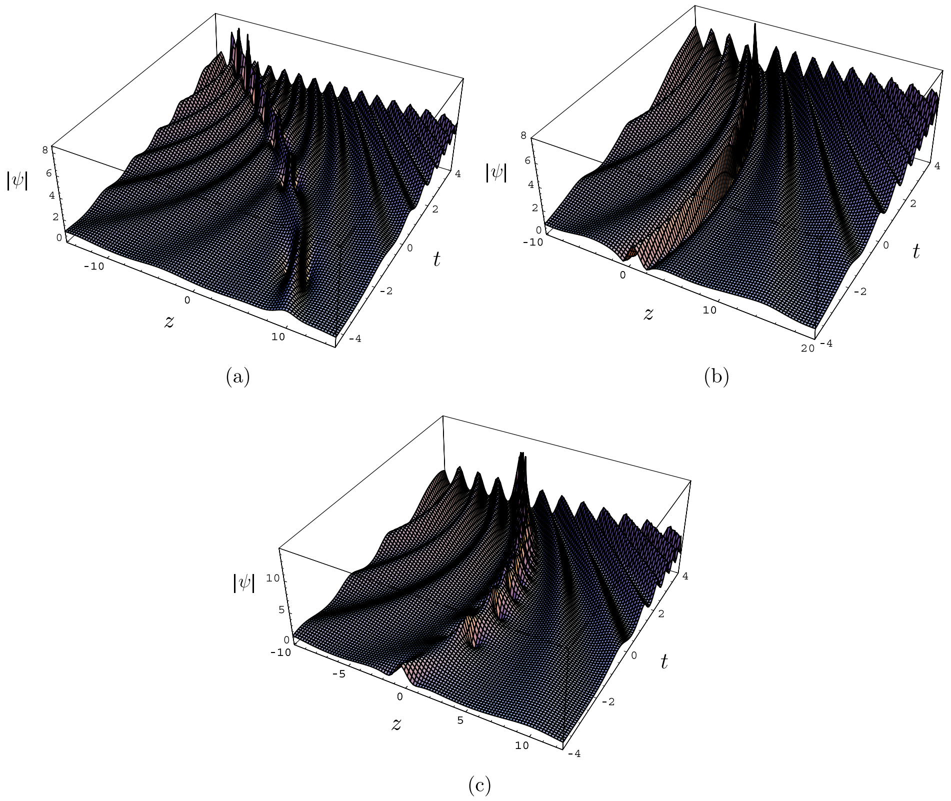


Figure 5. The soliton as a cnoidal wave background of equation (10). The parameters are $\alpha = 0.25$, $\beta = -0.1$, $C_1 = 0.1$, $C_2 = 2$, $v = 0$ for (a) Soliton moving crossing the cnoidal wave when $p = 1$, $k = 0.75$, $u = -0.28 + 0.43\mathrm{i}$, $M = 1.4$; (b) soliton moving parallel on the through of the cnoidal wave $p = 1$, $k = 0.9$, $u = 0.95\mathrm{i}$, $M = 1.5$; (c) soliton moving parallel on the crest of the cnoidal wave $p = 1.2$, $k = 0.9$, $u = 0.38\mathrm{i}$, $M = 1.5$.

$M = 1.5$ for three different values of $\alpha = 0.25$ (figure 2(a)), $\alpha = \cos(t)$ (figure 2(b)) and $\alpha = t$ (figure 2(c)). Figure 3 portrays the BEC soliton moving parallel on the crest of the cnoidal wave for the three different forms of $\alpha(t)$. Both the soliton and cnoidal wave propagate in a parabolic path when $\alpha = 0.25$, which is shown in figure 3(a). If $\alpha = \cos(t)$, the path of the soliton and cnoidal wave is an oscillating one as seen in figure 3(b). The path is inverse S-type when $\alpha = t$ (figure 3(c)). The other physical parameter values are $\beta = 0$, $C_1 = 0.1$, $C_2 = 0$, $v = 0$, $k = 0.9$, $p = 1.2$, $u = 0.38\mathrm{i}$ and $M = 1.5$.

3.2. Harmonic density profile ($\beta \neq 0, \alpha = 0$)

Having realized the influence of the linear density profile in the soliton dynamics on a cnoidal wave train background, we next proceed to explore the harmonic density profile. For this purpose, we assume that the value of α is equal to zero, i.e. $\alpha = 0$, under this condition, the INLS equation (1) is reduced to equation (3). The resulting equation (3) also describes a system of BECs trapped in harmonic potential. Recall that the last term in equation (3) corresponds to the feeding of the condensates from the non-equilibrium thermal clouds when $\beta < 0$. If $\alpha = 0$, i.e. $b(t) = 0$ when $C_1 = 0$. So $\ell = \mathrm{e}^{2\beta t}$, $T = \frac{1-\mathrm{e}^{-4\beta t}}{4\beta}$ and $\rho(t) = C_2\,\mathrm{e}^{2\beta t}$. Then from equation (10), we obtain the BEC soliton solution as a cnoidal wave background of equation (3), which is given by

$$\psi_H(z,t) = \left[\phi_c(Z,T) + 2i(\sigma - \sigma^*)\frac{q_1 q_2^*}{\sum_{m=1}^{2}|q_m|^2}\right] \times \exp[iaz^2 - 2\beta t], \tag{24}$$

where $\phi_c(Z,T) = p\,\mathrm{dn}(\chi,k)\,\mathrm{e}^{\mathrm{i}\xi}$ and subscript H in $\psi_H(z,t)$ indicates the harmonic density profile case. The condensate density is calculated by

$$N_H = \int_{-\infty}^{\infty}(|\psi_H(z,t)|^2 - |\psi_H(\pm\infty,t)|^2)\,\mathrm{d}z = N_L\,\mathrm{e}^{-4\beta t}. \tag{25}$$

From the above equation, it is worth mentioning that the condensate density grows exponentially as $\mathrm{e}^{-4\beta t}$ due to increase in the value of feeding term parameter β (β is negative). From equations (11) and (24), the trajectory of the superposed configuration of the soliton in the cnoidal wave for the harmonic density profile case is given by

$$\mathrm{Im}(\Delta) = \mathrm{Im}\left[p\delta(z\,\mathrm{e}^{-2\beta t} - C_2) + (\gamma - p\delta v)\left(\frac{1-\mathrm{e}^{-4\beta t}}{4\beta}\right)\right] = \text{constant}, \tag{26}$$

and the velocity of the soliton of the harmonic density profile with feeding of the condensate term can be written as

$$v_{sH} = -\left[\left(\frac{\mathrm{Im}(\gamma)}{\mathrm{Im}(p\delta)} - v\right)\mathrm{e}^{-2\beta t} + 2\beta z\right]. \tag{27}$$

Figure 4 illustrates the BEC soliton as a cnoidal wave background of equation (3) under the influence of the harmonic density profile. For the negative values of the harmonic density profile, the term β acts as linear gain. Hence the soliton as well as the cnoidal wave undergo compression wherein the value of $\beta = -0.1$. On the other hand, in another situation, i.e. in plasma physics, the last term in equation (3) corresponds to damping when $\beta > 0$. For this system, the number of electrons in the plasma decay exponentially as $e^{-4\beta t}$ owing to the increase of the value of the damping term parameter β (β is positive). Obviously, in this case, the width of the soliton and cnoidal wave get broadened.

3.3. General case: Linear and harmonic density profiles ($\alpha \neq \beta \neq 0$)

In the previous sub-sections, the influence of linear and harmonic density profiles has been investigated independently. Therefore, it is of paramount importance to study both profiles together. Thus, this sub-section discusses the most general case of equation (1) wherein the linear and harmonic density profiles are not zero (i.e. $\alpha \neq \beta \neq 0$). The outcome of the general case is portrayed in figure 5. In figure 5(a) the soliton interacts with the cnoidal wave, i.e. the soliton crosses the cnoidal wave owing to the parameter u and they traverse in a parabolic path. Besides, both cnoidal and soliton undergo compression/broadening for negative/positive values of β. For illustration purposes, we consider $\alpha = 0.25$, $\beta = -0.1$, $k = 0.75$, $p = 1$, $M = 1.4$ and $u(= -0.28+0.43\mathrm{i})$ is assumed to be complex in figure 5(a). As discussed above, we now consider another case wherein the soliton travels parallel to the cnoidal wave when the physical parameter u is purely imaginary. Figure 5(b) shows the BEC soliton moving parallel on the trough of the cnoidal wave for relatively higher values of u. Similarly, figure 5(c) portrays the BEC soliton moving parallel on the crest of the cnoidal wave for relatively lower values of u. In both the cases (figures 5(a) and (b)), both the waves undergo compression or broadening depending upon the physical parameter β. It is interesting to mention that, as discussed above, the other cases like $\alpha = \cos(t)$ and $\alpha = t$ can also be investigated.

From equations (10) and (11), the trajectory of the soliton in the cnoidal wave for the inhomogeneous system of equation (1) is given by

$$\mathrm{Im}(\Delta) = \mathrm{Im}\left[p\delta(z-\rho(t))\,e^{-2\beta t} + (\gamma - pv)\left(\frac{1-e^{-4\beta t}}{4\beta}\right)\right] = \text{constant}. \tag{28}$$

Note that, in equation (28), if the values of $\alpha(t) = \beta = C_1 = C_2 = 0$, then the trajectory of the soliton in the cnoidal wave approaches the homogeneous system.

4. Conclusion

In this work, we have investigated the bright solitons on a cnoidal wave train background for a system of BECs described by the inhomogeneous NLS equation including the linear and harmonic density profiles. By using the combination of Husimi's and Lens-type transformations, the inhomogeneous NLS equation has been reduced to a standard inhomogeneous NLS equation. Then, for the known integrability condition $a = \beta/2$, the standard inhomogeneous NLS was again reduced to a standard NLS equation. The superposed configuration of the soliton plus cnoidal wave solution has been found for the inhomogeneous NLS equation through the well-known NLS equation. Further, the influence of linear density and harmonic density profiles has been discussed in the dynamics of bright solitons with a cnoidal wave background. In a linear potential case, the three different forms of the linear density profiles have been investigated in detail. In the harmonic potential case, compression and broadening of the soliton as well as the cnoidal wave have been discussed for $\beta < 0$ and $\beta > 0$ respectively. Compression (broadening) occurs since the condensate density grows (electron density decays) exponentially with the increase in the value of β in negative (positive) sign. Besides, we have also analyzed the general case wherein both the density profiles have been considered. As expected, in the limiting case, the Jacobian elliptic function solution leads to the well-known soliton solution when the modulus equals 1.

Acknowledgments

The authors gratefully thank the anonymous referees for insightful comments. KP wishes to thank the IFCPAR (Ref.:IFC/3504-F/2005/2064), CSIR, DST and Ramanna Fellowship (DST), Government of India, for the financial support through major projects. KSN acknowledges the financial support from The Hong Kong Polytechnic University (through project G-YE30) and is also grateful to PKA Wai for valuable help.

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