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DYNAMICS OF \{\lambda \tanh(e^z) : \lambda \in \mathbb{R} \setminus \{0\}\}

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Abstract. In this paper, the dynamics of transcendental meromorphic functions in the one-parameter family
\[ M = \{ f_\lambda(z) = \lambda f(z) : f(z) = \tanh(e^z) \text{ for } z \in \mathbb{C} \text{ and } \lambda \in \mathbb{R} \setminus \{0\} \} \]
is studied. We prove that there exists a parameter value \( \lambda^* \approx -3.2946 \) such that the Fatou set of \( f_\lambda(z) \) is a basin of attraction of a real fixed point for \( \lambda > \lambda^* \) and, is a parabolic basin corresponding to a real fixed point for \( \lambda = \lambda^* \). It is a basin of attraction or a parabolic basin corresponding to a real periodic point of prime period 2 for \( \lambda < \lambda^* \). If \( \lambda > \lambda^* \), it is proved that the Fatou set of \( f_\lambda \) is connected and, is infinitely connected. Consequently, the singleton components are dense in the Julia set of \( f_\lambda \) for \( \lambda > \lambda^* \). If \( \lambda \leq \lambda^* \), it is proved that the Fatou set of \( f_\lambda \) contains infinitely many pre-periodic components and each component of the Fatou set of \( f_\lambda \) is simply connected. Finally, it is proved that the Lebesgue measure of the Julia set of \( f_\lambda \) for \( \lambda \in \mathbb{R} \setminus \{0\} \) is zero.

1. Introduction. Let \( f(z) \) be a transcendental meromorphic function in the complex plane \( \mathbb{C} \). Let \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) denote the extended complex plane. The central objects studied in the dynamics of a complex function \( f \) are its Fatou set and Julia set. The Fatou set of the function \( f \), denoted by \( \mathcal{F}(f) \), is defined as

\[ \mathcal{F}(f) = \left\{ z \in \hat{\mathbb{C}} : f^n(z) \text{ is defined for each } n = 0, 1, 2, \ldots \text{ and } \{f^n\}_{n=0}^\infty \text{ forms a normal family at a neighborhood of the point } z \right\}. \]

Then, the Julia set of \( f \), denoted by \( \mathcal{J}(f) \), is the complement of the Fatou set of \( f \) in the extended complex plane \( \hat{\mathbb{C}} \).

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The dynamics of transcendental meromorphic functions in the one-parameter family

$$\mathcal{M} = \{ f_\lambda(z) = \lambda f(z) : f(z) = \tanh(e^z) \text{ for } z \in \mathbb{C} \text{ and } \lambda \in \mathbb{R} \setminus \{0\} \}$$

is mainly studied in the present paper. The dynamics of the critically finite entire function $\lambda e^z$, ($\lambda \in \mathbb{C} \setminus \{0\}$) have been extensively studied and a number of interesting properties of the Julia set of $\lambda e^z$ are proved [3–5, 8, 10, 11, 15, 17, 21–24]. Devaney and Keen [12–14] studied the dynamics of meromorphic functions with constant/polynomial Schwarzian derivatives and in particular, the dynamics of functions in the one-parameter family $\{ \lambda \tan z : \lambda \in \mathbb{R} \setminus \{0\} \}$. Jiang [19], Keen and Kotus [20] furthered the study of dynamics of the tangent family. It is worth to note that the dynamics of $\lambda \tanh z$ is essentially same as the dynamics of $\lambda \tan(z)$, since $\tanh z$ and $\tan z$ are conformally conjugate.

A meromorphic function that takes real values only on the real line is called real meromorphic. The real meromorphic function $\tan z$ maps the upper half-plane into itself and it simplifies the determination of the dynamics of $\lambda \tanh z$, $\lambda \in \mathbb{R} \setminus \{0\}$. But, the function $\tanh(e^z)$ is not a real meromorphic function and its mapping properties are comparatively more complicated. Further, the function $\tanh(e^z)$ is a meromorphic function having non-rational Schwarzian derivative and the Nevanlinna order is infinite. However, the functions $\lambda e^z$ and $\lambda \tanh z$ have constant Schwarzian derivatives and finite orders. Even though the order of $f_\lambda(z) = \lambda \tanh(e^z)$ is infinite, it has only 3 finite asymptotic values, namely $\pm \lambda$ and 0. The asymptotic values of $e^z$ and $\tanh z$ are logarithmic that are the simplest kind of direct singularities of the respective inverse functions. Whereas the asymptotic values $\pm \lambda$ of $f_\lambda$ are logarithmic, and 0 is the indirect singularity of the inverse
function of $f_{\lambda}$. Thus, the properties of the function $f_{\lambda}$ differ in many ways from that of $\lambda e^z$ and $\lambda \tanh z$ and that motivates to explore the dynamics of $f_{\lambda}$.

It is well known that the bifurcation in the dynamics of functions in each of the families $\{ T_{\lambda}(z) = \lambda \tanh z : \lambda \in \mathbb{R} \setminus \{0\} \}$ and $\{ E_{\lambda}(z) = \lambda e^z : \lambda \in \mathbb{R} \setminus \{0\} \}$ occur at two critical parameter values. But, in the dynamics of functions in the family $\mathcal{M}$, we show that the bifurcation occurs only at one critical parameter. We mainly prove the following result on the dynamics of $f_{\lambda}$ in Section 4.

**Theorem 1.** Let $f_{\lambda} \in \mathcal{M}$. Let $\lambda^* = \frac{-1}{f'(x^*)}$ where $x^*$ is the unique real root of 
\[ \frac{x}{f(x)} + \frac{1}{f'(x)} = 0. \]

1. If $\lambda > \lambda^*$ then the Fatou set $\mathcal{F}(f_{\lambda})$ is equal to the basin of attraction $A(a_{\lambda})$ where $a_{\lambda}$ is the attracting real fixed point of $f_{\lambda}$.
2. If $\lambda = \lambda^*$ then the Fatou set $\mathcal{F}(f_{\lambda})$ is equal to the parabolic basin $P(x^*)$ where $x^*$ is the rationally neutral real fixed point of $f_{\lambda}$.
3. If $\lambda < \lambda^*$ then the Fatou set $\mathcal{F}(f_{\lambda})$ is equal to the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ of $f_{\lambda}$.

For the function $T_{\lambda}(z) = \lambda \tanh z$, the Fatou set of $T_{\lambda}$ is infinitely connected if $|\lambda| < 1$ and it contains only two simply connected components $\{z \in \mathbb{C} : \Re(z) < 0\}$ and $\{z \in \mathbb{C} : \Re(z) > 0\}$ if $|\lambda| \geq 1$. For the function $E_{\lambda}(z) = \lambda e^z$, every component of the Fatou set of $E_{\lambda}$ is simply connected if $\lambda < -e$, and the Fatou set of $E_{\lambda}$ itself is simply connected if $-e \leq \lambda \leq \frac{1}{e}$, and is empty if $\lambda > \frac{1}{e}$. In Section 5, the topological properties of the Fatou sets of $f_{\lambda}$ are investigated and the following two theorems are proved.
Theorem 2. Let $f_\lambda \in \mathcal{M}$ where $\lambda > \lambda^*$. Then, the Fatou set of $f_\lambda$ is connected and, is infinitely connected.

Theorem 3. Let $f_\lambda \in \mathcal{M}$ where $\lambda \leq \lambda^*$. Then,

1. The Fatou set of $f_\lambda$ contains infinitely many strictly pre-periodic (pre-periodic but not periodic) components.
2. Every component of the Fatou set is simply connected.

In Section 6, it is shown that the measure of the Julia set of $f_\lambda$ is zero.

2. General Properties of $f_\lambda$. In this section, we prove some basic results on the functions $f_\lambda \in \mathcal{M}$ that are relevant in the study of their dynamics. The function $f(z) = \tanh(e^z)$ is periodic of minimal period $2\pi i$ and maps the real line $\mathbb{R}$ onto $(0, 1)$. The poles of $f(z)$ are the zeros of $\cosh(e^z)$, and hence, they satisfy $e^{2\pi z} = -1 = e^{i\pi(2k+1)}$ for $k \in \mathbb{Z}$. Therefore, the set of poles of $f(z)$ is \( \{z = x + iy \in \mathbb{C} : x = \ln\left|\frac{\pi}{2}(2k + 1)\right| \text{ and } y = \frac{\pi}{2}(2l + 1) \text{ where } k \in \mathbb{Z} \text{ and } l \in \mathbb{Z}\} \). Further, all the poles are simple and lie in the right half-plane \( \{z \in \mathbb{C} : \Re(z) \geq \ln(\frac{\pi}{2})\} \). One can show that the Nevanlinna order of the function $f(z) = \tanh(e^z)$ is infinite.

Lemma 1. Let $g$ be a critically finite meromorphic function in $\mathbb{C}$ and $h$ be a non-constant critically finite entire function. Let $F(z) = g(h(z))$ be the composition function. If $a$ is a finite asymptotic value of $F(z)$ then either $a$ is an asymptotic value of $g$ or there exists $b \in \mathbb{C}$ such that $g(b) = a$ and $b$ is an asymptotic value of $h$. Consequently, the number of finite asymptotic values of the composite function $F$ is at most the sum of the number of finite asymptotic values of the individual functions $g$ and $h$. 
Proof. Let $\gamma : [0, \infty) \to \mathbb{C}$ be an asymptotic path corresponding to the asymptotic value $a$ for the function $F(z)$. Let $M$ denote the collection of all limit points of the set

$$\{h(\gamma(t_k)) : \{t_k\} \text{ is any sequence of positive real numbers which tends to } \infty \text{ as } k \to \infty\}.$$

Observe that $g(z) = a$ for every $z \in M$. Since $g$ is a non-constant meromorphic function, the set $M$ cannot have any limit point in $\mathbb{C}$. Therefore, $M$ is a discrete subset of $\mathbb{C}$. Now we claim that $M$ contains only one element in $\mathbb{C}$. If possible, the set $M$ contains more than one element in $\mathbb{C}$. Suppose that $m_1$ and $m_2$ are in $M$ with $m_1 \neq m_2$. Then, there exist open disks $B_1(m_1)$ and $B_2(m_2)$ such that $\overline{B_1(m_1)} \cap M = \{m_1\}$ and $\overline{B_2(m_2)} \cap M = \{m_2\}$. The curve $h(\gamma(t))$ intersects the disks $B_1(m_1)$ and $B_2(m_2)$ infinitely many times and also the boundaries $C_1 = \partial B_1(m_1)$ and $C_2 = \partial B_2(m_2)$ of these disks infinitely many times. Note that, if $\{h(\gamma(t)) : t \geq 0\} \cap C_i$ is a finite set $S$ (say), then $S \subset M$ which is a contradiction to $\overline{B_i(m_i)} \cap M = \{m_i\}$ for $i = 1, 2$. Suppose that $\{h(\gamma(t)) : t \geq 0\} \cap C_i$ will have a limit point, $l_i$ (say), since $C_1$ and $C_2$ are compact. This implies that $l_i \in M$ which is a contradiction to $\overline{B_i(m_i)} \cap M = \{m_i\}$ for $i = 1, 2$. Therefore, $M$ is a singleton set in $\mathbb{C}$.

If $M = \{b\}$ where $b \in \mathbb{C}$ then $a = g(b)$ and $b$ is an asymptotic value of $h(z)$. If $M = \{\infty\}$ then $a$ is an asymptotic value of $g(z)$. Therefore, in both the cases, $a$ corresponds either to an asymptotic value of $h$ or to that of $g$. This completes the proof.

The following proposition determines all the singular values of $f_\lambda \in \mathbb{M}$. 


Proposition 1. Let $f_\lambda \in \mathcal{M}$. Then $f_\lambda(z)$ has only three (finite) asymptotic values and no critical values.

Proof. Since $f'_\lambda(z) = \lambda e^z \text{sech}^2(e^z) \neq 0$ for any $z \in \mathbb{C}$, it follows that $f_\lambda(z)$ has no critical points and hence, it has no critical values.

Turning to asymptotic values, by Lemma 1, it follows that $f_\lambda(z)$ will have at most 3 finite asymptotic values, since $e^z$ has only one finite asymptotic value, namely, 0 and $\lambda \tanh(z)$ has two finite asymptotic values, namely, $\lambda$ and $-\lambda$.

If $\gamma_1(t) = -t$ for $t \in [0, \infty)$ then $\lim_{t \to \infty} f_\lambda(\gamma_1(t)) = 0$. If $\gamma_2(t) = t$ for $t \in [0, \infty)$ then $\lim_{t \to \infty} f_\lambda(\gamma_2(t)) = \lambda$. When $\gamma_3(t) = t + i\pi$ for $t \in [0, \infty)$, $\lim_{t \to \infty} f_\lambda(\gamma_3(t)) = -\lambda$. Therefore, 0 and $\pm \lambda$ are the three finite asymptotic values of $f_\lambda(z)$.

Two meromorphic functions $f_1, f_2 : \mathbb{C} \to \hat{\mathbb{C}}$ are called conformally conjugate if there is an analytic homeomorphism $\psi$ on $\hat{\mathbb{C}}$ such that $\psi(f_1(z)) = f_2(\psi(z))$ for all $z \in \hat{\mathbb{C}}$. If a conformal conjugacy $\psi$ exists between two transcendental meromorphic functions then the analytic homeomorphic map $\psi(z)$ will be of the form $\psi(z) = az + b$ where $a$ and $b$ are complex constants with $a \neq 0$. In the following, we show that no two functions $f_{\lambda_1}$ and $f_{\lambda_2}$ in $\mathcal{M}$ are conformally conjugate. Suppose that there exists an analytic homeomorphism $\psi(z) = az + b$ for all $z \in \hat{\mathbb{C}}$ between two functions $f_{\lambda_1}$ and $f_{\lambda_2}$ in $\mathcal{M}$ with $\lambda_1 \neq \lambda_2$. In the proof of Proposition 1, it is shown that the function $f_{\lambda_i}$ has three finite asymptotic values, namely, 0, $\lambda_i$ and $-\lambda_i$. Note that $\pm \lambda_i$ are the exceptional values of $f_{\lambda_i}$. Now, the conjugacy map $\psi$ is required to take the set $\{\lambda_1, -\lambda_1\}$ to $\{\lambda_2, -\lambda_2\}$. That is, either $\psi(\lambda_1) = \lambda_2$, $\psi(-\lambda_1) = -\lambda_2$ or $\psi(\lambda_1) = -\lambda_2$, $\psi(-\lambda_1) = \lambda_2$. It implies that $b = 0$ and $af_{\lambda_1}(z) = f_{\lambda_2}(az)$ for all $z \in \mathbb{C}$. Therefore, it follows that $af'_{\lambda_1}(0) = af'_{\lambda_2}(0)$ and $\lambda_1 = \lambda_2$ which is not true.
Define \( \phi : \mathbb{R} \to \mathbb{R} \) by \( \phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} \). Rewriting it,
\[
\phi(x) = \frac{x f'(x) + f(x)}{f(x) f'(x)} = \frac{1}{4 e^x \tanh(e^x)} \left(4 x e^x + e^{2x} - e^{-2x} \right).
\]
Letting \( \phi_1(x) = 4x e^x + e^{2x} - e^{-2x} \), we observe that \( \phi'_1(x) = 2e^x(2x + 2 + e^{2x} + e^{-2x}) = 2e^x \phi_2(x) \), where \( \phi_2(x) = 2x + 2 + e^{2x} + e^{-2x} \). The function \( \phi'_1(x) = 2 + 2e^x(e^{2x} - e^{-2x}) > 0 \) for \( x < 0 \). This implies that \( \phi_2(x) \) is strictly increasing for \( x < 0 \). Since \( \phi_2(x) \to -\infty \) as \( x \to -\infty \) and \( \phi_2(x) \to 2 + e^2 + e^{-2} > 0 \) as \( x \to 0 \), there exists a point \( x_2 < 0 \) such that \( \phi_2(x) < 0 \) for \( x < x_2 \), \( \phi_2(x_2) = 0 \) and \( \phi_2(x) > 0 \) for \( x_2 < x < 0 \) and consequently, \( \phi'_1(x) < 0 \) for \( x < x_2 \), \( \phi'_1(x_2) = 0 \) and \( \phi'_1(x) > 0 \) for \( x_2 < x < 0 \). Therefore, \( \phi_1(x) \) is decreasing for \( x < x_2 \) and, is increasing for \( x_2 < x < 0 \). This shows that the function \( \phi_1(x) \) attains the minimum value at the point \( x_2 \) and the minimum value \( \phi_1(x_2) \) is negative, because \( \phi_1(x) \to 0 \) as \( x \to -\infty \). Since \( \phi_1(x) \to e^2 - e^{-2} > 0 \) as \( x \to 0 \), there exists a unique point \( x^* \) with \( x_2 < x^* < 0 \) such that \( \phi_1(x) < 0 \) for \( x < x^* \), \( \phi_1(x) = 0 \) for \( x = x^* \) and \( \phi_1(x) > 0 \) for \( x^* < x < 0 \); and consequently, \( \phi(x) < 0 \) for \( x < x^* \), \( \phi(x) = 0 \) for \( x = x^* \) and \( \phi(x) > 0 \) for \( x^* < x < 0 \). Observe that \( \phi(x) > 0 \) for \( x \geq 0 \).

Define
\[
\lambda^* = \frac{x^*}{f(x^*)} = \frac{-1}{f'(x^*)} \quad (1)
\]
where \( x^* \) is the unique real root of the equation \( \phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} = 0 \). Numerically it is found that \( x^* \approx -1.0789 \) and \( \lambda^* \approx -3.2946 \).

3. Real Periodic Points. In this section, the real periodic points of \( f_\lambda \in \text{M} \) is investigated. In Proposition 2, it is proved that \( f_\lambda \) cannot have a real periodic point of prime period more than two. The existence and nature of the real fixed points is proved in Proposition 3. The existence and nature of the real periodic points of prime period 2 is analyzed in Proposition 4.
Proposition 2. Let $f_\lambda \in \mathcal{M}$. Then, $f_\lambda$ has no real periodic point of prime period more than two.

Proof. Let $g_\lambda(x) = f_\lambda(f_\lambda(x))$ for $x \in \mathbb{R}$. Then, the function $g_\lambda(x)$ is strictly increasing on $\mathbb{R}$, since $g'_\lambda(x) = \lambda f'(\lambda f(x)) > 0$ for all $x \in \mathbb{R}$. Suppose there exists a point $x_0$ such that $x_0$ is a real periodic point of prime period $p = 2$ for $g_\lambda$.

Since $g_\lambda(x_0) \neq x_0$, either $g_\lambda(x_0) > x_0$ or $g_\lambda(x_0) < x_0$. If $g_\lambda(x_0) > x_0$, it implies that $g'_{\lambda}^k(x_0) > g'_{\lambda}^{k-1}(x_0)$ for all $k > 1$. It makes $g'_{\lambda}(x_0) > x_0$ for every $k \in \mathbb{N}$ which is a contradiction to the fact that $g'_{\lambda}(x_0) = x_0$. A similar contradiction can well be realized by assuming $g_\lambda(x_0) < x_0$. Therefore, $g_\lambda(x)$ has no real periodic point of prime period $p = 2$. Since, $g_\lambda(x)$ can not have a real periodic point of prime period 2, it follows that the function $f_\lambda(x)$ can not have a real periodic point of prime period 4. Since $f_\lambda$ is a continuous real valued function and not possessing a real periodic point of prime period 4 on $\mathbb{R}$, by Sarkovskii’s theorem ( [9], Page 62), we conclude that $f_\lambda(x)$ can not have a real periodic point of prime period more than two. \qed

The function $f(x) = \tanh(e^x)$ is positive for all $x \in \mathbb{R}$. Since $f'(x) = e^x \text{sech}^2(e^x) > 0$ for all $x \in \mathbb{R}$, the function $f(x)$ is strictly increasing on $\mathbb{R}$. It is easy to see that $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 1$ as $x \to +\infty$. Now, we find the nature of the function $f''(x) = e^x \text{sech}^2(e^x)(1 - 2e^x \tanh(e^x))$ on $\mathbb{R}$. Observe that the function $\frac{d}{dx}(1 - 2e^x \tanh(e^x)) = -2e^x(e^x \text{sech}^2(e^x) + \tanh(e^x)) < 0$ for all $x \in \mathbb{R}$.

Therefore, the function $\psi(x) = 1 - 2e^x \tanh(e^x)$ is a strictly decreasing on $\mathbb{R}$. Since

$$\lim_{x \to -\infty} 1 - 2e^x \tanh(e^x) = 1$$
and

$$\lim_{x \to 0} 1 - 2e^x \tanh(e^x) = 1 - 2 \frac{e^2 - 1}{e^2 + 1} = 3 - \frac{e^2}{e^2 + 1} < 0,$$

it follows that there exists a point $\hat{x} < 0$ such that $\psi(x) > 0$ for $x < \hat{x}$, $\psi(x) = 0$ for $x = \hat{x}$ and $\psi(x) < 0$ for $x > \hat{x}$. Consequently $f''(x) = e^x \text{sech}^2(e^x)(1 -$
2e^x \tanh(e^x)) > 0 for x < \hat{x}, f''(x) = 0 for x = \hat{x} and f''(x) < 0 for x > \hat{x}. This shows that the function f'(x) increases in the interval (−∞, \hat{x}), decreases in the interval (\hat{x}, ∞) and attains the maximum value at the point \hat{x}. Also f'(x) → 0 as |x| → ∞. Define \hat{\lambda} as 1/f'(\hat{x}). It is numerically computed that \hat{x} ≈ −0.261 and \hat{\lambda} ≈ 2.233.

The existence and the nature of the real fixed points is proved in the following proposition.

**Proposition 3.** Let f_\lambda ∈ M.

1. If \lambda > \lambda^*, f_\lambda has a unique real fixed point a_\lambda (say) and that is attracting.
2. If \lambda = \lambda^*, f_\lambda has a unique rationally neutral real fixed point at x = x^*, where
   
x^* is the unique real root of \phi(x) = \frac{x}{f(x)} + \frac{1}{f'(x)} = 0.
3. If \lambda < \lambda^*, f_\lambda has a unique real fixed point r_\lambda (say) and that is repelling.

**Proof.** Set h_\lambda(x) = f_\lambda(x) − x = \lambda f(x) − x where f(x) = \tanh(e^x) for x ∈ \mathbb{R} and \lambda is a non-zero real parameter. Then, h'_\lambda(x) = \lambda f'(x) − 1 and h''_\lambda(x) = \lambda f''(x).

For all \lambda,

\[
\lim_{x \to -\infty} h_\lambda(x) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} h_\lambda(x) = -\infty.
\]

Since h_\lambda(x) is a continuous function on \mathbb{R}, it has a real zero. Consequently, the function f_\lambda has a real fixed point x_\lambda (say). Since f(x) > 0 for all x ∈ \mathbb{R}, the real fixed point of f_\lambda has the same sign as that of \lambda. If \lambda > 0, the function h'_\lambda(x) is increasing from the value −1 to the value h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) − 1 in the interval (−∞, \hat{x}] and it is decreasing from the value h'_\lambda(\hat{x}) to −1 in the interval [\hat{x}, ∞) where \hat{x} satisfies f''(\hat{x}) = 0. If \lambda < 0, the function h'_\lambda(x) is decreasing from the value −1 to the value h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) − 1 < 0 in the interval (−∞, \hat{x}] and it is increasing from the value h'_\lambda(\hat{x}) to −1 in the interval [\hat{x}, ∞). For \lambda < 0, it follows
that the function \( h_\lambda(x) \) is strictly decreasing and consequently, the real fixed point \( x_\lambda \) of \( f_\lambda \) is unique.

**Case (1): \( \lambda > \lambda^* \)**

**Subcase (a): \( \lambda \geq \hat{\lambda} \)**

In this case, the function \( h'_\lambda(x) \) is increasing from the value \(-1\) to the value \( h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 \geq \hat{\lambda} f'(\hat{x}) - 1 = 0 \) in the interval \(( -\infty, \hat{x} \)] and it is decreasing from the value \( h'_\lambda(\hat{x}) \) to \(-1\) in the interval \([\hat{x}, \infty)\). Therefore, there exist two points \( x_{1,\lambda} \) and \( x_{2,\lambda} \) (say) with \( x_{1,\lambda} \leq x_{2,\lambda} \) such that \( h'_\lambda(x) = 0 \) for \( x = x_{1,\lambda} \) and \( x = x_{2,\lambda} \). Further, \( h'_\lambda(x) < 0 \) for \( x \in ( -\infty, x_{1,\lambda} ) \cup ( x_{2,\lambda}, \infty ) \) and \( h'_\lambda(x) > 0 \) for \( x \in ( x_{1,\lambda}, x_{2,\lambda} ) \). If \( x_{2,\lambda} \leq 0 \), then \( -1 < h'_\lambda(x) < 0 \) for all \( x > 0 \). Therefore, it follows that the real fixed point \( x_\lambda \) (which is positive as \( \lambda > 0 \) in this case) of \( f_\lambda \) is unique and attracting.

When \( x_{2,\lambda} > 0 \), the function \( h_\lambda \) attains the maximum value at \( x = x_{2,\lambda} \) in \((0, \infty)\).

Since \( 0 < h_\lambda(0) < h_\lambda(x_{2,\lambda}) \) and \( h_\lambda(x) \) is decreasing in the interval \((x_{2,\lambda}, \infty)\), it follows that \( x_{2,\lambda} < x_\lambda \). Therefore, the real fixed point \( x_\lambda \) of \( f_\lambda \) is attracting for \( f_\lambda \). Rename the real fixed point \( x_\lambda \) as \( a_\lambda \) when \( \lambda \geq \hat{\lambda} \).

**Subcase (b): \( 0 < \lambda < \hat{\lambda} \)**

If \( 0 < \lambda < \hat{\lambda} \), the maximum value \( h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 < \hat{\lambda} f'(\hat{x}) - 1 = 0 \). It follows that \( -1 < h'_\lambda(x) = f'_\lambda(x) - 1 < 0 \) for all \( x \in \mathbb{R} \). Therefore, the real fixed point \( x_\lambda \) of \( f_\lambda \) is unique and it is an attracting fixed point for \( f_\lambda \). Rename the real fixed point \( x_\lambda \) as \( a_\lambda \).

**Subcase (c): \( -\hat{\lambda} < \lambda < 0 \)**

If \( -\hat{\lambda} < \lambda < 0 \), the minimum value \( h'_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 > -\hat{\lambda} f'(\hat{x}) - 1 = -2 \). It follows that \( -2 < h'_\lambda(x) = f'_\lambda(x) - 1 < -1 \) for all \( x \in \mathbb{R} \). Therefore, the real fixed point \( x_\lambda \) of \( f_\lambda \) is attracting for \( f_\lambda \). In this case, we rename \( x_\lambda \) as \( a_\lambda \).
Subcase (d): $\lambda^* < \lambda \leq -\hat{\lambda}$

The function $h^*_\lambda(x)$ is decreasing from the value $-1$ to the value $h^*_\lambda(\hat{x}) = \lambda f'(\hat{x}) - 1 \leq -\hat{\lambda}f'(\hat{x}) - 1 \leq -1$ in the interval $(-\infty, \hat{x}]$ and it is increasing from the value $h^*_\lambda(\hat{x})$ to $-1$ in the interval $[\hat{x}, \infty)$. Since $h^*_\lambda(\hat{x}) + 2 \leq 0$ for $\lambda^* < \lambda \leq -\hat{\lambda}$, there exist two points $y_{1, \lambda}$ and $y_{2, \lambda}$ (say) with $y_{1, \lambda} \leq y_{2, \lambda}$ such that $h^*_\lambda(x) + 2 = 0$ for $x = y_{1, \lambda}$ and $x = y_{2, \lambda}$. Further, $h^*_\lambda(x) + 2 > 0$ for $x \in (-\infty, y_{1, \lambda}) \cup (y_{2, \lambda}, \infty)$ and $h^*_\lambda(x) + 2 < 0$ for $x \in (y_{1, \lambda}, y_{2, \lambda})$. Now, the parameter $\lambda$ can be realized in two ways as $\lambda = \frac{-1}{f'(y_{1, \lambda})}$ and $\lambda = \frac{x_{\lambda}}{f'(x_{\lambda})}$ where $y_{1, \lambda}$ is the smaller root of $h^*_\lambda(x) + 2 = 0$ and $x_{\lambda}$ is the unique real fixed point of $f_\lambda$. It is noticed that $x^* < \hat{x} < 0$. Now we shall show that the points $x_{\lambda}$ and $y_{1, \lambda}$ are in the interval $[x^*, \hat{x}]$ and $x_{\lambda} < y_{1, \lambda}$. Since $\lambda^* < \lambda \leq -\hat{\lambda}$, we have $\frac{-1}{f'(x^*)} < \frac{-1}{f'(y_{1, \lambda})} \leq \frac{-1}{f'(\hat{x})}$. Using the fact that $\frac{-1}{f'}$ is strictly increasing in $(-\infty, \hat{x})$, we get

$$x^* < y_{1, \lambda} \leq \hat{x}.$$ 

For all $x < 0$, $\frac{d}{dx} \left( \frac{x}{f(x)} \right) > 0$ implies that $\frac{x}{f(x)}$ is strictly increasing in $\mathbb{R}^- = \{ x \in \mathbb{R} : x < 0 \}$. So, the inequality $\lambda^* < \lambda \leq -\hat{\lambda}$ gives $\frac{x^*}{f(x^*)} < \frac{x_{\lambda}}{f(x_{\lambda})} \leq \frac{-1}{f'(\hat{x})}$. Since $\hat{x} > x^*$, we have $\phi(\hat{x}) > 0$ and $\frac{\hat{x}}{f(\hat{x})} > \frac{-1}{f'(\hat{x})}$. Therefore, $\frac{x^*}{f(x^*)} < \frac{x_{\lambda}}{f(x_{\lambda})} \leq \frac{-1}{f'(\hat{x})} < \frac{\hat{x}}{f(\hat{x})}$, which gives that

$$x^* < x_{\lambda} < \hat{x}.$$ 

Since $\phi(y_{1, \lambda}) > 0$, it follows that $\frac{y_{1, \lambda}}{f'(y_{1, \lambda})} > \frac{-1}{f'(y_{1, \lambda})} = \frac{x_{\lambda}}{f(x_{\lambda})}$. Since the function $\frac{x}{f(x)}$ is increasing for $x < 0$, we get $x_{\lambda} < y_{1, \lambda}$. Now, the function $h^*_\lambda(x) + 2 > 0$ for $x < y_{1, \lambda}$. So, it follows that $-1 < f'_\lambda(x) < 0$ for $x < y_{1, \lambda}$ and in particular, $-1 < f'_\lambda(x_{\lambda}) < 0$. Therefore, the real fixed point $x_{\lambda}$ is attracting and rename it as $a_{\lambda}$.

Case (2): $\lambda = \lambda^*$

By definition $\lambda^* = \frac{x^*}{f(x^*)} = \frac{-1}{f'(x^*)}$. Since the function $\frac{x}{f(x)}$ is one-to-one in the
negative real axis, it follows that the real fixed point $x_\lambda$ is equal to $x^*$. The real fixed point $x^*$ is a rationally neutral fixed point, because $\lambda^* f'(x^*) = -1$.

**Case (3):** $\lambda < \lambda^*$

As in Subcase (d), the minimum value $h'_\lambda(\hat{x}) < -2$. Therefore, there exist two points $y_{1,\lambda}$ and $y_{2,\lambda}$ (say) with $y_{1,\lambda} < y_{2,\lambda}$ such that $h'_\lambda(x) + 2 = 0$ for $x = y_{1,\lambda}$ and $x = y_{2,\lambda}$. Further, $h'_\lambda(x) + 2 > 0$ for $x \in (-\infty, y_{1,\lambda}) \cup (y_{2,\lambda}, \infty)$ and $h'_\lambda(x) + 2 < 0$ for $x \in (y_{1,\lambda}, y_{2,\lambda})$. Here our intention is to show that the fixed point $x_\lambda$ lies in $(y_{1,\lambda}, y_{2,\lambda})$ where $|f'_\lambda(x)| > 1$. Arguing on the similar lines as in Subcase (d), one can get that $y_{1,\lambda} < x^* < \hat{x} < y_{2,\lambda}$ and $x_\lambda < x^* < \hat{x} < y_{2,\lambda}$ for $\lambda < \lambda^*$. Since $\phi(y_{1,\lambda}) < 0$, we get $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} < \frac{-1}{f'(y_{1,\lambda})}$. But, $\lambda = \frac{-1}{f'(y_{1,\lambda})} = \frac{x_\lambda}{f(x_\lambda)}$. Therefore, $\frac{y_{1,\lambda}}{f(y_{1,\lambda})} < \frac{x_\lambda}{f(x_\lambda)}$ and consequently $y_{1,\lambda} < x_\lambda$. Therefore, the fixed point $x_\lambda$ is repelling. Let us rename it as $r_\lambda$.


The existence and the nature of the real periodic points of prime period 2 is explored in the following proposition.

**Proposition 4.** Let $f_\lambda \in M$.

1. If $\lambda > \lambda^*$, $f^2_\lambda$ has only one real fixed point $a_\lambda$ which is an attracting real fixed point of $f_\lambda$.

2. If $\lambda = \lambda^*$, $f^2_\lambda$ has only one real fixed point $x^*$ which is a rationally neutral real fixed point of $f_\lambda$.

3. If $\lambda < \lambda^*$, $f^2_\lambda$ has exactly three real fixed points. One of the fixed points of $f^2_\lambda$ is $r_\lambda$ which is a repelling real fixed point of $f_\lambda$. The other two fixed points of $f^2_\lambda$ are the periodic points of (prime) period 2 of $f_\lambda$ and form an attracting or a parabolic 2-periodic cycle $\{a_{1,\lambda}, a_{2,\lambda}\}$ (say) with $a_{1,\lambda} < r_\lambda < a_{2,\lambda} < 0$. 
Proof. Case 1: $\lambda > \lambda^*$

If $\lambda > \lambda^*$, by Proposition 3(1), $f_\lambda(x)$ has a unique attracting fixed point $a_\lambda$ on the real line. The fixed point $a_\lambda$ of $f_\lambda$ is also a fixed point of $f_\lambda^2$. Now, we show that $f_\lambda^2$ has no other real fixed points.

![Graphs](image)

**Figure 1.** Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $(f_\lambda^2)'(x)$ for $\lambda > \lambda^*$

For $\lambda > 0$, $f_\lambda$ is strictly increasing on $\mathbb{R}$. If $f_\lambda(x) \neq x$ for a point $x \in \mathbb{R}$ then $f_\lambda^n(x) \neq x$ for any integer $n > 1$. Therefore, it follows that $f_\lambda (\lambda > 0)$ has no real periodic points of prime period $p \geq 2$.

Let $\lambda^* < \lambda < 0$. Suppose that there is a fixed point of $f_\lambda^2$ which is different from $a_\lambda$. As $f_\lambda$ has only one real fixed point, any fixed point other than $a_\lambda$ of $f_\lambda^2$ will be a 2-periodic cycle for $f_\lambda$. If $f_\lambda$ has more than one 2-periodic cycles then the outer most 2-periodic cycle is chosen for consideration. This is possible, because, if $f_\lambda$ has two different 2-periodic cycles $\{a, b\}$ with $a < b$ and $\{c, d\}$ with $c < d$, then it follows from the fact $f_\lambda$ is strictly decreasing for $\lambda < 0$ that the two different
2-periodic cycles satisfy $c < a < a_\lambda < b < d$ or $a < c < a_\lambda < d < b$. In the first case $\{c, d\}$ and in the second case $\{a, b\}$ is called the outer cycle.

Let $\{d_1, d_2\}$ be the outermost 2-periodic cycle of $f_\lambda$ such that $f_\lambda(d_1) = d_2$ and $f_\lambda(d_2) = d_1$ with $d_1 < d_2$. Set $D_1 = (\infty, d_1)$ and $D_2 = (d_2, \infty)$. Since $f_\lambda^n(x) > x$ for any $x \in D_1$, the sequence $\{f_\lambda^{2n}(x)\}$ will be a monotonically increasing sequence and $d_1 = \sup\{f_\lambda^{2n}(x) : x \in D_1 \text{ and } n \in \mathbb{N}\}$. Therefore, $f_\lambda^{2n}(x) \rightarrow d_1$ as $n \rightarrow \infty$. Similarly, $\{f_\lambda^{2n}(x)\}$ is a monotonically decreasing sequence converging to $d_2$ for any $x \in D_2$, since $f_\lambda^n(x) < x$ for $x \in D_2$ and $d_2 = \inf\{f_\lambda^{2n}(x) : x \in D_2 \text{ and } n \in \mathbb{N}\}$. This shows that the cycle $\{d_1, d_2\}$ can be either an attracting or a parabolic cycle. Note that $\lambda < d_1 < a_\lambda < d_2 < 0 < -\lambda$. This implies that $f_\lambda^{2n}(\lambda) \rightarrow d_1$, $f_\lambda^{2n}(0) \rightarrow d_2$ and $f_\lambda^{2n}(-\lambda) \rightarrow d_2$ as $n \rightarrow \infty$. Thus, all the singular values are attracted by the 2-periodic cycle $\{d_1, d_2\}$. It is shown in Proposition 3 that $a_\lambda$ is a real attracting fixed point of $f_\lambda$ for $\lambda > \lambda^*$. So, the basin of attraction $A(a_\lambda)$ of the attracting fixed point $a_\lambda$ must contain at least one singular value which is a contradiction to the fact that all three singular values tend either to $d_1$ or to $d_2$ under iterations of $f_\lambda^2$. Therefore, $f_\lambda^2$ can not have any real fixed point other than $a_\lambda$ if $\lambda^* < \lambda < 0$ (See Figure 1).

**Case 2: $\lambda = \lambda^*$:**

If $\lambda = \lambda^*$, by Proposition 3(2), $f_\lambda(x)$ has a unique rationally neutral fixed point $x^*$ on the real line. The fixed point $x^*$ of $f_\lambda$ is also a fixed point for $f_\lambda^2$. By similar arguments as in Case 1, one can show that $f_\lambda^2$ has no real periodic point of prime period 2 (See Figure 2).
DYNAMICS OF $\lambda \tanh(e^z)$

Figure 2. Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $(f_\lambda^2)'(x)$ for $\lambda = \lambda^*$

Case 3: $\lambda < \lambda^*$:

If $\lambda < \lambda^*$, by Proposition 3(3), $f_\lambda(x)$ has a unique repelling fixed point $r_\lambda$ on the

Figure 3. Graphs of (i) $f_\lambda^2(x) - x$ and (ii) $(f_\lambda^2)'(x)$ for $\lambda < \lambda^*$
real line. The fixed point $r_\lambda$ of $f_\lambda$ is also a fixed point for $f^2_\lambda$. Now, we show that
including $r_\lambda$, the function $f^2_\lambda$ has 3 fixed points on $\mathbb{R}$.

Let $x < r_\lambda$. Suppose that $f^2_\lambda(x) > x$. Since $f^2_\lambda(x)$ is strictly increasing on $\mathbb{R}$, it follows that $f^{2n}_\lambda(x) > f^{2(n-1)}_\lambda(x)$ for all $n \in \mathbb{N}$. But, the sequence $\{f^{2n}_\lambda(x)\}_{n>0}$ is bounded above by $r_\lambda$. By Bolzano theorem, the sequence $\{f^{2n}_\lambda(x)\}$ converges to a point $a$ (say). By the continuity of $f_\lambda$, it follows that the limit point $a$ will be a periodic point of prime period at most two. As $f_\lambda$ does not have any real fixed point other than $r_\lambda$, the limit point $a$ must be a periodic point of prime period 2. Similarly, the other possibility $f^2_\lambda(x) < x$ also leads to the same conclusion.

Therefore, $f_\lambda$ has a periodic point of prime period 2 on $\mathbb{R}$.

Now, we show that $f_\lambda$ has a unique periodic point of prime period 2 on $\mathbb{R}$. Suppose that $f_\lambda$ has more than one periodic point of prime period 2 on $\mathbb{R}$. Then, choose the outermost (in the sense defined earlier in Case 1) 2-periodic cycle of $f_\lambda$.

Let $\{o_{1\lambda}, o_{2\lambda}\}$ be the outermost 2-periodic cycle of $f_\lambda$ such that $f_\lambda(o_{1\lambda}) = o_{2\lambda}$ and $f_\lambda(o_{2\lambda}) = o_{1\lambda}$ with $o_{1\lambda} < o_{2\lambda}$. As shown in case of $\lambda \in (\lambda^*, 0)$, we can show that the 2-periodic cycle $\{o_{1\lambda}, o_{2\lambda}\}$ is either an attracting cycle or a parabolic cycle of $f_\lambda$ and the singular values 0 and $\pm \lambda$ are attracted by this cycle. Now, let us consider the inner most 2-periodic cycle $\{i_{1\lambda}, i_{2\lambda}\}$ (say) of $f_\lambda$ with $f_\lambda(i_{1\lambda}) = i_{2\lambda}$ and $f_\lambda(i_{2\lambda}) = i_{1\lambda}$ with $i_{1\lambda} < i_{2\lambda}$. Observe that $f_\lambda(x) \in (r_\lambda, i_{2\lambda})$ for $x \in (i_{1\lambda}, r_\lambda)$ and $f_\lambda(x) \in (i_{1\lambda}, r_\lambda)$ for $x \in (r_\lambda, i_{2\lambda})$. This gives that the sequence $\{f^{2n}_\lambda(x)\}$ is bounded for $x \in (i_{1\lambda}, i_{2\lambda})$. Since $f^2_\lambda$ is strictly increasing on $\mathbb{R}$, the sequence $\{f^{2n}_\lambda(x)\}$ is monotonic. Since $r_\lambda$ is repelling, $\{f^{2n}_\lambda(x)\} \rightarrow i_{1\lambda}$ as $n \rightarrow \infty$ for $x \in (i_{1\lambda}, r_\lambda)$ and $\{f^{2n}_\lambda(x)\} \rightarrow i_{2\lambda}$ as $n \rightarrow \infty$ for $x \in (r_\lambda, i_{2\lambda})$. This shows that the inner cycle $\{i_{1\lambda}, i_{2\lambda}\}$ is also either attracting or parabolic. But, there is no singular value that can be attracted by the inner cycle $\{i_{1\lambda}, i_{2\lambda}\}$, since all the singular values
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are already attracted by the outer most cycle. This rules out the existence of the inner most cycle $\{i_{1\lambda}, i_{2\lambda}\}$. Therefore, the function $f_\lambda$ has only one 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ (say) that is either attracting or parabolic on $\mathbb{R}$ if $\lambda < \lambda^*$ (See Figure 3).

This completes the proof. □

4. **Dynamics of $f_\lambda \in \mathcal{M}$**. The dynamics of the function $f_\lambda(z)$ for $z \in \mathbb{C}$ is investigated in the present section.

**Proposition 5.** Let $f_\lambda \in \mathcal{M}$. Then, the Fatou set of $f_\lambda$ does not contain wandering domain and Baker domain.

**Proof.** For critically finite meromorphic functions, the non-existence of wandering domains is proved in [2] and the non-existence of Baker domains is proved in [?] and [6]. Since the function $f_\lambda$ is critically finite by Proposition 1, it follows that the Fatou set of $f_\lambda$ does not contain wandering domain and Baker domain. □

We determine the dynamics of $f_\lambda$ on the real line in the following proposition.

**Proposition 6.** Let $f_\lambda \in \mathcal{M}$.

1. If $\lambda > \lambda^*$ then $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for all $x \in \mathbb{R}$ where $a_\lambda$ is the attracting real fixed point of $f_\lambda$.

2. If $\lambda = \lambda^*$ then $f_\lambda^n(x) \to x^*$ as $n \to \infty$ for all $x \in \mathbb{R}$ where $x^*$ is the rationally neutral real fixed point of $f_\lambda$.

3. If $\lambda < \lambda^*$ then $f_\lambda^{2n}(x) \to a_{1\lambda}$ as $n \to \infty$ for $x < r_\lambda$ and $f_\lambda^{2n}(x) \to a_{2\lambda}$ as $n \to \infty$ for $x > r_\lambda$ where $\{a_{1\lambda}, a_{2\lambda}\}$ is the attracting or parabolic real 2-periodic cycle and $r_\lambda$ is the repelling real fixed point of $f_\lambda$.

**Proof. Case 1: $\lambda > \lambda^*$**

By Proposition 3(1) and Proposition 4(1), the function $f_\lambda(z)$ has a unique real
attracting fixed point $a_\lambda$ and $f_\lambda^2$ has no fixed point other than $a_\lambda$ on the real line. It is noted that $f_\lambda^2$ is strictly increasing and bounded on $\mathbb{R}$. Observe that $f_\lambda^2(x) > x$ for $x < a_\lambda$. It implies that $\{f_\lambda^{2n}(x)\}$ is a monotonically increasing bounded sequence and hence convergent. By continuity of $f_\lambda^2$, it follows that the limit point of $\{f_\lambda^{2n}(x)\}$ is a fixed point of $f_\lambda^2$ and therefore it equals to the only such point, namely, $a_\lambda$. Therefore, $f_\lambda^{2n}(x) \to a_\lambda$ as $n \to \infty$ for $x < a_\lambda$. Similarly, the same conclusion follows for $x > a_\lambda$ since $f_\lambda^2(x) < x$ for $x > a_\lambda$. Therefore, $\lim_{n \to \infty} f_\lambda^{2n}(x) = a_\lambda$ for all $x \in \mathbb{R}$. Since $a_\lambda$ is an attracting fixed point of the continuous function $f_\lambda$, it is concluded that $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for all $x \in \mathbb{R}$.

**Case 2:** $\lambda = \lambda^*$

By Proposition 3(2) and Proposition 4(2), the function $f_\lambda(x)$ has a unique rational neutral real fixed point $x^*$ and $f_\lambda^2$ has no fixed point other than $x^*$ on the real line. Since $f_\lambda^2$ is a strictly increasing, bounded function on $\mathbb{R}$ with $f_\lambda^2(x) > x$ for $x < x^*$ and $f_\lambda^2(x) < x$ for $x > x^*$, it follows by similar arguments as in the previous case that $f_\lambda^{2n}(x) \to x^*$ as $n \to \infty$ for $x \in \mathbb{R}$. Since $f_\lambda$ is continuous and $x^*$ is a fixed point of $f_\lambda$, it is concluded that $f_\lambda^n(x) \to x^*$ as $n \to \infty$ for all $x \in \mathbb{R}$.

**Case 3:** $\lambda < \lambda^*$

On the real line $\mathbb{R}$, the function $f_\lambda$ has a unique repelling real fixed point $r_\lambda$ by Proposition 3(3), and has a unique attracting or parabolic 2-periodic cycle $\{a_{1\lambda}, a_{2\lambda}\}$ with $a_{1\lambda} < r_\lambda < a_{2\lambda} < 0$ by Proposition 4(3). Observe that $f_\lambda^2(x) > x$ for $x < a_{1\lambda}$. Since $f_\lambda^2$ is strictly increasing on $\mathbb{R}$ and $f_\lambda^2(a_{1\lambda}) = a_{1\lambda}$, it follows that the sequence $\{f_\lambda^{2n}(x)\}$ is a monotonically increasing sequence and $\sup\{f_\lambda^{2n}(x) : n \in \mathbb{N}\} = a_{1\lambda}$ for $x \leq a_{1\lambda}$. This gives that $\lim_{n \to \infty} f_\lambda^{2n}(x) = a_{1\lambda}$ for $x \leq a_{1\lambda}$. Since $f_\lambda^2(x) < x$ for $a_{1\lambda} < x < r_\lambda$, the sequence $\{f_\lambda^{2n}(x)\}$ is monotonically decreasing and bounded below by $a_{1\lambda}$. Therefore, $f_\lambda^{2n}(x) \to a_{1\lambda}$ as $n \to \infty$ for $a_{1\lambda} < x < r_\lambda$. For $x \in$
DYNAMICS OF $\lambda \tanh(e^z)$ (r$\lambda$, a$\lambda$), the function $f^2_{\lambda}$ satisfies $f^2_{\lambda}(x) > x$. Consequently, the sequence $\{f^{2n}_{\lambda}(x)\}$ is monotonically increasing and converges to $a_{2\lambda}$. When $x \geq a_{2\lambda}$, the sequence $\{f^n_{\lambda}\}$ is decreasing and bounded below by $a_{2\lambda}$, since $f^2_{\lambda}(x) < x$ for $x > a_{2\lambda}$ and $f^2_{\lambda}(a_{2\lambda}) = a_{2\lambda}$. Therefore, $f^{2n}_{\lambda}(x) \to a_{2\lambda}$ as $n \to \infty$ for $x \geq a_{2\lambda}$ which completes the proof.

In the following, we present the proof of Theorem 1.

Proof of Theorem 1.

Case 1: $\lambda > \lambda^*$

By Proposition 3(1), the function $f_{\lambda}(z)$ has a unique real attracting fixed point $a_{\lambda}$ on the real line. Let $A(a_{\lambda}) = \{z \in \hat{\mathbb{C}} : f^n_{\lambda}(z) \to a_{\lambda} \text{ as } n \to \infty\}$ be the basin of attraction of the real attracting fixed point $a_{\lambda}$. Since by Proposition 6(1), the real line $\mathbb{R}$ is in the basin of attraction $A(a_{\lambda})$ and in particular, all the singular values $\{\lambda, -\lambda, 0\}$ and their forward orbits are in $A(a_{\lambda})$.

The Fatou set of $f_{\lambda}(z)$ has no basin of attraction other than $A(a_{\lambda})$. To see this, assume, if possible, $A(z_{\lambda})$ is a basin of attraction of an attracting periodic point $z_{\lambda} \neq a_{\lambda}$. Obviously, $A(z_{\lambda}) \cap A(a_{\lambda}) = \emptyset$. But, $A(z_{\lambda})$ contains at least one singular value and its forward orbit. This contradicts the fact all the singular values and its forward orbits are contained in $A(a_{\lambda})$, since $A(z_{\lambda}) \cap A(a_{\lambda}) = \emptyset$ for $z_{\lambda} \neq a_{\lambda}$.

The Fatou set of $f_{\lambda}(z)$ cannot contain a parabolic domain. For, if the Fatou set of $f_{\lambda}(z)$ contains a parabolic domain $U$, then $U$ must contain at least one singular value, which leads to a contradiction that all singular values are in $A(a_{\lambda})$.

Again, the Fatou set of $f_{\lambda}(z)$ cannot contain a Siegel disk or a Herman ring. For, if possible, the Fatou set of $f_{\lambda}(z)$ contains a Siegel disk or a Herman ring, then the boundary of Siegel disk / Herman ring is contained in the closure of the forward
orbits of all singular values of \( f_\lambda(z) \). But all the singular values and their forward orbits are contained in \( A(a_\lambda) \), giving a contradiction.

By Proposition 5, the Fatou set of \( f_\lambda(z) \) does not contain Baker domains and wandering domains. Therefore, the Fatou set of \( f_\lambda(z) \) is equal to the basin of attraction \( A(a_\lambda) \) of the attracting real fixed point \( a_\lambda \) if \( \lambda > \lambda^* \).

**Case 2:** \( \lambda = \lambda^* \)

The function \( f_\lambda(z) \) has a unique rationally neutral real fixed point \( x^* \) on the real line by Proposition 3(2). Let \( P(x^*) = \{ z \in \hat{\mathbb{C}} \; : \; f_\lambda^n(z) \to x^* \text{ as } n \to \infty \} \) be the parabolic basin corresponding to the rationally neutral real fixed point \( x^* \). By Proposition 6(2), it follows that the real line \( \mathbb{R} \) and in particular, all the singular values \( \{\lambda, -\lambda, 0\} \) and their forward orbits are in the parabolic basin \( P(x^*) \). Now, the Fatou set of \( f_\lambda(z) \) for \( \lambda = \lambda^* \) does not contain any other parabolic domain \( U \) other than \( P(x^*) \). If the Fatou set of \( f_{\lambda^*}(z) \) contains any other parabolic domain \( U (\neq P(x^*)) \), then \( U \) must contain at least one singular value which is not possible.

Since all singular values are in \( P(x^*) \), the Fatou set of \( f_{\lambda^*}(z) \) can not contain a basin of attraction. The proofs of the fact that the Fatou set of \( f_\lambda(z) \) for \( \lambda = \lambda^* \) does not contain Siegel disk, Herman ring, Baker domain and wandering domain are similar to that of Case 1. Thus, all the possible stable domains other than the components of \( P(x^*) \) are ruled out and hence the Fatou set of \( f_{\lambda^*}(z) \) equals the parabolic basin \( P(x^*) \) corresponding to the rationally neutral real fixed point \( x^* \).

**Case 3:** \( \lambda < \lambda^* \)

By Proposition 4(3), the function \( f_\lambda(z) \) has an attracting or a parabolic real 2-periodic cycle \( \{a_{1\lambda}, a_{2\lambda}\} \) with \( a_{1\lambda} < r_\lambda < a_{2\lambda} < 0 \) where \( r_\lambda \) is the unique repelling real fixed point. Let us denote the basin of attraction of the attracting real 2-periodic cycle or the parabolic basin corresponding to the real parabolic 2-periodic
cycle as
\[ A = \{ z \in \mathbb{C} : f_\lambda^{2n}(z) \to a_{1\lambda} \text{ or } f_\lambda^{2n}(z) \to a_{2\lambda} \text{ as } n \to \infty \} . \]

By Proposition 6(3), it follows that the real line \( \mathbb{R} \) except the point \( r_\lambda \) and in particular, all the singular values \( \{ \lambda, -\lambda, 0 \} \) and their forward orbits are in \( A \). By proceeding in the same lines of arguments as in Case 1 or Case 2, we get that Fatou set of \( f_\lambda(z) \) does not contain Herman ring, Siegel disk, Baker domain, wandering domain and any basin of attraction or parabolic basin other than \( A \). Therefore, the Fatou set of \( f_\lambda(z) = A \) for \( \lambda < \lambda^* \).

Theorem 1 gives the following characterization of the Julia set of \( f_\lambda(z) \) which is computationally useful to generate the pictures of the Julia sets.

**Corollary 1.** Let \( f_\lambda \in \mathcal{M} \).

1. If \( \lambda > \lambda^* \) then the Julia set \( J(f_\lambda) \) is the complement of the basin of attraction \( A(a_\lambda) \) where \( a_\lambda \) is the attracting real fixed point of \( f_\lambda \).
2. If \( \lambda = \lambda^* \) then the Julia set \( J(f_\lambda) \) is the complement of the parabolic basin \( P(x^*) \) where \( x^* \) is the rationally neutral real fixed point of \( f_\lambda \).
3. If \( \lambda < \lambda^* \) then the Julia set \( J(f_\lambda) \) is the complement of the basin of attraction or the parabolic basin corresponding to the attracting or the parabolic real 2-periodic cycle \( \{ a_{1\lambda}, a_{2\lambda} \} \).

5. **Topology of the Fatou Components.** In the present section, the proofs of Theorems 2 and 3 are mainly provided. Some preliminary observations on the Fatou set of \( f_\lambda \) are made in Propositions 7 and 8.

**Proposition 7.** Let \( f_\lambda \in \mathcal{M} \). Then,
1. The Fatou set of $f_\lambda$ contains the left half-plane $H_\lambda = \{ z \in \mathbb{C} : \Re(z) < M_\lambda \}$ where $M_\lambda$ is a real number depending on $\lambda$.

2. The Fatou set of $f_\lambda$ contains the horizontal lines $L_{2k+1} = \{ x + i(2k + 1)\pi : x \in \mathbb{R} \}$ for every integer $k$. Further, there exists a real number $\delta \in (0, \frac{\pi}{2})$ depending upon $\lambda$ such that the strip $S_{2k+1} = \{ z \in \mathbb{C} : |\Im(z) - (2k + 1)\pi| < \delta, \Re(z) \geq M_\lambda \}$ is contained in the Fatou set.

Proof. 1. For every $f_\lambda \in \mathcal{M}$, the point $z = 0$ is always either in the basin of attraction or in the parabolic domain by Proposition 6. Since $z = 0$ is in the Fatou set of $f_\lambda$, there exists a disk $D_r(0) = \{ z \in \mathbb{C} : |z| < r \}$ for some $r > 0$ such that $D_r(0) \subset \mathcal{F}(f_\lambda)$.

Since $e^z$ maps the left half-plane $\{ z \in \mathbb{C} : \Re(z) < a \}$ where $a \in \mathbb{R}$ onto a punctured disk $D^*(0) = \{ z \in \mathbb{C} : 0 < |z| < e^a \}$ and $\tanh(0) = 0$, we can find a real number $M_\lambda$ depending on $\lambda$ such that the left half-plane $H_\lambda = \{ z \in \mathbb{C} : \Re(z) < M_\lambda \}$ is mapped inside the open ball $D_r(0) \subset \mathcal{F}(f_\lambda)$ by the map $w = f_\lambda(z)$. Therefore, the Fatou set of $f_\lambda$ contains the left half-plane $H_\lambda = \{ z \in \mathbb{C} : \Re(z) < M_\lambda \}$.

2. The function $e^z$ maps the horizontal lines $L_{2k+1} = \{ x + i(2k + 1)\pi : x \in \mathbb{R} \}$ where $k \in \mathbb{Z}$, onto the negative real axis $\{ x \in \mathbb{R} : x < 0 \}$. The function $\lambda \tanh(x)$ maps the negative real axis into a subset of the real axis. By Proposition 6, if $\lambda > \lambda^*$, the real line $\mathbb{R}$ is contained in the Fatou set of $f_\lambda$. Therefore, it follows that the horizontal lines $L_{2k+1} = \{ x+i(2k+1)\pi : x \in \mathbb{R} \}$ where $k \in \mathbb{Z}$, are in the Fatou set of $f_\lambda$ for $\lambda > \lambda^*$.

If $\lambda \leq \lambda^* < 0$, the function $\lambda \tanh(x)$ maps the negative real axis into a subset of the positive real axis. By Proposition 6, if $\lambda \leq \lambda^*$, the positive
real axis is contained in the Fatou set of \( f_\lambda \). This gives that the horizontal lines \( L_{2k+1} = \{ x + i(2k + 1)\pi : x \in \mathbb{R} \} \) where \( k \in \mathbb{Z} \), are in the Fatou set of \( f_\lambda \) for \( \lambda \leq \lambda^* \).

It is already shown that \(-\lambda\) lies in the Fatou set of \( f_\lambda \). So, there exists a disc \( D_r(-\lambda) \) with center at \(-\lambda\) and radius \( r \) such that \( D_r(-\lambda) \) is a subset of the Fatou set. One can find \( a \lambda < 0 \) depending on \( \lambda \) so that \( \lambda \tanh z \) maps the half-plane \( \tilde{H} = \{ z : \Re(z) < \tilde{M}_\lambda \} \) into \( D_r(-\lambda) \).

Remark 1. For \( \lambda > \lambda^* \), it also follows that the Fatou set of \( f_\lambda \) contains the horizontal lines \( L_{2k} = \{ x + i 2k\pi : x \in \mathbb{R} \} \) where \( k \in \mathbb{Z} \) by the same arguments used in proving the second part of the above proposition.

A maximally connected subset of the Julia set is called a component of the Julia set. We prove in the following proposition that the Julia set of \( f_\lambda \) for \( \lambda > \lambda^* \) can not contain an unbounded component.
Proposition 8. Let $f_\lambda \in M$. If $\lambda > \lambda^*$ then every component of $\mathcal{J}(f_\lambda) \cap \mathbb{C}$ is bounded.

Proof. Let, on the contrary, $\gamma$ be an unbounded component of $\mathcal{J}(f_\lambda) \cap \mathbb{C}$. Then a sequence $t_n$ can be found on $\gamma$ such that $\lim_{n \to \infty} t_n = \infty$. It follows from Proposition 7 and Remark 1 that $\gamma$ lies in a horizontal strip bounded by $L_k$ and $L_{k+1}$ for some $k \in \mathbb{Z}$ and the set $\{\Re(t_n) : n \in \mathbb{N}\}$ is unbounded. Now, observe that the image curve $\gamma_1 = e^{\gamma}$ of $\gamma$ is unbounded under the mapping $w = e^z$.

If $\gamma_1$ intersects $L_k$ for some $k \in \mathbb{Z}$ then the map $\lambda \tanh(z)$ takes each such intersecting point to a real number which is in the Fatou set. In this way, there is a point common to $\gamma$ and the Fatou set of $f_\lambda$ which is a contradiction. Therefore, $\gamma_1$ lies in some horizontal strip bounded by two consecutive $L_k$’s. Since $\gamma_1$ is unbounded, there exists a sequence $s_n$ on $\gamma_1$ such that $\lim_{n \to \infty} \Re(s_n) = \infty$ or $\lim_{n \to \infty} \Re(s_n) = -\infty$. But in both the cases, $\lambda \tanh(s_n)$ tends to an asymptotic value of $f_\lambda$ as $n \to \infty$. Since all the three asymptotic values lie in the Fatou set, there is a sequence $\{z_n\}_{n>0}$ on $\gamma$ such that $e^{z_n} = s_n$ and $f_\lambda(z_n)$ is a subset of the Fatou set for sufficiently large $n$. By the complete invariance of the Fatou set, there are points $z_n$ on $\gamma$ which are in the Fatou set. It gives a contradiction. Therefore, any component of $\mathcal{J}(f_\lambda) \cap \mathbb{C}$ is bounded. 

In the following, the proof of Theorem 2 is given.

Proof of Theorem 2.

Let $V$ be a component of the Fatou set of $f_\lambda$ different from the immediate basin of attraction $IM(a_\lambda)$ of the attracting fixed point $a_\lambda$. Then, there exists a natural number $k$ such that $f_\lambda^k(V) \subseteq IM(a_\lambda)$. Let $W = f_\lambda^{k-1}(V)$. If $U_1$ and $U_2$ are two Fatou components of a meromorphic function $f$ such that $f : U_1 \to U_2$, then
U_2 \setminus f(U_1) contains at most two points [18]. The two exceptional values ±λ of f_λ lie in IM(a_λ). Therefore, it follows that f_λ(W) = IM(a_λ) \setminus \{λ, −λ\}. Let D_r(λ) be a disk of radius r > 0 with center λ such that D_r(λ) is contained in IM(a_λ). Let U(r) be a component of f_λ^{-1}(D_r(λ)) in W. If r_2 < r_1 < r then there are components U(r_2) of f_λ^{-1}(D_{r_2}(λ)) and U(r_1) of f_λ^{-1}(D_{r_1}(λ)) in U(r) ⊂ W such that U(r_2) ⊂ U(r_1). Note that U(r) is unbounded, since there is only one logarithmic singularity of f_λ^{-1} over λ and for that ∩_{r>0}(U(r)) = ∅ [7]. Thus, there are at least two unbounded components, namely, W and IM(a_λ) of the Fatou set. Consequently, the boundary of any of these two unbounded components is an unbounded component of β(f_λ) ∩ C. But, it is not possible by Proposition 8. Therefore, the Fatou set of f_λ for λ > λ^* contains only one component and hence, the Fatou set is connected.

It is shown in [1] that the connectivity of an invariant Fatou component is either 1, 2 or ∞, 2 being the case for Herman rings. For λ > λ^*, the Fatou set of f_λ is equal to the basin of attraction of the attracting fixed point a_λ and the connectivity of the Fatou set is either 1 or ∞. If the connectivity of the Fatou set is 1, then the Julia set is connected and there is an unbounded component of β(f_λ) ∩ C. But this is impossible by Proposition 8. Therefore, the Fatou set of f_λ for λ > λ^* is infinitely connected.

As a consequence of Proposition 8 and the infinite connectivity of J(f_λ) for λ > λ^*, we make the following remark on the Julia set of f_λ for λ > λ^*.

**Remark 2.** Let w be a pre-pole of f_λ. If it is not a singleton component of the Julia set then there will be a component γ of the Julia set that contains w and f_λ^k(γ) ⊂ β(f_λ) is a component containing the point z = ∞ for some natural number
k. But, it is not possible for $\lambda > \lambda^*$ by Proposition 8. Thus, every pre-pole is a singleton component of the Julia set of $f_\lambda$ for $\lambda > \lambda^*$. Since pre-poles are dense in $\mathcal{J}(f_\lambda)$, we conclude that the singleton components of the Julia set are dense in the Julia set of $f_\lambda$ for $\lambda > \lambda^*$. It can also be concluded from the previous theorem and using Theorem (A) in [16]. It is not known that non-singleton components exist or not in the Julia set of $f_\lambda$ for $\lambda > \lambda^*$.

Let $I_1$ be a component of the Fatou set containing the interval $(-\infty, a_1\lambda)$ when $\lambda < \lambda^*$. When $\lambda = \lambda^*$, let $I_1$ denote the Fatou component containing the interval $(-\infty, x^*)$. Let $I_2$ denote the Fatou component containing $f_\lambda(I_1)$. We use these notations in the following lemma which is required to prove Theorem 3.

**Lemma 2.** Let $f_\lambda \in \mathcal{M}$ with $\lambda \leq \lambda^*$. Let $V$ be a component of the Fatou set $\mathcal{F}(f_\lambda)$ of $f_\lambda$. If $\gamma$ is a Jordan curve in $V$ and the bounded component $B$ of $\hat{\mathbb{C}} \setminus \gamma$ intersects the Julia set then $B$ does not contain any pole of $f_\lambda$.

**Proof.** Since $V$ is a Fatou component, $f_\lambda(V)$ is contained in a Fatou component, say $V_1$. Let $\gamma$ be a Jordan curve in $V$ and the bounded component $B$ of $\hat{\mathbb{C}} \setminus \gamma$ intersects the Julia set.

Suppose that $V_1$ is different from $I_1$. Let us assume, on the contrary that $B$ contains a pole. Then, $B_1 = f_\lambda(B)$ contains $\{z : |z| > M\}$ for some $M > 0$. Since $I_1$ is unbounded, $B_1$ intersects $I_1$. It means that there are points in $B$ whose $f_\lambda$-images belong to $I_1$. Consequently, there is a Fatou component, say $W$ in $B$ such that $f_\lambda(W) \subseteq I_1$. In [18], it has been proved that for any meromorphic function $f : A_1 \to A_2$, the cardinality of the set $A_2 \setminus f(A_1)$ is at most two where $A_1$ and $A_2$ are two Fatou components of $f$. Using this result, it follows that $E = I_1 \setminus f_\lambda(W)$ contains at most two points. Since $\lambda \in I_1$, there exists a neighborhood $N_\lambda$ of the
point \( \lambda \) which is completely contained in \( I_1 \) and \( N_\lambda \cap E = \{ \lambda \} \). Therefore, there is a component of \( f_\lambda^{-1}(N_\lambda) \) in \( W \). As there is only one singularity lying over \(-\lambda\) and it is logarithmic, every component of \( f_\lambda^{-1}(N_\lambda) \) is unbounded \([7]\). Consequently, \( W \) is unbounded and \( B \) is also unbounded which is not true. Therefore, it follows that \( B \) contains no pole of \( f_\lambda \).

Suppose that \( V \) is a Fatou component such that \( f_\lambda(V) \subset I_1 \). Since \( I_2 \) is unbounded, \(-\lambda \in I_2 \) and, there is only one singularity lying over \(-\lambda\) and it is logarithmic, the same arguments given in the previous paragraph with \( I_1 \) replaced by \( I_2 \) are applied to conclude that \( B \) contains no pole of \( f_\lambda \). It completes the proof. \( \square \)

Now, we present the proof of Theorem 3.

**Proof of Theorem 3.**

1. By Theorem 1, it follows that the Fatou set \( \mathcal{F}(f_\lambda) \) for \( \lambda < \lambda^* \) is equal to the basin of attraction or the parabolic domain corresponding to the attracting or the parabolic real 2-periodic cycle \( \{ a_{1\lambda}, a_{2\lambda} \} \) of \( f_\lambda \). Let \( \{ a_{1\lambda}, a_{2\lambda} \} \) be attracting cycle. Let \( IM(a_{1\lambda}) \) be the component of the Fatou set containing the point \( a_{1\lambda} \) and \( IM(a_{2\lambda}) \) be the component of the Fatou set containing the point \( a_{2\lambda} \). Then, \( (-\infty, r_{\lambda}) \subset IM(a_{1\lambda}) \) and \( (r_{\lambda}, \infty) \subset IM(a_{2\lambda}) \). Let \( L_{2k} = \{ x + i \, 2k\pi : x \in \mathbb{R} \} \) where \( k \in \mathbb{Z} \) and \( k \neq 0 \). Then, \( f_\lambda : L_{2k} \to (\lambda, 0) \) is a bijection and it maps \( L_{2k}^- = \{ x + i \, 2k\pi : -\infty < x < r_{\lambda} = f_\lambda^{-1}(r_{\lambda}) \} \) and \( L_{2k}^+ = \{ x + i \, 2k\pi : r_{\lambda} = f_\lambda^{-1}(r_{\lambda}) < x < \infty \} \) to \( (r_{\lambda}, 0) \) and \( (\lambda, r_{\lambda}) \) respectively. It gives that \( L_{2k}^+ \) and \( L_{2k}^- \) lie in two different components of the Fatou set. It is clear that some left half-plane \( H_\lambda \), all horizontal lines \( L_{2k+1} = \{ x + i \, (2k+1)\pi : x \in \mathbb{R} \} \) for \( k \in \mathbb{Z} \) and \( L_{2k}^- = \{ x + i \, 2k\pi : -\infty < x < r_{\lambda} \} \)
are in $IM(a_{1\lambda})$. Further, $L_{2k}^+$ lies in a component, $W_k$ (say) of the Fatou set which is different from $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$. For each non-zero integer $k$, we can find such component $W_k$ which contains the line $L_{2k}^+$ and $W_k \cap W_l = \emptyset$ for $k \neq l$. These components $W_k$’s are pre-periodic but not periodic.

If $\{a_{1\lambda}, a_{2\lambda}\}$ is a parabolic cycle then there will be two different components of Fatou set containing $(-\infty, a_{1\lambda})$ and $(a_{2\lambda}, \infty)$. Considering them as $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$, it can be observed that $IM(a_{1\lambda})$ contains some half-plane $H_\lambda$ and horizontal lines $L_{2k+1}$. Similar arguments as in previous paragraph gives the existence of infinitely many Fatou components $W_k$ containing $L_{2k}^+ = \{x + i 2k\pi : f_\lambda^{-1}(a_{1\lambda}) < x < \infty\}$ for non-zero integer $k$ which are different from $IM(a_{1\lambda})$ and $IM(a_{2\lambda})$. These components are pre-periodic but not periodic.

For $\lambda = \lambda^*$, the proof is similar to the case $\lambda < \lambda^*$.

2. Let $V$ be any component of the Fatou set of $f_\lambda$ for $\lambda \leq \lambda^*$. Suppose that $V$ is not simply connected. Let $\gamma$ be a Jordan curve in $V$ for which the bounded component $U$ of $\hat{\mathbb{C}} \setminus \gamma$ contains at least one component of $\hat{\mathbb{C}} \setminus V$. Set $U_n = f_\lambda^n(U)$ for $n = 0, 1, 2, \ldots$. By Lemma 2, it follows that $U$ does not contain any pole. Since the boundary of $U$ also does not contain any pole, the component $U_1 = f_\lambda(U)$ is a bounded domain. Also, the boundary of $U_1$ is a subset of $f_\lambda(\partial U)$. Since the boundary $\partial U$ of $U$ is the Jordan curve $\gamma$ which is in the Fatou set, the image $f_\lambda(\partial U)$ is in a Fatou component, and hence, $\partial U_1$ is in a Fatou component. If $U_1$ does not contain a pole, the boundary of $U_2$ lies in a Fatou component by repeating the above arguments. As $U_\cap \partial(f_\lambda) \neq \emptyset$, after finite number of steps, we can find a natural number $n_0$ for which $U_{n_0}$ contains a pole which gives a contradiction to Lemma 2. Therefore, it is
concluded that the component $V$ of the Fatou set of $f_\lambda$ for $\lambda \leq \lambda^\ast$ is simply connected.

\[\square\]

**Remark 3.** For $\lambda \leq \lambda^\ast$, all the singular values of $f_\lambda$ are in the immediate basin of attraction or in the petals of the parabolic domain which are not completely invariant.

6. **Measure of $\mathfrak{J}(f_\lambda)$**. In this section, the (Lebesgue) measure of the Julia set of $f_\lambda \in \mathcal{M}$ is computed.

Let $m(A)$ denote the measure of $A \subset \hat{\mathbb{C}}$ and $D_r(z)$ denote the disc of radius $r$ with center at $z$. A subset $E$ of $\hat{\mathbb{C}}$ is said to be thin at $\infty$ if its density is bounded away from 1 in all sufficiently large discs, that is, if there exist positive $R_0$ and $\epsilon$ such that, for all complex $z$ and every disc $D_r(z) = \{w : |w - z| < r\}$, $r > R_0$,

\[
density(E, D_r(z)) = \frac{m(E \cap D_r(z))}{m(D_r(z))} < 1 - \epsilon.
\]

For a given meromorphic function $f$, let

\[P^*_f = \{z : \text{for some } n \in \mathbb{N} \text{ some branch of } f^{-n} \text{ has a singularity at } z\}
\]

and $P_f = P^*_f \setminus \{\infty\}$.

The following proposition is due to Stallard [25].

**Proposition 9.** Let $f$ be a meromorphic function and $d(\overline{P_f}, \mathfrak{J}(f)) > 0$ where $\overline{P_f}$ is the closure of $P_f$ in $\mathbb{C}$. If $E$ is a measurable completely invariant subset of $\mathfrak{J}(f)$ such that $E$ is thin at $\infty$, then $m(E) = 0$. In particular, the Julia set has measure zero if it is thin at $\infty$.

**Theorem 4.** Let $f_\lambda \in \mathcal{M}$. Then, the Julia set of $f_\lambda$ has measure zero.
Fatou set of $\mathbb{R}^J$ it is enough to show that disc or a parabolic domain. It gives that $d(P_{f_\lambda}, J(f_\lambda)) > 0$. In view of Proposition 9, it is enough to show that $J(f_\lambda)$ is thin at $\infty$ in order to show that the measure of $J(f_\lambda)$ is zero.

Let $M \equiv M(\lambda)$ and $\delta \equiv \delta(\lambda)$ be two real numbers such that $H_\lambda = \{z \in \mathbb{C} : \Re(z) < M\}$ and $S_{2k+1} = \{z \in \mathbb{C} : |\Im(z) - (2k + 1)\pi| < \delta, \Re(z) \geq M\}$ are in the Fatou set of $f_\lambda$ which is possible by Proposition 7.

Now, consider the square $S(z, r) = \{w : |\Re(w) - \Re(z)| < \frac{\sqrt{2}}{2}, |\Im(w) - \Im(z)| < \frac{\sqrt{2}}{2}\}$ having its sides parallel to the co-ordinate axes and it is inscribed in the disc $D(z, r)$ with center at $z$ and radius $r$. For a rectangle $R$ having its sides parallel to co-ordinate axes with vertical side length $2\pi$ and horizontal side length $h$, $R \cap \mathcal{F}(f_\lambda) \supset R \cap (\bigcup_{k \in \mathbb{Z}} S_{2k+1})$. It implies that $m(R \cap \mathcal{F}(f_\lambda)) > m(R \cap (\bigcup_{k \in \mathbb{Z}} S_{2k+1})) > 2\delta h$. If $j = [\frac{\sqrt{2}}{2}]$ is the greatest integer not exceeding $\frac{\sqrt{2}}{2}$ then $S(z, r)$ will contain $j$ different rectangles each having its sides parallel to co-ordinate axes with vertical side length $2\pi$ and horizontal side length $r\sqrt{2}$. It gives that $m(\mathcal{F}(f_\lambda) \cap S(z, r)) > j2\delta r\sqrt{2} \geq (\frac{\sqrt{2}}{2} - 1)(2\delta r\sqrt{2}) = \frac{2\delta r^2}{\pi} - 2\delta r\sqrt{2}$. Consequently, $m(\mathcal{F}(f_\lambda) \cap D_r(z)) > \frac{2\delta r^2}{\pi} - 2\delta r\sqrt{2} = 2\delta(\frac{r^2}{\pi} - r\sqrt{2})$ and

**density**$(\mathcal{F}(f_\lambda), D_r(z)) = \frac{m(\mathcal{F}(f_\lambda) \cap D_r(z))}{m(D_r(z))} > \frac{2\delta}{\pi r^2} \left(\frac{r^2}{\pi} - r\sqrt{2}\right)$.

Now, $\text{density}(\mathcal{F}(f_\lambda), D_r(z)) > \frac{2\delta}{\pi} \left(\frac{1}{\pi} - \frac{\sqrt{2}}{r}\right) > \frac{\delta}{\pi}$ for $r > 2\sqrt{2}\pi$.

Letting $\epsilon = \frac{\delta}{\pi}$ and $R_0 = 2\sqrt{2}\pi$, it is concluded that $\text{density}(\mathcal{F}(f_\lambda), D_r(z)) > \epsilon$ for all $z \in \mathbb{C}$ and all $r > R_0$. Since $\text{density}(\mathcal{F}(f_\lambda), D_r(z)) + \text{density}(J(f_\lambda), D_r(z)) = 1$, it follows that $\text{density}(J(f_\lambda), D_r(z)) < 1 - \epsilon$ for all $z \in \mathbb{C}$ and all $r > R_0$.

Therefore, the Julia set of $f_\lambda$ is thin at $\infty$ which completes the proof. \hfill $\square$

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DYNAMICS OF $\lambda \tanh(e^z)$


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