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New constructions of two slim dense near hexagons

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Abstract: We provide a geometrical construction of the unique slim dense near hexagon with parameters $(s, t, t_2) = (2, 5, \{1, 2\})$. Using this construction, we construct the rank 3 symplectic dual polar space $DSp(6, 2)$ which is the unique slim dense near hexagon with parameters $(s, t, t_2) = (2, 6, 2)$. Both near hexagons are constructed from two copies of the unique generalized quadrangle with parameters (2,2).

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1. Introduction

A *partial linear space* is a point-line geometry $S = (P, L)$ with 'point-set' P and 'line-set' L of subsets of P of size at least 2 such that any two distinct points are contained in at most one line. Two distinct points x and y are *collinear*, written as $x \sim y$, if there is a line containing them. In that case we denote this line by xy. We write $x \nsim y$ if two distinct points x and y are not collinear. For $x \in P$ and $A \subseteq P$, we define $x^\perp = \{x\} \cup \{y \in P : x \sim y\}$ and $A^\perp = \bigcap_{x \in A} x^\perp$. Note that $xy = \{x, y\}^\perp$ if $x \sim y$. A subset of P is a *subspace* of S if any line containing at least two of its points is contained in it. A subspace of S is *singular* if each pair of its distinct points is collinear. A *geometric hyperplane* of S is a subspace, different from the empty set and P, that meets every line non-trivially. The graph $\Gamma(P)$ with vertex set P, in which two distinct vertices are *adjacent* if they are collinear in S, is the *collinearity graph* of S. The diameter of $\Gamma(P)$ is called the *diameter* of S. If $\Gamma(P)$ is connected, we say that S is *connected*.

A *near polygon* [11] is a connected partial linear space of finite diameter satisfying the following near-polygon property (NP): *For each point-line pair (x, l) with $x \notin l$, there is a unique point in l nearest to x.* If the diameter of S is n, then S is called a *near $2n$-gon*, and for $n = 3$ a *near hexagon*. A near 0-gon is a point and a near 2-gon is a line. An important class of near polygons is the the class of generalized $2n$-gons. The class of the near 4-gons coincides with the class of the *generalized quadrangles*.

Let $S = (P, L)$ be a near polygon. A subspace C of S is *convex* if every shortest path in $\Gamma(P)$ between two points of C is entirely contained in C. A *quad* is a convex subspace of S of diameter 2 such that no point of it is adjacent to all other points of it. The points and the lines contained in a quad define a generalized quadrangle. We denote by $d(x, y)$ the *distance* in $\Gamma(P)$ between two points x and y of S. Let x_1 and x_2 be two points of S with $d(x_1, x_2) = 2$. If x_1 and x_2 have at least two common neighbours y_1 and y_2 such that one of the lines $x_i y_j$ contains at least three points, then x_1 and x_2 are contained in a unique quad ([11], Proposition 2.5, p.10). A near polygon is called *dense* if every line contains at least three points and every pair of points at distance 2 have at least two common neighbours. A structure theory for dense near polygons can be found in [2].

Let $S = (P, L)$ be a dense near $2n$-gon. Then, the number $t + 1$ of lines containing a given point of S is independent of the point ([2], Lemma 19, p.152). Let $t_2 = \{|\{x, y\}^\perp| - 1 : x, y \in P, d(x, y) = 2\}$. We say that S has *parameters* (s, t, t_2) if each line of S contains $s + 1$

points, each point is contained in $t+1$ lines and t_2 is as above. If $n = 2$, then $t_2 = \{t\}$, though t_2 may have more than one element in general. A near 4-gon with parameters $(s, t, \{t\})$ is written as a (s, t)-GQ. If Q is a quad of S, then for $x \in P$, either

(i) there is a unique point $y \in Q$ (depending on x) collinear with x and $d(x, z) = d(x, y) + d(y, z)$ for all $z \in Q$; or
(ii) $d(x, Q) = 2$ and the set $\mathcal{O}_x = \{y \in Q : d(x, y) = 2\}$ is an ovoid of Q.

([11], Proposition 2.6, p.12). We say that the quad Q is *classical* if (i) holds for each $x \in P$. If every quad in S is classical, then the near hexagon S is said to be *classical*.

Let $S = (P, L)$ be a polar space of rank n (see [3]). The *dual polar space of rank* n associated with S is the point-line geometry $DS = (P', L')$, whose point set P' is the collection of all maximal singular subspaces of S; and a line of DS is the collection of all maximal singular subspaces of S containing a specific singular subspace of S of co-dimension 1. These geometries were characterized in terms of points and lines by Cameron [4]. The dual polar spaces of rank n are precisely the classical dense near $2n$-gons ([4], Theorem 1, p.75).

A near polygon is *slim* if each line contains exactly three points. In that case, if x and y are two collinear points, we define $x * y$ by $xy = \{x, y, x * y\}$. All slim dense near hexagons are classified by Brouwer et al. [1]. We refer to [6]–[8] for new classification results concerning slim dense near polygons.

Theorem 1.1. ([1], *Theorem 1.1, p.349) Let $S = (P, L)$ be a slim dense near hexagon. Then, P is finite and S is isomorphic to one of the eleven near hexagons with parameters as given below:*

$\|P\|$	t	t_2	$\|P\|$	t	t_2
891	20	$\{4\}$	135	6	$\{2\}$
759	14	$\{2\}$	105	5	$\{1, 2\}$
729	11	$\{1\}$	81	5	$\{1, 4\}$
567	14	$\{2, 4\}$	45	3	$\{1, 2\}$
405	11	$\{1, 2, 4\}$	27	2	$\{1\}$
243	8	$\{1, 4\}$	–	–	–

The slim dense near hexagon on 135 points with parameters $(s, t, t_2) = (2, 6, 2)$ is the symplectic dual polar space of rank 3 over the field F_2 with two elements. We denote this near hexagon by $DSp(6, 2)$ and by $\mathbb{H}_3$ the near hexagon on 105 points with parameters $(s, t, t_2) = (2, 5, \{1, 2\})$. In the following, we present three known descriptions of the near hexagon $\mathbb{H}_3$. Another description of this near hexagon is given in ([1], p.352) using the notion of a Fischer space.

Example 1.2. *([1], p.355) Let X be a set of size 8. Let S be the partial linear space whose point set consists of all partitions of X into four 2-subsets, and lines are the collections of three points which (regarded as partitions) share two given disjoint 2-subsets. Then S is a slim dense near hexagon isomorphic to $\mathbb{H}_3$.*

Example 1.3. *([1], p.352) Let $Q(6,2)$ be a non-singular quadric in $PG(6,2)$ and let $DQ(6,2)$ be the associated dual polar space. Let Π be a hyperplane of $PG(6,2)$ intersecting $Q(6,2)$ in a non-singular hyperbolic quadric $Q^+(5,2)$. The set of all maximal subspaces of $Q(6,2)$ which are not contained in $Q^+(5,2)$ is a subspace of $DQ(6,2)$ and the induced subgeometry is isomorphic to the near hexagon $\mathbb{H}_3$. (Note that $DQ(6,2)$ is isomorphic to $DSp(6,2)$.)*

The next construction (Example 1.5) is due to Bart De Bruyn ([5], p.51). We first note down some facts about $(2,2)$-GQs which play an important role in Example 1.5 below as well as in our constructions in the next section.

Let $S=(P,L)$ be a $(2,2)$-GQ. Then S is unique up to an isomorphism ([9], 5.2.3, p.78). In the notation of [9], S is isomorphic to the classical generalized quadrangle $W(2)\simeq Q(4,2)$.

A combinatorial model of a $(2,2)$-GQ: Let $\Omega=\{1,2,3,4,5,6\}$. An *edge* of Ω is a 2-subset of Ω. A *factor* of Ω is a set of three pair-wise disjoint edges. Let $\mathcal{E}$ be the set of all edges and $\mathcal{F}$ be the set of all factors of Ω. Then $|\mathcal{E}|=|\mathcal{F}|=15$. Taking $\mathcal{E}$ as the point-set and $\mathcal{F}$ as the line-set, the pair $(\mathcal{E},\mathcal{F})$ is a $(2,2)$-GQ.

A *k-arc* of points of S is a set of k pair-wise non-collinear points of S. A k-arc is *complete* if it is not contained in a $(k+1)$-arc. A point x is a *center* of a k-arc if x is collinear with every point of it. A 3-arc is called a *triad.* We refer to [10] for a detailed study of k-arcs of a $(2,t)$-GQ. We repeatedly use the following result, mostly without mention.

Lemma 1.4. *([10], Proposition 3.2, p.161) Let $S=(P,L)$ be a (2,2)-GQ and T be a triad of points of S. Then*

(i) $|T^{\perp}|=1$ *or* 3*;*
(ii) $|T^{\perp}|=1$ *if and only if T is an incomplete triad if and only if T is contained in a unique (2,1)-subGQ of S;*
(iii) $|T^{\perp}|=3$ *if and only if T is a complete triad.*

Dually, we can define a k-arc of lines of S and center of such a k-arc. Since $W(2)$ is self-dual ([9], 3.2.1, p.43), Lemma 1.4 holds for a triad of lines also. From the combinatorial model above, it can be seen that any two non-collinear points (respectively; disjoint lines) of S are contained in a unique complete triad of points (respectively; lines) of S.

Example 1.5. *([5], p.51) Let $S = (P, L)$ be the $(2, 2)$-GQ. A partial linear space $S' = (P', L')$ can be constructed from S as follows. Set*

$$P' = \{(x, y) \in P \times P : x = y \text{ or } x \sim y\}.$$

There are four types of lines in L'. For each line $l = \{x, y, z\}$ of S, we have the following lines of S' (up to a permutation of the points of l):

(i) $\{(x, x), (y, y), (z, z)\}$,
(ii) $\{(x, x), (x, y), (x, z)\}$,
(iii) $\{(x, y), (y, z), (z, x)\}$.

Let $T = \{k, m, n\}$ be an incomplete triad of lines of S such that $T^\perp = \{l\}$. We may assume that $k \cap l = \{x\}$, $m \cap l = \{y\}$ and $n \cap l = \{z\}$. Let x' be an arbitrary point of $k \setminus \{x\}$. Let y' (respectively, z') be the unique point of $m \setminus \{y\}$ (respectively, $n \setminus \{z\}$) not collinear with x'. Then the lines of fourth type are:

(iv) $\{(x, x'), (y, y'), (z, z')\}$.

Then S' is a slim dense near hexagon isomorphic to $\mathbb{H}_3$. (It can be seen that the set $\{x', y', z'\}$ is a complete triad of points of S.)

2. New Constructions

Here, we provide new geometrical constructions for the near hexagons $DSp(6, 2)$ and $\mathbb{H}_3$ from two copies of the unique (2,2)-GQ. In fact, we first construct $\mathbb{H}_3$ and then construct $DSp(6, 2)$ in which $\mathbb{H}_3$ is embedded as a geometric hyperplane.

2.1. **Construction of $\mathbb{H}_3$.** In this construction the point set is the same as in Example 1.5 (up to an isomorphism). Let $S = (P, L)$ and $S' = (P', L')$ be two (2,2)-GQs. Let $x \leftrightarrow x'$, $x \in P, x' \in P'$, be an isomorphism between S and S'. We define a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ as follows. The point set is

$$\mathcal{P} = \{(x, y') \in P \times P' : y' \in x'^\perp\},$$

and the lines are of the form

$$\{(x, u'), (y, v'), (z, w')\},$$

where $T = \{x, y, z\}$ is either a line or a complete triad and $T'^\perp = x'^\perp \cap y'^\perp \cap z'^\perp = \{u', v', w'\}$.

Theorem 2.1. *The partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a slim dense near hexagon with parameters $(s, t, t_2) = (2, 5, \{1, 2\})$.*

2.2. **Construction of $DSp(6, 2)$.** Let $S = (P, L)$, $S' = (P', L')$ and $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be as in the construction of $\mathbb{H}_3$. We define a partial linear space $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ as follows. The point set is

$$\mathbb{P} = \mathcal{P} \cup P \cup P',$$

and the line set $\mathbb{L}$ consists of the lines in $\mathcal{L}$ together with the following collection $\mathbb{L}_1$ of lines:

($\mathbb{L}_1$) : The collection $\mathbb{L}_1$ consists of lines of the form $\{x, (x, u'), u'\}$ for every point $(x, u') \in \mathcal{P}$.

Theorem 2.2. *The partial linear space $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ is a slim dense near hexagon with parameters $(s, t, t_2) = (2, 6, 2)$*

An immediate consequence of Theorems 2.1 and 2.2 is that the near hexagon $\mathbb{H}_3$ is a geometric hyperplane of the near hexagon $DSp(6, 2)$.

3. Proof of Theorem 2.1

Let $\alpha = (x, u')$ and $\beta = (y, v')$ be two distinct points of $\mathcal{S}$. By the construction of lines of $\mathcal{S}$, $\alpha \sim \beta$ if and only if $x \neq y$, $u' \neq v'$, $u' \in y'^{\perp}$ and $v' \in x'^{\perp}$. Let α and β be distinct non-collinear points of $\mathcal{S}$. Then one of the following possibilities occur:

($A1$) $x = y, u' \neq v'$;
($A2$) $x \neq y, u' = v'$;
($A3$) $x \neq y, u' \neq v'$, $u' \notin y'^{\perp}$ and $v' \notin x'^{\perp}$;
($A4$) $x \neq y, u' \neq v'$ and either $u' \in y'^{\perp}$ and $v' \notin x'^{\perp}$ or $u' \notin y'^{\perp}$ and $v' \in x'^{\perp}$.

Lemma 3.1. *Assume that ($A1$) or ($A2$) holds. Then $|\{\alpha, \beta\}^{\perp}| \geq 2$.*

Proof. Assume that ($A1$) holds. Then $x' \in \{u', v'\}^{\perp}$. If $u' \sim v'$, we may assume that $v' \neq x'$. So $x'v' = u'v'$. Then $(v, u' * v')$ and $(x * v, u' * v')$ are in $\{\alpha, \beta\}^{\perp}$. If $u' \nsim v'$, let $\{u', v', w'\}$ be the complete triad of S' containing u' and v'. Then (a, w') and (b, w') are in $\{\alpha, \beta\}^{\perp}$, where $\{a', b'\} = \{u', v'\}^{\perp} \setminus \{x'\}$. A similar argument applies if ($A2$) holds. □

Lemma 3.2. *Assume that ($A3$) holds. Then $|\{\alpha, \beta\}^{\perp}| \geq 3$.*

Proof. If $x \sim y$ and $u' \sim v'$, then $\{x', y', u', v'\}$ defines a quadrangle in S'. Then (u, x'), (v, y') and $(u * v, x' * y')$ are in $\{\alpha, \beta\}^{\perp}$.

If $x \sim y$ and $u' \nsim v'$, let $T' = \{u' * x', v' * y', z'\}$ be the complete triad of S' containing $u' * x'$ and $v' * y'$. Then $u', v' \in T'^{\perp}$ and $x' * y' \notin T'$.

Now, $x' * y' \sim z'$, because $x' * y' \nsim u' * x'$, $x' * y' \nsim v' * y'$ and T' is a complete triad. Then $(u*x, x'), (v*y, y')$ and $(z, x'*y')$ are in $\{\alpha, \beta\}^\perp$.

By a similar argument, if $x \nsim y$ and $u' \sim v'$ then $(u, u'*x'), (v, v'*y')$ and $(u*v, z')$ are in $\{\alpha, \beta\}^\perp$, where $\{u'*x', v'*y', z'\}$ is the complete triad of S' containing $u'*x'$ and $v'*y'$.

Now, assume that $x \nsim y$ and $u' \nsim v'$. If $u' = x'$ and $v' = y'$, then $(a, a'), (b, b')$ and (c, c') are in $\{\alpha, \beta\}^\perp$, where $\{u', v'\}^\perp = \{a', b', c'\}$ in S'. We may assume that $v' \neq y'$. Then the complete triads $\{x', y'\}^\perp$ and $\{u', v'\}^\perp$ of S' intersect in the point $w' = v' * y'$. This fact is independent of whether $u' = x'$ or not. Let $\{x', y'\}^\perp = \{a', b', w'\}$ and $\{u', v'\}^\perp = \{p', q', w'\}$ in S'. Since $a' \nsim w'$, $b' \nsim w'$ and $\{p', q', w'\}$ is a complete triad of S', each of a' and b' is collinear with exactly one of p' and q'. Similarly, each of p' and q' is collinear with exactly one of a' and b'. So, we may assume that $a' \sim p'$ and $b' \sim q'$. Then $(p, a'), (q, b')$ and (w, w') are contained in $\{\alpha, \beta\}^\perp$. □

Lemma 3.3. *Assume that* $(A4)$ *holds. Then* $d(\alpha, \beta) = 3$.

Proof. We may assume that $u' \in y'^\perp$ and $v' \notin x'^\perp$. Suppose that $d(\alpha, \beta) = 2$ and $(z, w') \in \{\alpha, \beta\}^\perp$. Then $z \notin \{x, y\}$, $w' \notin \{u', v'\}$, $u', v' \in z'^\perp$ and $w' \in \{x', y'\}^\perp$. Let $T' = \{x', y', z'\}$. Then T' is either a line or a complete triad of S', because x, y and z are pair-wise distinct and $u', w' \in T'^\perp$ with $u' \neq w'$. Since $v' \in \{y', z'\}^\perp$, it follows that $v' \in T'^\perp$ and $v' \in x'^\perp$, a contradiction to our assumption.

So $d(\alpha, \beta) \neq 2$. Now, choose $w' \in \{x', y'\}^\perp$ with $w' \neq u'$. Then $\alpha \sim (y, w')$ and $d((y, w'), \beta) = 2$ by Lemma 3.1. Hence $d(\alpha, \beta) = 3$. □

As a consequence of the above results, we have

Corollary 3.4. *The diameter of* $\mathcal{S}$ *is 3.*

We next prove that the near-polygon property (NP) is satisfied in $\mathcal{S}$. Let $\mathtt{L} = \{\alpha, \beta, \gamma\}$ be a line and θ be a point of $\mathcal{S}$. Let $\alpha = (x, u'), \beta = (y, v'), \gamma = (z, w')$ and $\theta = (p, q')$. Then $T = \{x, y, z\}$ is either a line or a complete triad of S and $T'^\perp = \{u', v', w'\}$. Any two collinear points of $\mathcal{S}$ have only one common neighbour. So, if θ has distance 1 from two points of $\mathtt{L}$, then it is itself a point of $\mathtt{L}$.

Proposition 3.5. *If* θ *has distance 2 from two points of* $\mathtt{L}$*, then it is collinear with the third point of* $\mathtt{L}$.

Proof. Let $d(\theta, \alpha) = d(\theta, \beta) = 2$. We prove $\theta \sim \gamma$ by showing that $p \neq z, q' \neq w'$, $q' \in z'^\perp$ and $w' \in p'^\perp$.

If $p = x$ and $q' \neq v'$ (respectively, $p \neq x$ and $q' = v'$), then $d(\theta, \beta) = 2$ (respectively, $d(\theta, \alpha) = 2$) yields $v' \notin x'^\perp$, a contradiction. So $p = x$

if and only if $q' = v'$. Similarly, $p = y$ if and only if $q' = u'$. Thus, if $p \in \{x, y\}$, then $p \neq z, q' \neq w'$, $q' \in z'^{\perp}$ and $w' \in p'^{\perp}$.

If $p \notin \{x, y\}$, then the above argument implies that $p \neq z$ and $q' \neq w'$. Also, $d(\theta, \alpha) = d(\theta, \beta) = 2$ yields $x', y' \notin q'^{\perp}$ and $u', v' \notin p'^{\perp}$. This implies that $q' \in z'^{\perp}$ and $w' \in p'^{\perp}$. $\square$

Proposition 3.6. *If θ has distance 3 from two points of $\mathtt{L}$, then it has distance 2 to the third point of $\mathtt{L}$.*

Proof. Let $d(\theta, \alpha) = d(\theta, \beta) = 3$. We prove $d(\theta, \gamma) = 2$. By Lemma 3.1, we may assume that $p \neq z$ and $q' \neq w'$. This, together with $d(\theta, \alpha) = d(\theta, \beta) = 3$, implies that $p' \notin T'$, $q' \notin T'^{\perp}$. We show that $q' \notin z'^{\perp}$ and $w' \notin p'^{\perp}$. This would complete the proof.

Suppose that $q' \in z'^{\perp}$. Since $q' \notin T'^{\perp}$, $q' \notin x'^{\perp}$ and $q' \notin y'^{\perp}$. Then, $d(\theta, \alpha) = d(\theta, \beta) = 3$ yields $u', v' \in p'^{\perp}$. This implies that $p' \in \{u', v'\}^{\perp} = T'$, a contradiction. A similar argument shows that if $w' \in p'^{\perp}$, then $q' \in T'^{\perp}$, a contradiction. $\square$

Proof of Theorem 2.1. Propositions 3.5 and 3.6 together with Corollary 3.4 imply that $\mathcal{S}$ is a near hexagon. By Lemmas 3.1 and 3.2, $\mathcal{S}$ is dense. Since $|\mathcal{P}| = 105$, Theorem 1.1 completes the proof.

Thus, quads in $\mathcal{S}$ are $(2, 1)$ or $(2, 2)$-GQs. In fact, it can be shown that equality holds in Lemmas 3.1 and 3.2.

4. Proof of Theorem 2.2

By the construction of lines of $\mathbb{S}$, no two points of P, as well as of P', are collinear in $\mathbb{S}$. Further, if $x \in P$ and $u' \in P'$, then $x \sim u'$ if and only if $(x, u') \in \mathcal{P}$, or equivalently, $u' \in x'^{\perp}$ in S'. Let α and β be two distinct non-collinear points of $\mathbb{S}$ with $\alpha \in P \cup P'$. Then one of the following possibilities occur:

$(B1)$ $\alpha = x$ and $\beta = y$ for some $x, y \in P$ with $x \neq y$;
$(B2)$ $\alpha = u'$ and $\beta = v'$ for some $u', v' \in P'$ with $u' \neq v'$;
$(B3)$ $\alpha = x \in P$ and $\beta = u' \in P'$ with $u' \notin x'^{\perp}$;
$(B4)$ $\alpha = x \in P$ and $\beta = (y, v') \in \mathcal{P}$ with $x \neq y$ and $v' \in x'^{\perp}$ in S';
$(B5)$ $\alpha = u' \in P'$ and $\beta = (y, v') \in \mathcal{P}$ with $u' \neq v'$ and $y \in u^{\perp}$ in S;
$(B6)$ $\alpha = x \in P$ and $\beta = (y, v') \in \mathcal{P}$ with $x \neq y$ and $v' \notin x'^{\perp}$ in S';
$(B7)$ $\alpha = u' \in P'$ and $\beta = (y, v') \in \mathcal{P}$ with $u' \neq v'$ and $y \notin u^{\perp}$ in S.

Lemma 4.1. *Assume that $(B1)$ or $(B2)$ holds. Then $|\{\alpha, \beta\}^{\perp}| \geq 3$ in $\mathbb{S}$.*

Proof. If $(B1)$ holds, then $w' \in \{x, y\}^{\perp}$ in $\mathbb{S}$ for each $w' \in \{x', y'\}^{\perp}$ in S'. So $|\{\alpha, \beta\}^{\perp}| \geq 3$. Similarly, if $(B2)$ holds then $|\{\alpha, \beta\}^{\perp}| \geq 3$. $\square$

Lemma 4.2. *Assume that* $(B3)$ *holds. Then* $d(\alpha, \beta) = 3$.

Proof. Clearly $d(\alpha, \beta) \geq 3$ since $u' \notin x'^{\perp}$. Let $v' \in \{u', x'\}^{\perp}$ in S'. Then x, v', v, u' is a path of length 3 in $\Gamma(\mathbb{P})$. So $d(\alpha, \beta) = 3$. □

Lemma 4.3. *Assume that* $(B4)$ *or* $(B5)$ *holds. Then* $|\{\alpha, \beta\}^{\perp}| \geq 3$.

Proof. Assume that $(B4)$ holds. If $x \sim y$ in S, then $v' \in x'y'$ in S'. If $v' = x'$, then $v', (x, y')$ and $(x, v' * y')$ are in $\{\alpha, \beta\}^{\perp}$. If $v' \neq x'$, then $v', (x, x')$ and $(x, v' * x')$ are in $\{\alpha, \beta\}^{\perp}$.

If $x \nsim y$ in S, let $\{x', y'\}^{\perp} = \{u', v', w'\}$ in S'. Then $v', (x, u')$ and (x, w') are in $\{\alpha, \beta\}^{\perp}$. A similar argument applies if $(B5)$ holds. □

Lemma 4.4. *Assume that* $(B6)$ *or* $(B7)$ *holds. Then* $d(\alpha, \beta) = 3$.

Proof. Assume that $(B6)$ holds. Suppose that $\theta \in \{\alpha, \beta\}^{\perp}$. Then $\theta \neq v'$, since $v' \notin x'^{\perp}$ in S'. So $\theta = (x, w')$ for some $w' \in x'^{\perp}$. Then $\theta \sim \beta$ implies that $v' \in x'^{\perp}$ in S', a contradiction. So $d(\alpha, \beta) \neq 2$. Now, $y \sim \beta$ and $d(\alpha, y) = 2$ (Lemma 4.1). So $d(\alpha, \beta) = 3$. A similar argument can be applied if $(B7)$ holds. □

Note that the embedding of $\mathcal{S}$ into $\mathbb{S}$ is isometric. As a consequence of the above results of this section together with Corollary 3.4, we have

Corollary 4.5. *The diameter of* $\mathbb{S}$ *is 3.*

We prove that property (NP) is satisfied in $\mathbb{S}$.

Proposition 4.6. *Let* $\mathtt{L}$ *be a line of* $\mathbb{S}$ *of type* $(\mathbb{L}_1)$ *and* α *be a point of* $\mathbb{S}$ *not contained in* $\mathtt{L}$. *Then* $\mathtt{L}$ *contains a unique point nearest to* α.

Proof. Let $\mathtt{L} = \{x, \beta, u'\}$ where $\beta = (x, u') \in \mathcal{P}$. Let $\alpha = v' \in P'$. Then $v' \neq u'$ and $d(\alpha, u') = 2$ (Lemma 4.1). Now $d(\alpha, \beta) = 2$ or 3 according as $x \in v^{\perp}$ in S or not. In the first case, $\alpha \sim x$, and in the latter case, $d(\alpha, x) = 3$ (Lemma 4.2). A similar argument holds if $\alpha \in P$.

Let $\alpha = (y, v') \in \mathcal{P}$. If $x = y$, then $u' \neq v'$ and $x \in \{u, v\}^{\perp}$ in S. So $\alpha \sim x$ and $d(\alpha, \beta) = d(\alpha, u') = 2$ (Lemmas 3.1 and 4.3). Similarly, if $u' = v'$ then $\alpha \sim u'$ and $d(\alpha, \beta) = d(\alpha, x) = 2$. Assume that $x \neq y$ and $u' \neq v'$. If $\alpha \sim \beta$, then $u' \in y'^{\perp}$ and $v' \in x'^{\perp}$ in S'. So $d(\alpha, x) = d(\alpha, u') = 2$ (Lemma 4.3). If $d(\alpha, \beta) = 2$, then $u' \notin y'^{\perp}$ and $v' \notin x'^{\perp}$ in S'. By Lemma 4.4, $d(\alpha, x) = d(\alpha, u') = 3$. If $d(\alpha, \beta) = 3$, then either $u' \in y'^{\perp}$ and $v' \notin x'^{\perp}$, or $u' \notin y'^{\perp}$ and $v' \in x'^{\perp}$ in S'. Then $d(\alpha, x) = 3$ and $d(\alpha, u') = 2$ in the first case, and $d(\alpha, x) = 2$ and $d(\alpha, u') = 3$ in the latter. □

Now, let $\mathtt{L} = \{\beta, \theta, \gamma\} \in \mathcal{L}$ be a line of $\mathbb{S}$ and $\alpha \in P \cup P'$. We take $\beta = (x, u'), \theta = (y, v')$ and $\gamma = (z, w')$.

Proposition 4.7. *If α has distance 2 from two points of* $\mathtt{L}$*, then it is collinear with the third point of* $\mathtt{L}$*.*

Proof. Let $\alpha = q' \in P'$ and $d(\alpha, \beta) = d(\alpha, \theta) = 2$. Then $q' \notin \{u', v'\}$ and $x, y \in q^{\perp}$ in S. Thus, $q' \in \{x', y'\}^{\perp} = \{u', v', w'\}$ in S'. So $q' = w'$ and $\alpha \sim \gamma$. A similar argument holds if $\alpha \in P$. □

Proposition 4.8. *If α has distance 3 from two points of* $\mathtt{L}$*, then it has distance 2 to the third point of* $\mathtt{L}$*.*

Proof. Let $\alpha = q' \in P'$ and $d(\alpha, \beta) = d(\alpha, \theta) = 3$. Then $q' \notin \{u', v'\}$ and $x, y \notin q^{\perp}$ in S. So $q' \neq w'$ and $q' \in z'^{\perp}$ in S'. The latter follows from the fact that $\{x, y, z\}$ is a line or a complete triad of S. Thus, $d(\alpha, \gamma) = 2$. A similar argument holds if $\alpha \in P$. □

Proof of Theorem 2.2. By the results of this section together with Theorem 2.1, $\mathbb{S}$ is a slim dense near hexagon. Since $|\mathcal{P}| = 135$, Theorem 1.1 completes the proof.

References

[1] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink, Near polygons and Fischer spaces, *Geom. Dedicata* **49** (1994), no. 3, 349–368.

[2] A. E. Brouwer and H. A. Wilbrink, The structure of near polygons with quads, *Geom. Dedicata* **14** (1983), no. 2, 145–176.

[3] F. Buekenhout and E. Shult, On the foundations of polar geometry, *Geom. Dedicata* **3** (1974), 155–170.

[4] P. J. Cameron, Dual polar spaces, *Geom. Dedicata* **12** (1982), no. 1, 75–85.

[5] B. De Bruyn, A new geometrical construction for the near hexagon with parameters $(s, t, T_2) = (2, 5, \{1, 2\})$, *J. Geom.* **78** (2003), no. 1–2, 50–58.

[6] B. De Bruyn, "Near Polygons", Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.

[7] B. De Bruyn and P. Vandecasteele, Near polygons with a nice chain of sub near polygons, *J. Combin. Theory Ser. A* **108** (2004), 297–311.

[8] B. De Bruyn and P. Vandecasteele, The classification of the slim dense near octagons, *European J. Combin.* **28** (2007), 410–428.

[9] S. E. Payne and J. A. Thas, "Finite Generalized Quadrangles," Research Notes in Mathematics, 110, Pitman (Advanced Publishing Program), Boston, MA, 1984.

[10] B. K. Sahoo, Generalized quadrangles of order $(2, t)$, *Pragmatic Algebra*, 157–170, SAS Int. Publ., Delhi, 2006.

[11] E. Shult and A. Yanushka, Near n-gons and line systems, *Geom. Dedicata* **9** (1980), no. 1, 1–72.