LANGLANDS DECOMPOSITION OF AFFINE KAC-MOODY ALGEBRAS

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Langlands decompositions of affine Kac-Moody algebras have been obtained by the method of direct determination as introduced by Cornwell for Lie algebras. This method is particularly helpful in the case of lower rank algebras. The involutive automorphisms required for such a study are obtained from the Satake diagrams of the corresponding algebras. This has been well illustrated by taking $A^{(1)}_3$ (untwisted) and $A^{(2)}_4$ (twisted) as representative examples.

Key Words: Kac-Moody algebra; Dynkin diagrams; Satake diagrams

1. INTRODUCTION

Now it is beyond doubt that Kac-Moody algebras [1, 2] more particularly the affine versions have wide physical applications in the context of integrable systems [3], two-dimensional field theories and string theories [4] etc. The representation theory of such Kac-Moody algebras runs almost parallel with that of Lie algebras. Already the highest weight representations of these algebras have been discussed in great detail. Cornwell [5] has introduced the method of direct determination of Iwasawa decomposition of Lie algebras, which plays main role in the construction of unitary irreducible representations. We have already applied this technique to obtain Iwasawa decomposition of various types of Kac-Moody algebras and superalgebras [6-8]. The method of direct determination can also be extended very easily to give the Langlands decomposition [9] of all the parabolic subalgebras, which form an essential part in the construction of various unitary irreducible representations. Keeping this in mind, in this communication we have obtained the Langlands decompositions of affine Kac-Moody algebras taking $A^{(1)}_3$ and $A^{(2)}_4$ as illustrative examples. The involutive automorphisms [10] required for such studies have been obtained from their corresponding Satake diagrams [11, 13]. The Satake diagrams are nothing but modified Dynkin diagrams which are used in the
classification of the real forms of Lie algebras as well as the associated symmetric spaces [14]. In recent times, these symmetric spaces have found application in the quantum integrable systems [3] and random matrix models [12] that have been studied in various quantum transport problems.

The organization of the paper is as follows: In section II, we give an introduction to direct determination of Langlands decomposition and briefly outlined the procedures for the construction of Satake diagrams of Kac-Moody algebras. In Section-III, we have applied this method to find out the Langlands decomposition of $A^{(1)}_3$ and $A^{(2)}_4$ respectively. Section-IV contains few concluding remarks.

II(A). LANGLANDS DECOMPOSITION OF AFFINE KAC-MOODY ALGEBRAS

The notion of direct determination of Iwasawa decomposition [6, 8] of Lie algebra is extended to the Langlands decomposition of parabolic subalgebras. Let $g_R$ be a real Kac-Moody algebra generated from its compact real form $g_K$ by an involutive automorphism defined with respect to the Cartan subalgebra $h$ of $g$, which is the complexification of $g_R$.

The following commutation relations are satisfied by the elements of $g$:

$$[h, e_\alpha] = \alpha(h) e_\alpha, h \in h, \alpha \in R$$

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ 0, & \text{otherwise} \end{cases}$$

$$[e_\alpha, e_{-\alpha}] = h_\alpha, h_\alpha \in h.$$  \hspace{1cm} (2.1)

Here $R$ denotes the set of roots of $g$ with respect to $h$ and the Killing form is defined by $B(e_\alpha, e_{-\alpha}) = -1$. Here $\alpha(h) = B(h, h_\alpha)$. The compact real form $g_K$, which we get using Cartan involution is given by $i h_\alpha, \alpha = \alpha_0, \alpha_1, \ldots, \alpha_r$ and $i (e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha})$ for all $\alpha \in R$.

Let $K$ be the maximal compact subalgebra of $g_R$ defined in such a way that $a \in K$ iff $a \in g_R$ and $\sigma a = a$ where $\sigma$ is an involutive automorphism of $g_R$. Let $P$ be the subspace such that $a \in P$ iff $a \in g$ and $\sigma a = -a$.

Thus, $K$ and $P$ are given by

$$K = \{ i h_\alpha, \text{for } \alpha = \alpha_0, \alpha_1, \ldots, \alpha_r \text{ and } (e_\alpha + e_{-\alpha}), \}$$

$$i (e_\alpha - e_{-\alpha}) \text{ for all } \alpha \text{ with } \exp \alpha(h) = +1 \}$$

$$P = \{ i (e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}) \text{ for all } \alpha \text{ with } \exp \alpha(h) = -1 \}$$

(2.2)

Let ‘$A$’ be the maximal abelian subalgebra of $P$ with dimension $m$ and $M$ be the centralizer of $A$ in $K$. Thus, $A$ may be taken to have a basis consisting of the elements of the form $i (e_\alpha + e_{-\alpha})$. Let $R_A$ denote the set of positive roots $\alpha$ appear in this way in $A$. Similarly, $M$ may be taken to have a basis consisting of the elements of the form $(e_\alpha + e_{-\alpha})$, with the set of positive root $\alpha$ appearing this way in $M$, being denoted by $R_M$, together possibly with some elements of $h \cap g$. If $h'' \in h \cap g$
is an element of $\mathcal{M}$, then $\alpha(h'') = 0$ for all $\alpha \in RA \cup RM$. Complexification of $A \oplus \mathcal{M}$ together with the derivation $d'$ gives a Cartan subalgebra $h'$ of $\mathfrak{g}$ with basis $h'_0, h'_1, \ldots h'_r$ and $d'$.

Now there exists an inner automorphism [9] $V : h' \to h$ i.e.

$$h_j = V h'_j, \text{ where } V = \prod_{\alpha} V_{\alpha}, \alpha \in R,$$

$$V_{\alpha} = \exp[ad\{ia_{\alpha}(e_{\alpha} - e_{-\alpha})\}] \quad \text{and} \quad a_{\alpha} = \frac{\pi}{\{8(\alpha, \alpha)\}^{1/2}} \quad (2.3)$$

Let $\Delta^+$ be the set of positive roots, then $h_{\alpha} = \sum_{j=0}^r b_j(\alpha)h_j$. (2.4)

Thus $\alpha \in \Delta^+$ iff $b_j(\alpha) > 0$ where $j$ is the least index such that $b_j(\alpha) \neq 0$. The positive roots can be again divided into the following classes:

(i) $\Delta^+ = \{\alpha | \alpha \in \Delta^+, \alpha(h) \neq \alpha(V_{\sigma}V^{-1}h) \text{ for all } h \in h\}$, (2.5)

(ii) $\Delta^- = \{\alpha | \alpha \in \Delta^+, \alpha(h) = \alpha(V_{\sigma}V^{-1}h) \text{ for all } h \in h\}$ (2.6)

Let the subalgebra $\tilde{N}$ be spanned by the elements $V^{-1}e_{\alpha}, \alpha \in \Delta^+$ and $N = \tilde{N} \cap \mathfrak{g}$, where $\tilde{N}$ and $N$ are the nilpotent subalgebras of $\mathfrak{g}_R$ and $\mathfrak{g}$ respectively. Thus the Iwasawa decomposition of $\mathfrak{g}_R$ is given by

$$\mathfrak{g}_R = K \oplus \mathcal{A} \oplus N, \quad (2.7)$$

where $\oplus$ indicates the direct sum of vector spaces rather than a sum of mutually commuting Lie subalgebras.

Now, a minimal parabolic subalgebra of $\mathfrak{g}_R$ is defined to be any subalgebra that is conjugate to

$$P_I = M \oplus \mathcal{A} \oplus N. \quad (2.8)$$

A general parabolic subalgebra of $\mathfrak{g}_R$ may be defined to be any subalgebra of $\mathfrak{g}_R$ that contains a minimal parabolic subalgebra of $\mathfrak{g}_R$. There exist $2^m$ conjugacy classes of parabolic subalgebras of $\mathfrak{g}_R$ and in each such class there is a standard parabolic subalgebra $P_{\theta}$, which can be obtained in the following way:

Let $\Sigma$ be the set of roots $\lambda$ for $A$ and $\Psi$ be the set of positive roots in $\Sigma$ where $\psi = \{\lambda_1, \lambda_2, \ldots \}$. Let $\theta$ denote the subset of $\Psi$ and $< \theta >$ the set of roots in $\Sigma$ which arises as a linear combination of roots in $\theta$. Define $< \theta >_{+} = \Sigma_{+} \cap < \theta >$, where $\Sigma_{+}$ and $\Sigma_{-}$ denote the positive and negative roots in $\Sigma$. Let $A_{+}(\theta), A_{-}(\theta)$ and $N(\theta)$ denote the subspace of $A$ corresponding to $< \theta >_{+}, < \theta >_{-}$ and $\Sigma_{+} - < \theta >_{\pm}$ respectively.

Now define

$$A_{\theta} = \{ h \in A | \lambda(h) = 0 \text{ for all } \lambda \in \theta \}. \quad (2.9)$$
Now for each \( \lambda \in \theta \) construct \( Q_\lambda \) such that \( Q_\lambda \in VA \) and is a linear combination of all the elements \( h_\alpha \in h \) for which the restriction of \( \alpha \) to \( VA \) is \( \lambda \).

Again, let \( A(\theta) \) be the orthogonal component of \( A_\theta \) in \( A \) with respect to the Cartan Killing form, then the Langlands decomposition of parabolic subalgebra of \( g_R \) is

\[
P_\theta = M_\theta \oplus A_\theta \oplus N_\theta
\]

where
\[
M_\theta = M \oplus N_+(\theta) \oplus N_-(\theta) \oplus A(\theta).
\]

(2.10)

A real Cartan subalgebra \( h'_r \) is said to be \( \sigma \) invariant if

\[
h'_r = (h'_r \cap K) \oplus (h'_r \cap P).
\]

(2.11)

A parabolic subalgebra \( P_\theta \) is said to be cuspidal, if there exists an \( \sigma \)-invariant real Cartan subalgebra \( h'_r \) such that

\[
A_\theta = h'_r \cap P.
\]

(2.12)

This shows that the minimal parabolic algebra is cuspidal.

II(B). SATAKE DIAGRAMS OF AFFINE KAC-MOODY ALGEBRAS

Each Kac-Moody algebra \( g \) determines \( g_R \) (real form of \( g \)) where \( g \) is the complexification of \( g_R \) i.e. \( g = g_R \oplus ig_R \) (direct sum). Such a real form \( g_R \) determines a mapping \( C : g \to g_r \). The mapping \( C \) has the following properties:

(i) \([CX, CY] = C[X, Y]\) for \( X, Y \in g \).

(ii) \( C \) is an involution i.e. \( C^2 = I_g \).

(iii) \( C \) is semilinear, i.e. \( C(\pi X + \mu Y) = \bar{\pi}C(X) + \bar{\mu}C(Y) \) for \( X, Y \in g \) and \( \mu, \pi \in C \).

(2.14)

A bijection \( C : g \to g \) with the above properties is called conjugation of \( g \). Conversely any conjugation of \( g \) determines uniquely a real algebra, \( g_R = \{ X \in g : CX = X \} \) such that \( g = g_R + ig_R \). Hence we have a canonical one to one correspondence between conjugation of \( g \) and real forms of \( g \). Let \( C \) be the conjugation of \( g \) defined by \( g_R \), so that \( C(X + iY) = (X - iY) \) for \( X, Y \in g_R \). \( C \) acts on the root system as follows:

For each root \( \alpha \in R \), we define \( \sigma(\alpha) \) such that

\[
\sigma(\alpha(h)) = \bar{\alpha}(C(h)), \ h \in h.
\]

(2.15)

Then we have,

\[
C(g_\alpha) = g_{-\sigma(\alpha)}.
\]

(2.16)

The mapping \( \alpha \to -\sigma(\alpha) \) extends by linearity to an involutary isometry under which \( R - R_0 \) is stable and \( R_0 \) is the set of roots \( \alpha \in R \), such that \( \sigma(\alpha) = \alpha \). We have \( \alpha + \sigma(\alpha) \notin R \) for all
\( \alpha \in R \). Therefore, we are led to consider pairs \((R, \sigma)\), where \(R\) is a restricted root system and \(\sigma\) is an involutary isometry such that \(\sigma(R) = R\). Each Kac-Moody algebra \(g\), therefore determines a normal pair \((R, \sigma)\), which determines \(g_R\).

The construction of Satake diagrams associated with real Kac-Moody algebras from the Dynkin diagram of the corresponding complex Kac-Moody algebras proceeds as follows:

Let \(R\) be the root system of affine Kac-Moody algebra. For \(\alpha \in R\), let \(\bar{\alpha} = \alpha - \sigma(\alpha)\), where \(\sigma\) is the involutive automorphism of \(R\). Let us introduce \(R^{-} = \{\bar{\alpha} \mid \bar{\alpha} \neq 0, \alpha \in R\}\). Further let \(B_-(\text{resp. } B)\) denote the basis of \(R_-\) (resp. \(R\)) and \(B_0\) be a basis of \(R_0\), then \(B_0 = B \cap R_0\) is a basis of \(R_0\). Let \(B_+ = B/B_0 = \{\alpha_i\}\) and \(B_0 = \{\beta_i\}\), then

\[
-\sigma(\alpha_i) = \alpha_{\pi(i)} + \sum \eta_{il} \beta_l
\]

(2.17)

where \(\pi\) is the involutive permutation of \(\{0, 1, 2 \ldots r\}\) and \(\eta_{il}\) are non-negative integers.

We can now associate with \(B\) its Satake diagrams as follows:

In the Dynkin diagrams of \(B\), denote the roots \(\alpha_i\) by white dot \(\circ\) as usual and the roots \(\beta_l\) by black dot \(\blacklozenge\). If \(\pi(i) = k\), indicate this by \(\circ \quad \blacklozenge\). The Satake diagrams determine the involution \(\sigma\) of \(R\) uniquely. We note that \(\sigma(\beta_l) = \beta_l\) and if \(\alpha \in R\) then \(\alpha + \sigma(\alpha) \notin R\).

In finite dimensional cases, the Satake diagrams of simple Lie algebras determines the real forms of these algebras uniquely up to isomorphism and also determine the associated symmetric space. In a similar manner we can also construct Satake diagrams from the Dynkin diagrams of all affine Kac-Moody algebras, which we hope, will provide one way of classification of real forms of these algebras and will also determine the associated symmetric space (if it exists for an infinite setting). Recently symmetric spaces have got wide application in quantum transport problems, quantum integrable systems and random matrix models etc. Thus we hope these new types of symmetric spaces may play important role in such type of studies in future. The same method can also be applied to Lie superalgebra case. In an earlier paper Satake diagrams (super) has been successfully used to determine the real forms of simple Lie superalgebras [16], which have one to one correspondence with real forms determined by Parker [15]. The same method can be used in the case of Kac-Moody superalgebra.

III(A). LanguLands Decomposition of \(A_3^{(1)}\)

The Cartan matrix of is \(A_3^{(1)}\), the four simple roots of \(A_3^{(1)}\) are \(\alpha_0, \alpha_1, \alpha_2\) and \(\alpha_3\). The possible Satake diagrams of \(A_3^{(1)}\) along with their root automorphisms are depicted in Table I.
Let us consider the involutive automorphism of $A_3^{(1)}$ determined by any one of the Satake diagrams, say (vi) of table I.

The simple root automorphisms are given by

$$
\begin{align*}
\sigma(\alpha_1) &= \alpha_1, -\sigma(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3, \\
\sigma(\alpha_3) &= \alpha_3, -\sigma(\alpha_0) = \alpha_0 + \alpha_1 + \alpha_3.
\end{align*}
$$

(3.1)
So, we have
\[ \sigma(\alpha_2) = \alpha_3, -\sigma(\delta - \alpha_1 - \alpha_2 - \alpha_3) = \delta - \alpha_2 \]
(3.2)

where
\[ \alpha_0 = \delta - (\alpha_1 + \alpha_2 + \alpha_3). \]

So, the positive roots are given by
\[ \Delta = \{ \alpha_1, \alpha_2, \alpha_3, \pm \alpha_1 + n\delta, \pm \alpha_2 + n\delta, \pm \alpha_3 + n\delta, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \}
\[ \pm (\alpha_1 + \alpha_2) + n\delta, \pm (\alpha_2 + \alpha_3) + n\delta, \pm (\alpha_1 + \alpha_2 + \alpha_3) + n\delta, n\delta, n \in Z^+ \}. \]
(3.3)

We can apply simple root automorphisms to find out the automorphism of other roots and we see that the positive roots can be separated into two categories i.e.
\[ \exp \alpha(h) = +1 \;
\]
for
\[ \alpha = \begin{cases} 
\alpha_1, \alpha_2, \alpha_3, \pm \alpha_1 + n\delta, \pm \alpha_2 + n\delta, \pm \alpha_3 + n\delta, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \\
\pm (\alpha_1 + \alpha_2) + n\delta, \end{cases} \]
(3.4)

\[ \exp \alpha(h) = -1 \;
\]
for
\[ \alpha = \begin{cases} 
\alpha_2, -\alpha_1 + n\delta, \alpha_3, +n\delta, \alpha_2 + \alpha_3, \pm (\alpha_2 + \alpha_3) + n\delta, \\
\pm (\alpha_1 + \alpha_2 + \alpha_3) + n\delta, n\delta \end{cases}. \]
(3.5)

Thus for \( A_3^{(1)} \), \( K \) and \( P \) are given by
\[ K = \{ i\hbar_\alpha, \text{for } \alpha = \alpha_0, \alpha_1, \alpha_2, \alpha_3 \text{ and } (e_\alpha + e_-\alpha), i(e_\alpha - e_-\alpha) \text{ for } \alpha \text{ given by eqn (3.4)} \} \]
(3.6)

\[ P = \{ i(e_\alpha + e_-\alpha), (e_\alpha - e_-\alpha) \text{ for } \alpha \text{ given by eqn (3.5)} \}. \]
(3.7)

We now select a maximal abelian subalgebra, \( \mathcal{A}' \) which is two-dimensional and may be chosen to have basis elements
\[ h_0' = i\{ e_{\alpha_3} + e_-\alpha_2 \}, h_1' = i\{ e_{\alpha_1 + \alpha_2 + \alpha_3 + n\delta} + e_-\alpha_1 - e_{\alpha_2 + \alpha_3 + n\delta} \}. \]
(3.8)

So, we have \( R_A = \{ \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 + n\delta \}, R_M \) is empty. \( \mathcal{M}' \) is two-dimensional and its basis elements are given by
\[ -i\hbar_2' = h_{(m+n)\delta}, = i\hbar_3' = i(h_{\alpha_1} - h_{\alpha_3}) \]
(3.9)

Note that \( h_0', h_1', h_2', h_3' \) together with a scaling element \( d' \) are the elements of Cartan subalgebra \( h' \). The inner automorphism of \( A_3^{(1)} \) is given by
\[ V = \prod_{\alpha} V_\alpha \text{ for all } \alpha \in R_A \cup R_M, \]
(3.10)

\[ V = V_{\alpha_2} V_{\alpha_1 + \alpha_2 + \alpha_3 + m\delta} \]
\[ = \exp\{ ad[i a_{\alpha_2} (e_{\alpha_2} - e_-\alpha_2)] \} \exp\{ ad[i a_{\alpha_1 + \alpha_2 + \alpha_3 + m\delta} (e_{\alpha_1 + \alpha_2 + \alpha_3 + m\delta} - e_-\alpha_1 - e_{\alpha_2 + \alpha_3 + m\delta})] \} \]
(3.11)
\[ a_{\alpha_2} = \frac{\pi}{[t^n 8(\alpha_2, \alpha_2)]^{(1/2)}} \]

\[ a_{\alpha_1 + \alpha_2 + \alpha_3 + m\delta} = \frac{\pi}{[t^n 8(\alpha_1 + \alpha_2 + \alpha_3 + m\delta, \alpha_1 + \alpha_2 + \alpha_3 + m\delta)]^{(1/2)}} \]

where \( t^n \) represents the complex variable.

Applying this to Cartan subalgebra of \( h' \), we get

\[ h_0 = -h_{\alpha_2} \]
\[ h_1 = -(h_{\alpha_1} + h_{\alpha_2} + h_{\alpha_3} + h_{(m+n)\delta}) \]
\[ h_2 = -h_{(m+n)\delta} \]
\[ h_3 = -(h_{\alpha_1} - h_{\alpha_3}) \]

With respect to this Cartan subalgebra, the set of positive roots is given by

\[ \Delta^+ = \left\{ \alpha_1 - \alpha_2, \alpha_3, \alpha_1 + m\delta, -\alpha_2 \pm m\delta, \alpha_1 \pm m\delta, -(\alpha_1 + \alpha_2), -(\alpha_2 + \alpha_3), -(\alpha_1 + \alpha_2 + \alpha_3), -(\alpha_1 + \alpha_2 + \alpha_3 + m\delta), m\delta, m \in \mathbb{Z}^+ \right\} \]

(3.15)

Now this can be divided into two categories \( \Delta^+_+ \) and \( \Delta^+_\), where

\[ \Delta^+_+ = \{ -m\delta \} \]

(3.16)

and

\[ \Delta^+_\ = \Delta^+ / \Delta^- \]

(3.17)

Now choose the fundamental root system,

\[ \Psi = \{ \lambda_1, \lambda_2 \} \]

(3.18)

where

\[ \lambda_1 = \delta - (\alpha_1 + \alpha_2 + \alpha_3), \alpha_1 \]

(3.19)

\[ \lambda_2 = \alpha_2, \alpha_3. \]

So that using equation (3.15), we write

\[ \Sigma^+ = \{ \lambda_1, \lambda_2, (-1 \pm 2m)\lambda_1 + (-1 \pm 2m)\lambda_2, (-2 \pm 2m)\lambda_2 \pm 2m\lambda_1, (-1 \pm 2m)\lambda_1 + (-2 \pm 2m)\lambda_2 \}. \]

(3.20)

It follows that \( \lambda_1 \) is the restriction of \( \alpha_0, \alpha_1 \) to \( V_A \), whereas \( \lambda_2 \) is the restriction of \( \alpha_2, \alpha_3 \).

Now since \( \dim A = m = 2 \), so there are four standard parabolic sub-algebras, namely the minimal parabolic sub-algebra, \( g \) itself and two others which will now be determined:
Case (I) — Choose
\[ \theta = \{ \lambda_2 \}. \]  
(3.21)

Then \( V_A\theta \) has generator \( \lambda_\alpha \) with
\[ \alpha = \alpha_1 + \alpha_2 + \alpha_3 + n\delta. \]  
(3.22)

So that \( A_\theta \) has generator \( i(e_\alpha + e_{-\alpha}) \), where
\[ \alpha = \alpha_1 + \alpha_2 + \alpha_3 + n\delta. \]  
(3.23)

As \( Q_{\lambda_2} = h_{\lambda_2} = h_{\alpha_2} \), so \( A(\theta) \) has generator \( i(e_{\alpha_2} + e_{-\alpha_2}) \).  
(3.24)

Now, as \( \langle \theta \rangle = \{ \lambda_2 \} \). So \( \langle \theta \rangle_+ = -\Sigma_+ \cap \langle \theta \rangle = \{ \lambda_2 \} \) and \( \tilde{N}_-(\theta) \) is generated by the element
\[ V^{-1}e_{\alpha_2} = \frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}). \]  
(3.25)

and as \( \langle \theta \rangle_+ = \Sigma_+ \cap \langle \theta \rangle = \{-\lambda_2\} \), So \( \tilde{N}_+(\theta) \) is generated by the element
\[ V^{-1}e_{\alpha_2} = -\frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}). \]  
(3.26)

and \( \tilde{N}(\theta) \) is generated by elements
\[ V^{-1}e_{\alpha_1 \pm (m+n)\delta} = \pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) A e_{-(\alpha_2 + \alpha_3) \mp (n \pm (m+n)\delta)} + i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) e_{\alpha_1 \pm (m+n)\delta} \]
\[ \pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) B e_{\alpha_3 \mp m\delta} - i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) C e_{\alpha_1 + \alpha_2 \pm (m+n)\delta}, \]  
(3.27)

where
\[ A = \text{Sgn}(N_{\alpha_1 + \alpha_2 + \alpha_3 - \alpha_2 - \alpha_3}), \]
\[ B = \text{Sgn}(N_{\alpha_1, \alpha_2} N_{\alpha_1 + \alpha_2 + \alpha_3, - \alpha_3}), \]
\[ C = \text{Sgn}(N_{\alpha_1, \alpha_2}). \]  
(3.28)

\[ V^{-1}e_{-\alpha_3 \pm (m+n)\delta} = \pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) D e_{-\alpha_1 - \alpha_2 \mp (n \pm (m+n))\delta} + i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) e_{\alpha_3 \pm (m+n)\delta} \]
\[ \pm \left( \frac{1}{1 + t^{2n}} \right) E e_{-\alpha_1 \mp (n \pm (m+n))\delta} - i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) F e_{\alpha_2 + \alpha_3 \pm (m+n)\delta}. \]  
(3.29)

where
\[ D = \text{Sgn}(N_{\alpha_1 + \alpha_2 + \alpha_3, - \alpha_1 - \alpha_2}), \]
\[ E = \text{Sgn}(N_{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 N_{\alpha_3, \alpha_2}}), \]
\[ F = \text{Sgn}(N_{\alpha_3, \alpha_2}). \]  
(3.30)
\[ V^{-1} e^{-(\alpha_1 + \alpha_2) \pm (m+n)\delta} = \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) e^{-(\alpha_1 + \alpha_2) \pm ((m+n)\delta} \]
\[
\pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) G e^{\alpha_3 \pm (n\pm(m+n))\delta} \\
\pm i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) H e^{-\alpha_1 \pm (m+n)\delta} \\
\pm i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) I e^{\alpha_2 + \alpha_3 \pm (n\pm(m+n))\delta}. \quad (3.31)\]

where
\[ G = Sgn\left(N_{\alpha_1 + \alpha_2 + \alpha_3, -(\alpha_1 + \alpha_2)}\right), \]
\[ H = Sgn\left(N_{\alpha_2, -(\alpha_1 + \alpha_2)}\right), \]
and
\[ I = Sgn\left(N_{\alpha_2, -(\alpha_1 + \alpha_2)}N_{\alpha_1 + \alpha_2 + \alpha_3, -\alpha_1}\right). \quad (3.32)\]

\[ V^{-1} e^{-(\alpha_2 + \alpha_3) \pm (m+n)\delta} = \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) e^{-(\alpha_2 + \alpha_3) \pm (m+n)\delta} \]
\[
\pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) J e^{\alpha_1 \pm (n\pm(m+n))\delta} \\
\pm i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) K e^{-\alpha_2 \pm (m+n)\delta} \\
\pm i \frac{1}{2^{1/2}} \left( \frac{1}{1 + t^{2n}} \right) L e^{(\alpha_2 + \alpha_1) \pm (n\pm(m+n))\delta}. \quad (3.33)\]

where
\[ J = Sgn\left(N_{\alpha_1 + \alpha_2 + \alpha_3, -(\alpha_2 + \alpha_3)}\right), \]
\[ K = Sgn\left(N_{\alpha_2, -(\alpha_2 + \alpha_3)}\right), \]
and
\[ L = Sgn\left(N_{\alpha_2, -(\alpha_2 + \alpha_3)}N_{\alpha_1 + \alpha_2 + \alpha_3, -\alpha_1}\right). \quad (3.34)\]

Consequently the basis elements of \( N(\theta) \) are given by

\[ V^{-1} e^{-(\alpha_1 + \alpha_2 + \alpha_3) \pm (m+n)\delta} = -\frac{i}{2} \left( h_{(\alpha_1 + \alpha_2 + \alpha_3) \pm m\delta} \right) \\
-\frac{1}{2} \left( e^{(\alpha_1 + \alpha_2 + \alpha_3) \pm (m+n)\delta} - e^{-(\alpha_1 + \alpha_2 + \alpha_3) \pm (m+n)\delta} \right). \quad (3.35)\]
\[
\frac{1}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) A \left( e_{-(\alpha_2 + \alpha_3) \pm (n \pm (m+n))} - e_{(\alpha_2 + \alpha_3) \pm (n \pm (m+n))} \right) \\
\pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) \left( e_{\alpha_1 \pm (m+n)} - e_{\alpha_1 \pm (m+n)} \right) \\
\pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) B \left( e_{-\alpha_3 \pm m \delta} - e_{\alpha_3 \pm m \delta} \right) \\
- \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) C \left( e_{\alpha_1 + \alpha_2 \pm (m+n)} - e_{-\alpha_1 - \alpha_2 \pm (m+n)} \right), \\
\pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) D \left( e_{-\alpha_1 - \alpha_2 \pm (n \pm (m+n))} - e_{\alpha_1 + \alpha_2 \pm (n \pm (m+n))} \right) \\
\pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) \left( e_{\alpha_3 \pm (m+n)} - e_{-\alpha_3 \pm (m+n)} \right) \\
\pm \left( \frac{1}{1 + \ell^{2n}} \right) E \left( e_{-\alpha_1 \pm (n \pm (m+n))} - e_{\alpha_1 \pm (n \pm (m+n))} \right) \\
- \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) F \left( e_{\alpha_2 + \alpha_3 \pm (m+n)} - e_{-\alpha_2 - \alpha_3 \pm (m+n)} \right), \\
\frac{1}{2^{1/2}} \left( \frac{1}{2 + \ell^{2n}} \right) \left( e_{-\alpha_1 + \alpha_2 \pm (m+n)} - e_{\alpha_1 + \alpha_2 \pm (m+n)} \right) \\
\pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) G \left( e_{\alpha_3 \pm (n \pm (m+n))} - e_{-\alpha_3 \pm (n \pm (m+n))} \right) \\
\pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) H \left( e_{-\alpha_1 \pm (m+n)} - e_{\alpha_1 \pm (m+n)} \right) \\
\pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) I \left( e_{\alpha_2 + \alpha_3 \pm (n \pm (m+n))} - e_{-\alpha_2 - \alpha_3 \pm (n \pm (m+n))} \right), \\
\frac{1}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) \left( e_{-\alpha_2 + \alpha_3 \pm (m+n)} - e_{\alpha_2 + \alpha_3 \pm (m+n)} \right) \\
\pm \frac{1}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) J \left( e_{\alpha_1 \pm (n \pm (m+n))} - e_{-\alpha_1 \pm (n \pm (m+n))} \right) \\
\pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) K \left( e_{-\alpha_3 \pm (m+n)} - e_{\alpha_3 \pm (m+n)} \right) \\
\pm \frac{i}{2^{1/2}} \left( \frac{1}{1 + \ell^{2n}} \right) L \left( e_{\alpha_2 + \alpha_1 \pm (n \pm (m+n))} - e_{-\alpha_2 + \alpha_1 \pm (n \pm (m+n))} \right),
\]
\[-\frac{i}{2}(h_{(\alpha_1+\alpha_2+\alpha_3)^\pm m\delta}) - \frac{1}{2}(e_{(\alpha_1+\alpha_2+\alpha_3)^\pm (m+n)\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)^\pm (m+n)\delta}). \quad (3.36)\]

and the basis elements of $M_\theta$ may be taken to be
\[
\frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}),
\]

\[
\frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - i(e_{\alpha_2} - e_{-\alpha_2}), i(e_{\alpha_2} + e_{-\alpha_2}), i(h_{\alpha_1} - h_{\alpha_2}). \quad (3.37)
\]

Clearly $i\hbar_{\alpha_2}, i(e_{(\alpha_1+\alpha_2+\alpha_3)^\pm n\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)^\pm n\delta}), i(h_{\alpha_1} - h_{\alpha_2})$ are the generators of a real $Z$-invariant Cartan subalgebra $h'_c$ for which $h'_c \cap P = A_\theta$. So the parabolic subalgebra is cuspidal and $P_\theta = M_\theta \oplus A_\theta \oplus N_\theta$.

**Case (II)** — Here, choose

\[\theta = \{\lambda_1\} \quad (3.38)\]

In this case, $V A_\theta$ has generator

\[(h_{\alpha_2} + h_{\alpha_1+\alpha_2+\alpha_3+n\delta}). \quad (3.39)\]

So that $A_\theta$ has generator

\[i(e_{\alpha_2} + e_{-\alpha_2}) + i(e_{\alpha_1+\alpha_2+\alpha_3+n\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)^\pm n\delta}). \quad (3.40)\]

As $Q_{\lambda_1}$ is a linear combination of $\alpha_2$ and $\alpha_1 + \alpha_2 + \alpha_3 + n\delta$, so $A(\theta)$ has generator

\[\{i(e_{\alpha_1+\alpha_2+\alpha_3+n\delta} - e_{-(\alpha_1+\alpha_2+\alpha_3)^\pm n\delta}) - i(e_{\alpha_2} + e_{-\alpha_2})\}. \quad (3.41)\]

Moreover as, $\langle \theta \rangle_- = \{\lambda_1\}$, so elements of $\tilde{N}_-(\theta)$ is generated by

\[V^{-1} e_{\alpha_1} = \frac{1}{2} e_{\alpha_1} - \frac{i}{2} A_1 e_{-\alpha_2-\alpha_3-n\delta} - \frac{i}{2} B_1 e_{\alpha_1+\alpha_2} - \frac{1}{2} C_1 e_{-\alpha_3-n\delta} \quad (3.42)\]

where

\[A_1 = Sgn(N_{\alpha_1+\alpha_2+\alpha_3, -(\alpha_2+\alpha_3)}),\]

\[B_1 = Sgn(N_{\alpha_1, \alpha_2}),\]

and

\[C_1 = Sgn(N_{\alpha_1, \alpha_2 N_{-\alpha_3, (\alpha_1+\alpha_2+\alpha_3)}}, \quad (3.43)\]

and as $\langle \theta \rangle_+ = \{-\lambda_1\}$, so elements of $\tilde{N}_+(\theta)$ is generated by

\[V^{-1} e_{-\alpha_1} = \frac{1}{2} e_{-\alpha_1} - \frac{i}{2} A_2 e_{\alpha_2+\alpha_3+n\delta} - \frac{i}{2} B_2 e_{-\alpha_1-\alpha_2} - \frac{1}{2} C_2 e_{\alpha_3+n\delta} \quad (3.44)\]

where

\[A_2 = Sgn(N_{\alpha_1+\alpha_2+\alpha_3, -\alpha_1}),\]

\[B_2 = Sgn(N_{-\alpha_1, \alpha_1-\alpha_2}),\]

and

\[C_2 = Sgn(N_{-\alpha_1, -\alpha_2 N_{\alpha_1+\alpha_2+\alpha_3}, \alpha_1-\alpha_2}). \quad (3.45)\]
and elements of $\tilde{N}(\theta)$ are given by

$$V^{-1}e^{-\alpha_2 \pm (m+n)\delta} = -\frac{1}{2}(e^{-\alpha_2 \pm (m+n)\delta}$$

$$-e_{\alpha_2 \pm (m+n)\delta} - i(e_{\alpha_2 \pm (m+n)\delta} - e^{-\alpha_2 \pm (m+n)\delta})$$

and $V^{-1}e^{-\alpha_3 \pm (m+n)\delta}$,

$$V^{-1}e^{-\alpha_1 \pm (m+n)\delta}, V^{-1}e^{-\alpha_2 + \alpha_3 \pm (m+n)\delta}$$

and $V^{-1}e^{-\alpha_1 + \alpha_2 + \alpha_3 \pm (m+n)\delta}$, all of which are mentioned in Case I. (3.46)

The elements of $N(\theta)$ can be known by considering the elements $\tilde{N} \cap g$ and basis elements of $M_\theta$ may be chosen in a similar way as mentioned in case-I.

As there does not exist two mutually commuting linearly independent generators in $M_\theta \cap K$, so this parabolic subalgebra is not cuspidal and $P_\theta = M_\theta \oplus A_\theta \oplus N_\theta$. (3.47)

III(B). Langlands Decomposition of $A_4^{(2)}$

The Cartan matrix of $A_4^{(2)}$ is

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$ The three simple roots are $\alpha_0, \alpha_1$ and $\alpha_2$. The possible Satake diagrams of $A_4^{(2)}$ along with their root automorphism are given in table II.

Table II: Satake diagram and Involutive automorphism of $A_4^{(2)}$.

<table>
<thead>
<tr>
<th>Dynkin diagram</th>
<th>Satake diagrams</th>
<th>Involutive automorphism</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0 \alpha_1 \alpha_2$</td>
<td>$\alpha_0 \alpha_1 \alpha_2$</td>
<td>$-\sigma(\alpha_3) = \alpha_3$, $-\sigma(\alpha_1) = \alpha_1$, $-\sigma(\alpha_2) = \alpha_2$</td>
</tr>
<tr>
<td>$\alpha_0 \alpha_1 \alpha_2$</td>
<td>$\alpha_0 \alpha_1 \alpha_2$</td>
<td>$\sigma(\alpha_0) = \alpha_0$, $\sigma(\alpha_1) = \alpha_1$, $-\sigma(\alpha_2) = \alpha_2 + 4\alpha_0 + 4\alpha_1$.</td>
</tr>
</tbody>
</table>

Let us consider the involutive automorphism of $A_4^{(2)}$ determined by any one of the Satake diagrams, say (ii) of table II. The simple root automorphisms are given by

$$-\sigma(\alpha_2) = \alpha_2 + 4\alpha_1 + 4\alpha_0,$$

$$\sigma(\alpha_1) = \alpha_1,$$

$$\sigma(\alpha_0) = \alpha_0.$$
In terms of roots $\alpha$ and purely imaginary root $\delta$ the basic root automorphisms can be written as

$$-\sigma(\alpha_2) = \alpha_2 + 4\alpha_1 + 4\alpha_0, \quad \sigma(\alpha_1) = \alpha_1,$$

$$\sigma\left(\frac{1}{2}(\delta - 2\alpha_1 - \alpha_2)\right) = \frac{1}{2}(\delta - 2\alpha_1 - \alpha_2),$$

where

$$\alpha_0 = \frac{1}{2}(\delta - 2\alpha_1 - \alpha_2). \quad (4.2)$$

So, the positive roots of $A_4^{(2)}$ are given by

$$\Delta = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \pm \alpha_1 + n\delta, \pm \alpha_2 + 2n\delta, \pm (2\alpha_1 + \alpha_2) + 2n\delta,$$

$$\pm (\alpha_1 + \alpha_2) + n\delta, \frac{1}{2}(\pm \alpha_2 + (2n - 1)\delta),$$

$$\frac{1}{2}(\pm (2\alpha_1 + \alpha_2) + (2n - 1)\delta), n\delta, n \in \mathbb{Z}\}. \quad (4.3)$$

We can apply the simple root automorphism to find out automorphism of other roots and we see that the positive roots can be separated into two categories, i.e.

$$\exp \alpha(h) = +1 \text{ for } \alpha = \{\alpha_1, -\alpha_1 + n\delta, -\alpha_2 + 2n\delta, -(2\alpha_1 + \alpha_2) + 2n\delta,$$

$$-(\alpha_1 + \alpha_2) + n\delta, \frac{1}{2}(-\alpha_2 + (2n - 1)\delta), \frac{1}{2}(-(2\alpha_1 + \alpha_2) + (2n - 1)\delta)\}, \quad (4.4)$$

$$\exp \alpha(h) = -1 \text{ for } \alpha = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + n\delta, \alpha_2 + 2n\delta, (2\alpha_1 + \alpha_2) + 2n\delta,$$

$$(\alpha_1 + \alpha_2) + n\delta, \frac{1}{2}(\alpha_2 + (2n - 1)\delta), \frac{1}{2}((2\alpha_1 + \alpha_2) + (2n - 1)\delta), n\delta\}. \quad (4.5)$$

Thus for $A_4^{(2)}$, $K$ and $P$ are given by

$$K = \{i h_\alpha \text{ for } \alpha = \alpha_0, \alpha_1, \alpha_2, (e_\alpha + e_{-\alpha}), i(e_{\alpha - e_\alpha}), \text{ for } \alpha \text{ given by eq}(4.4)\}, \quad (4.6)$$

and $$P = \{i(e_\alpha + e_{-\alpha}), (e_\alpha - e_{-\alpha}), \text{ for } \alpha \text{ given by eq}(4.5)\}. \quad (4.7)$$

We now select a maximal abelian subalgebra, $A$ which is two-dimensional and may be chosen to have basis elements,

$$h'_0 = i(e_{\alpha_2} + e_{-\alpha_2}) \quad (4.8)$$

$$h'_1 = i[e(2\alpha_1 + \alpha_2) + e_{-(2\alpha_1 + \alpha_2)}]. \quad (4.9)$$

So, we have $R_A = \{\alpha_2, 2\alpha_1 + \alpha_2\}$ and $R_M$ is empty and $M$ is one-dimensional and its basis element is given by

$$-ih'_2 = n\delta. \quad (4.10)$$

Note that $h'_0, h'_1, h'_2$ together with the scaling element $d'$ forms the Cartan subalgebra $h'$. 
The required inner automorphism of $A^{(2)}_4$ is
\begin{equation}
V = V_{\alpha_2} \cdot V_{2\alpha_1 + \alpha_2}
\end{equation}
where
\begin{align*}
V_{\alpha_2} & = \exp[\text{adj} \{ ia_{\alpha_2}(e_{\alpha_2} - e_{-\alpha_2}) \}] \\
& \quad \exp[\text{adj} \{ ia_{2\alpha_1 + \alpha_2}(e_{2\alpha_1 + \alpha_2} - e_{-2\alpha_1 - \alpha_2}) \}]
\end{align*}
and
\begin{align*}
a_{\alpha_2} & = \frac{\pi}{8(\alpha_1, \alpha_2)}(1/2) \quad a_{2\alpha_1 + \alpha_2} = \frac{\pi}{8(2\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2)}(1/2).
\end{align*}

Applying this to the Cartan subalgebra $h'$ of $A^{(2)}_4$, we obtain
\begin{align*}
h_0 & = -h_{\alpha_2}, \\
h_1 & = -(2h_{\alpha_1} + h_{\alpha_2}), \\
and \quad h_2 & = -h_n\delta.
\end{align*}

With respect to this Cartan-subalgebra, the set of positive roots is given by
\begin{align*}
\Delta^+ & = \left\{ \alpha_1, -\alpha_2, -(\alpha_1 + \alpha_2), -(2\alpha_1 + \alpha_2), \alpha_1 \pm n\delta, -\alpha_2 \pm 2n\delta, \\
& \quad -(2\alpha_1 + \alpha_2) \pm 2n\delta, -(\alpha_1 + \alpha_2) \pm n\delta, \\
& \quad \frac{1}{2}(\alpha_2 \pm (2n - 1)\delta), \frac{1}{2}(\alpha_1 \pm (2n - 1)\delta) \right\}.
\end{align*}

The sets $\Delta_+^+$ and $\Delta^+$ can be written as
\begin{align*}
\Delta_+^+ & = \Delta^+ / \Delta_+^\mp \\
and \quad \Delta^+ & = \{-n\delta\}.
\end{align*}

Now Choose the fundamental root system
\begin{equation}
\Psi = \{\lambda_1, \lambda_2\},
\end{equation}
where
\begin{align*}
\lambda_1 & = \frac{1}{2}(\delta - (2\alpha_1 + \alpha_2)), \alpha_1 \\
and \quad \lambda_2 & = \alpha_2.
\end{align*}

It follows that $\lambda_1$ is the restriction of $\frac{1}{2}(\delta - (2\alpha_1 + \alpha_2))$, $\alpha_1$, and $\lambda_2$ is the restriction of $\alpha_2$. So that using these in eqn. (4.15), we can write
\begin{align*}
\Sigma_+ & = \left\{ (1 \pm 4n)\lambda_1 \pm n\lambda_2, (-1 \pm 2n)\lambda_2 \pm 8n\lambda_1, (-2 \pm 8n)\lambda_1 \\
& \quad + (-1 \pm 2n)\lambda_2, (-1 \pm 4n)\lambda_1 + (-1 \pm n)\lambda_2, \\
& \quad \frac{1}{2}[(-1 \pm (2n - 1))\lambda_2 \pm 4(2n - 1)\lambda_1], \frac{1}{2}[-2 \pm 4((2n - 1))\lambda_1 + (-1 \pm (2n - 1))\lambda_2] \right\}
\end{align*}

Since $\dim A = m = 2$, there are four conjugacy classes of parabolic subalgebras, namely the minimal parabolic subalgebra, $g$ itself and two others which will now be constructed.
Case (I) — Now, choose the non-empty proper subset $\theta$ of $\Psi_1$ as

$$\theta = \{\lambda_2\}. \quad (4.22)$$

Then $V A_{\theta}$ has generator $h_\alpha$ with

$$\alpha = 2\alpha_1 + \alpha_2. \quad (4.23)$$

So $A_{\theta}$ has generator $(e_\alpha + e_{-\alpha})$ with

$$\alpha = 2\alpha_1 + \alpha_2. \quad (4.24)$$

Now as $\theta \lambda_2 = h \lambda_2 = h \alpha_2$, so $A(\theta)$ has generator

$$i(e_\alpha + e_{-\alpha}). \quad (4.25)$$

Moreover, as $\langle \theta \rangle = \{\lambda_2\}$ $\tilde{N}_-(\theta)$ is generated by

$$V^{-1} e_\alpha = \frac{1}{2} (e_\alpha + e_{-\alpha}) - \frac{i}{2} h_\alpha. \quad (4.26)$$

Then, as $\langle \theta \rangle = \{-\lambda_2\}$. So $\tilde{N}_+(\theta)$ is generated by

$$V^{-1} e_{-\alpha} = -\frac{1}{2} (e_\alpha + e_{-\alpha}) - \frac{i}{2} h_\alpha. \quad (4.27)$$

and $\tilde{N}_\theta$ is generated by

$$V^{-1} e_{\alpha_1} = -\frac{i}{2} A_1 e_{-\alpha_1 - \alpha_2} + \frac{1}{2} e_{\alpha_1} - \frac{i}{2} B_1 e_{-\alpha_1} - \frac{i}{2} C_1 e_{\alpha_1 + \alpha_2}, \quad (4.28)$$

where

\[
\begin{align*}
A_1 &= \text{Sgn}(N_{2\alpha_1 + \alpha_2, \alpha_1 - \alpha_2}), \\
B_1 &= \text{Sgn}(N_{\alpha_1, \alpha_2} N_{2\alpha_1 + \alpha_2, \alpha_1}), \\
C_1 &= \text{Sgn}(N_{\alpha_1, \alpha_2})
\end{align*}
\]

and

$$V^{-1} e_{-(\alpha_1 + \alpha_2)} = \frac{1}{2} e_{-\alpha_1 - \alpha_2} - \frac{i}{2} A_2 e_\alpha - \frac{i}{2} B_2 e_{-\alpha_2} - \frac{1}{2} C_2 e_{\alpha_1 + \alpha_2}. \quad (4.30)$$

where

\[
\begin{align*}
A_2 &= \text{Sgn}(N_{2\alpha_1 + \alpha_2, \alpha_1 - \alpha_2}), \\
B_2 &= \text{Sgn}(N_{\alpha_2, \alpha_1 - \alpha_2}), \\
C_2 &= \text{Sgn}(N_{\alpha_2, \alpha_1 - \alpha_2} N_{2\alpha_1 + \alpha_2, \alpha_1 - \alpha_2})
\end{align*}
\]
\[ V^{-1} e_{-2\alpha_1 - \alpha_2} = e_{-2\alpha_1 - \alpha_2}. \]  
\[ V^{-1} e_{\alpha_1 \pm n\delta} = \frac{1}{2} e_{\alpha_1 \pm n\delta} - \frac{i}{2} A_3 e_{-\alpha_1 - \alpha_2 \pm n\delta} \]
\[- \frac{i}{2} B_3 e_{\alpha_1 + \alpha_2 \pm n\delta} - \frac{1}{2} C_3 e_{-\alpha_1 \pm n\delta}, \]  
(4.32)

where

\[ A_3 = Sgn(N_{\alpha_1 - \alpha_2 \pm n\delta, 2\alpha_1 + \alpha_2}), \]
\[ B_3 = Sgn(N_{\alpha_2, \alpha_2 \pm n\delta}), \]
\[ C_3 = Sgn(N_{\alpha_2, \alpha_1 \pm n\delta, N_{2\alpha_1 + \alpha_2, -\alpha_1 \pm n\delta}}). \]  
(4.33)

\[ V^{-1} e_{-\alpha_2 \pm 2n\delta} = \frac{1}{2^{1/2}} e_{-\alpha_2 \pm 2n\delta} - \frac{i}{2^{1/2}} A_4 e_{\pm 2n\delta}, \]  
(4.36)

where

\[ A_4 = Sgn(N_{\alpha_2, -\alpha_2 \pm 2n\delta}). \]  
(4.37)

\[ V^{-1} e_{-2(\alpha_1 + \alpha_2) \pm 2n\delta} = \frac{1}{2^{1/2}} e_{-(2\alpha_1 + \alpha_2) \pm 2n\delta} - \frac{i}{2^{1/2}} A_5 e_{\pm 2n\delta}, \]  
(4.38)

where

\[ A_5 = Sgn(N_{2\alpha_1 + \alpha_2, -(2\alpha_1 + \alpha_2) \pm 2n\delta}). \]  
(4.39)

\[ V^{-1} e_{-(\alpha_1 + \alpha_2) \pm n\delta} = \frac{1}{2} e_{-\alpha_1 - \alpha_2 \pm n\delta} - \frac{i}{2} A_6 e_{\alpha_1 \pm n\delta} \]
\[- \frac{i}{2} B_6 e_{-\alpha_1 \pm n\delta} - \frac{1}{2} C_6 e_{\alpha_1 + \alpha_2 \pm n\delta}, \]  
(4.40)

where

\[ A_6 = Sgn(N_{2\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2 \pm n\delta}), \]
\[ B_6 = Sgn(N_{\alpha_2, -\alpha_1 - \alpha_2 \pm n\delta}), \]
\[ C_6 = Sgn(N_{\alpha_2, \alpha_2 - \alpha_1 \pm n\delta, N_{2\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2 \pm n\delta}}). \]  
(4.41)

\[ V^{-1} e_{\frac{1}{2}\{-(\alpha_2)\pm(2n-1)\delta\}} = \frac{1}{2^{1/2}} e_{\frac{1}{2}\{-(\alpha_2)\pm(2n-1)\delta\}} \]
\[- \frac{i}{2^{1/2}} A_7 e_{\frac{1}{2}(\alpha_2)\pm\frac{1}{2}(2n-1)\delta}, \]  
(4.42)

where

\[ A_7 = Sgn\left(N_{\alpha_2, \frac{1}{2}(\alpha_2)\pm\frac{1}{2}(2n-1)\delta}\right). \]  
(4.43)
$$V^{-1}e^{\frac{i}{2}(-2\alpha_1 + \alpha_2) \pm (2n-1)\delta} = \frac{1}{2^{1/2}} e^{\frac{i}{2}(-2\alpha_1 + \alpha_2) \pm (2n-1)\delta}$$

$$-\frac{i}{2^{1/2}} A_8 \ e^{-\frac{i}{2}(2\alpha_1 + \alpha_2) \pm \frac{1}{2}(2n-1)\delta},$$

where

$$A_8 = Sgn \left( N_{2\alpha_1 + \alpha_2} \right).$$

(4.44)

It follows that the basis elements of \( N_0 \) may be taken to be

$$-\frac{i}{2} A_1 (e_{-\alpha_1} - e_{\alpha_1} + a_2 + e_{\alpha_1}) + \frac{1}{2} (e_{\alpha_1} - e_{-\alpha_1})$$

$$-\frac{1}{2} B_1 (e_{-\alpha_1} - e_{\alpha_1}) - \frac{i}{2} C_1 (e_{\alpha_1} + a_2 - e_{-\alpha_1} - a_2),$$

$$\frac{1}{2} (e_{-\alpha_1} - a_2 - e_{\alpha_1} + a_2) - \frac{i}{2} A_2 (e_{\alpha_1} - e_{-\alpha_1}) - \frac{i}{2} B_2 (e_{-\alpha_1} - e_{\alpha_1})$$

$$-\frac{i}{2} C_2 (e_{\alpha_1} + a_2 - e_{-\alpha_1} - a_2), (e_{-\alpha_1} - a_2 - e_{\alpha_1} + a_2),$$

$$\frac{1}{2} (e_{\alpha_1} \pm n\delta - e_{-\alpha_1} \pm n\delta) - \frac{i}{2} A_3 (e_{-\alpha_1} - a_2 \pm n\delta - e_{\alpha_1} + a_2 \pm n\delta) - \frac{i}{2} B_3 (e_{\alpha_1} + a_2 \pm n\delta - e_{-\alpha_1} - a_2 \pm n\delta)$$

$$\frac{1}{2} C_3 (e_{-\alpha_2} \pm n\delta - e_{\alpha_2} \pm n\delta), \frac{1}{2^{1/2}} (e_{-\alpha_1} \pm n\delta - e_{\alpha_1} \pm n\delta) - \frac{i}{2^{1/2}} A_4 (e_{\pm 2n\delta} - e_{\mp 2n\delta}),$$

$$\frac{1}{2^{1/2}} (e_{-(2\alpha_1 + \alpha_2) \pm 2n\delta} - e_{(2\alpha_1 + \alpha_2) \pm 2n\delta})$$

$$-\frac{i}{2^{1/2}} A_5 (e_{\pm 2n\delta} - e_{\mp 2n\delta}), \frac{1}{2} (e_{-\alpha_1} - a_2 \pm n\delta - e_{\alpha_1} + a_2 \pm n\delta) - \frac{i}{2} A_6 (e_{\alpha_1} \pm n\delta - e_{-\alpha_1} \pm n\delta) - \frac{i}{2} B_6 (e_{-\alpha_1} \pm n\delta - e_{\alpha_1} \pm n\delta)$$

$$-\frac{1}{2} C_6 (e_{\alpha_1} + a_2 \pm n\delta - e_{-\alpha_1} - a_2 \pm n\delta),$$

$$\frac{1}{2^{1/2}} \left( e^{\frac{i}{2}(-\alpha_2 \pm (2n-1)\delta)} - e^{-\frac{i}{2}(-\alpha_2 \pm (2n-1)\delta)} \right)$$
\[
-\frac{i}{2^{7/2}} A_7 \left( e_{\frac{1}{2}}(\alpha_2 \pm (2n-1)\delta) - e_{-\frac{1}{2}}(\alpha_2 \pm (2n-1)\delta) \right),
\]
\[
-\frac{1}{2^{7/2}} \left( e_{\frac{1}{2}}((-2\alpha_1 + \alpha_2) \pm (2n-1)\delta) - e_{-\frac{1}{2}}((-2\alpha_1 + \alpha_2) \pm (2n-1)\delta) \right),
\]
\[
-\frac{i}{2^{7/2}} A_8 \left( e_{-\frac{1}{2}}((2\alpha_1 + \alpha_2) \pm \frac{1}{2}(2n-1)\delta) - e_{\frac{1}{2}}((2\alpha_1 + \alpha_2) \pm \frac{1}{2}(2n-1)\delta) \right),
\]
(4.45)

and the basis element of \( M_\theta \) may be taken to be
\[
n\delta, \frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - \frac{i}{2} h_{\alpha_2} - \frac{1}{2}(e_{\alpha_2} - e_{-\alpha_2}) - \frac{i}{2} h_{-\alpha_2} - i(e_{\alpha_2} + e_{-\alpha_2}).
\]
(4.46)

Since \( h'_\tau \cap P = A_\theta \), so this parabolic subalgebra \( P_\theta \) is cuspidal and
\[
P_\theta = M_\theta \oplus A_\theta \oplus N_\theta
\]
(4.47)

Case (II) — Here choose the non-empty proper subset \( \theta \) of \( \Psi_1 \) as
\[
\theta = \{\lambda_1\}.
\]
(4.48)

Then, \( VA_\theta \) has generator
\[
(h_{\alpha_2} + h_{2\alpha_1 + \alpha_2}).
\]
(4.49)

So that \( A_\theta \) has generator
\[
i(e_{\alpha_2} + e_{-\alpha_2}) + i(e_{2\alpha_1 + \alpha_2} + e_{-(2\alpha_1 + \alpha_2)}).
\]
(4.50)

As \( Q_{\lambda_1} = h_{\alpha_1} = \frac{1}{2} h_{2\alpha_1 + \alpha_2} - \frac{1}{2} h_{\alpha_2} \), so that the generator of \( A(\theta) \) may be taken to be,
\[
A(\theta) = \{i(e_{2\alpha_1 + \alpha_2} + e_{-(2\alpha_1 + \alpha_2)}) - i(e_{\alpha_2} - e_{-\alpha_2})\}.
\]
(4.51)

Moreover, as \( \langle \theta \rangle_1 = \{\lambda_1\} \), So \( \tilde{N}_-(\theta) \) is generated by
\[
V^{-1} e_{-\alpha_1} = \frac{1}{2} e_{-\alpha_1} - \frac{1}{2} A_9 e_{\alpha_1 + \alpha_2} - \frac{i}{2} B_9 e_{-\alpha_1 - \alpha_2} - \frac{1}{2} C_9 e_{\alpha_1},
\]
(4.52)

where
\[
A_9 = Sgn(N_{-\alpha_1, 2\alpha_1 + \alpha_2}),
\]
\[
B_9 = Sgn(N_{\alpha_2, -\alpha_1 - \alpha_2}),
\]
and
\[
C_9 = Sgn(N_{\alpha_2, -\alpha_1 - \alpha_2} N_{2\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2}).
\]
(4.53)

and as \( \langle \theta \rangle_+ = \{\lambda_1\} \), so \( \tilde{N}_+(\theta) \) is generated by \( V^{-1} e_{\alpha_1} \) which is as mentioned in eq. (4.28) and \( \tilde{N}_\theta \) is generated by elements, \( V^{-1} e_{-\alpha_1}, V^{-1} e_{-(\alpha_1 + \alpha_2)}, V^{-1} e_{-(2\alpha_1 + \alpha_2)}, V^{-1} e_{\alpha_1 \pm n\delta}, V^{-1} e_{-\alpha_2 \pm 2n\delta}, \)
\[
V^{-1} e_{-(\alpha_1 + \alpha_2) \pm n\delta}, V^{-1} e_{\frac{1}{2}(-\alpha_2 \pm (2n-1)\delta)}, V^{-1} e_{\frac{1}{2}[-(2\alpha_1 + \alpha_2) \pm (2n-1)\delta]}.
\]
(4.54)
All of which are already mentioned in eq. (4.27), eq. (4.29–4.44) as in case (a).

Preceding in the same way as in Case-I the elements of $N_\theta$ and $M_\theta$ may be generated.

Clearly

$$n\delta_i (e_{2\alpha_1 + \alpha_2} - e_{-(2\alpha_1 + \alpha_2)}) + i(e_{\alpha_2} - e_{-\alpha_2}), (e_{-\alpha_1} - e_{\alpha_2}) - iA_9(e_{\alpha_1 + \alpha_2} - e_{-(\alpha_1 + \alpha_2)})$$

$$-iB_9(e_{-(\alpha_1 + \alpha_2)} - e_{\alpha_1 + \alpha_2}) - C_9(e_{\alpha_1} - e_{-\alpha_1}), (e_{\alpha_1} - e_{-\alpha_1}) - iA_1(e_{-(\alpha_1 + \alpha_2)} - e_{(\alpha_1 + \alpha_2)})$$

are generators of a real $\mathbb{Z}$-invariant Cartan-subalgebra $h'$ for which $h' \cap P = A_\theta$. So this parabolic subalgebra is also cuspidal and

$$P_\theta = M_\theta \oplus A_\theta \oplus N_\theta. \quad (4.55)$$

IV. CONCLUSION

We have presented the Langlands decomposition of lower rank affine Kac-Moody algebras $A^{(1)}_3$ (untwisted) and $A^{(2)}_4$ (twisted) in detail. Use of Satake diagrams facilitates Iwasawa decomposition, which in turn leads to Langland decomposition. The minimal parabolic subalgebra being cuspidal leads to an induced representation. The Satake diagrams can be studied in a different angle altogether. They can be used to classify the real forms of affine Kac-Moody algebras and automatically give the Dynkin diagram of reduced root system, which are nothing but Dynkin diagrams of the associated symmetric space (if it can be defined for a infinite dimensional algebra). Such studies are currently in progress.

REFERENCES


